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MODAL LOGIC FOR 3D INCIDENCE GEOMETRY

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Three sorted geometrical space of incidence is represented as equivalent uni-sorted objects structure of incidences, and a modal logic using three equivalence relations and a difference relation is used to axiomatize that class. Completeness theorem is proven.

Keywords: Modal Logic of Irreflexivity, Incidence Geometry, Equivalence Relations, Axiomatization, Completeness Theorem.

1. INTRODUCTION

The incidence 3D geometry consists of 3 different sorts of objects: points, lines and plains and 3 relations called incidences. We introduce an equivalent one-sort geometrical structure, called a structure of incidence, which is suitable for modal considerations. The approach is the same as in the papers of Balbiani et al. [1], [2] but extended to 3D geometry.

In the beginning we present the 3D geometrical space of incidence and the one-sort geometrical structure of incidence. The category of incidence spaces corresponds closely to the geometry and its properties and its semantics is taken from it. One-sort geometrical structure of incidence is a structure which contains only one sort objects and 3 equivalence relations. Each object can play as a point, a line or a plane at the same time, and the incidence relations are expressed as composition of the equivalence relations. The equivalence of the category of the incidence spaces and the category of the structures of incidence is proven by defining functors from the incidence spaces to the structures of incidence and from the structures of incidence to the incidence spaces.

The one-sort objects based structures of incidence with 3 relations of equivalence and the difference relation are suitable for frames of a modal logic. The language of the modal logic contains 3 unary modalities for the equivalence relations and the difference operator representing inequality. The semantics uses the structures of incidence for frames of this logic.

The deductive system uses well-known rules as *Modus Ponens*, *Generalization* for each modal symbol, and the irreflexivity rule proposed by Gabbay [3]. The completeness proof does not use the *irreflexivity* rule directly but replaces it with an *infinitary* rule deductively equivalent to it. For that equivalent deductive system we prove the completeness. The completeness is proved using maximal consistent theories and the canonical frame and model. The completeness of the original system is a consequence of the deductive equivalence between these two rules.

The geometrical modal logic is derived from the minimal one with adding several axioms for each property of the incidence frames. Each property of the incidence frames is axiomatized and it is a canonical property. So the proposed finite axiomatization is the axiomatization of the logic which frames are the structures of incidence.

2. INCIDENCE GEOMETRY AND INCIDENCE FRAMES

First we show briefly the category of 3D geometrical incidence space. It is consisted of points, lines and planes, and the relations: a point belongs to a line, a line lays into a plane, and a point lays into a plane. So these relations are called incidence relations. Another relation which is also important is the difference between 2 points, 2 lines and 3 planes.

The incidence frames are explained afterward and the relations of incidences are replaced with 3 equivalence relations and the difference. The definition of the incidence frames and the equivalence between incidence frames and the category of incidence geometry is the topic of this first chapter.

2.1. THE CATEGORY OF INCIDENCE SPACES AND THE INCIDENCE GEOMETRY

Definition 2.1. Incidence space *we call any multi-sort system of the type* $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$, *where:*

- Po *is a non-empty set which elements are called* ***points***. *We note them with upper case Latin letters.*

- Li *is a non-empty set which elements are called* ***lines***. *We note them with lower case Latin letters.*

- *Pl is a non-empty set which elements are called* **planes**. *We note them with Greek letters.*

- $\varepsilon_{1.2} \subseteq Po \times Li, \varepsilon_{1.2}$ *is a two-sort relation between points and lines. It says that a point is into a line.*

- $\varepsilon_{1.3} \subseteq Po \times Pl, \varepsilon_{1.3}$ *is a two-sort relation between points and planes. It says that a point lays onto a plane.*

- $Po \cap Li = \emptyset, Po \cap Pl = \emptyset, Li \cap Pl = \emptyset$. *There aren't any objects that are points and lines. points and planes or planes or lines.*

- *Po. Li. Pl.* $\varepsilon_{1.2}$ *and* $\varepsilon_{1.3}$ *must have the properties (geometry axioms) below:*

1. $(\exists X \in Po, \exists Y \in Po)(X \neq Y)$. *There are at least 2 different points.*

2. $(\forall X \in Po, \forall Y \in Po)(\exists z \in Li)(X\varepsilon_{1,2}z \wedge Y\varepsilon_{1,2}z)$. *For each 2 points there is a line which goes through them. The points are incident with the line.*

3. $(\forall X \in Po, \forall Y \in Po)(\forall z \in Li, \forall t \in Li)(X \neq Y \wedge X\varepsilon_{1,2}z \wedge Y\varepsilon_{1,2}z \wedge X\varepsilon_{1,2}t \wedge Y\varepsilon_{1,2}t \Rightarrow z = t)$. *For 2 different points there is maximum one line which is incident with them.*

4. $(\forall z \in Li)(\exists X \in Po, \exists Y \in Po)(X \neq Y \wedge X\varepsilon_{1,2}z \wedge Y\varepsilon_{1,2}z)$. *For each line there are at least 2 different points that are incident with the line.*

5. $(\forall z \in Li)(\exists X \in Po)(\neg X \in z)$. *For each line there is a point that is not incident with the line.*

6. $(\forall X \in Po, \forall Y \in Po, \forall Z \in Po)(\exists \alpha \in Pl)(X\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\alpha \wedge Z\varepsilon_{1,3}\alpha)$. *For each 3 points there is a plane that is incident with them.*

7. $(\forall X \in Po, \forall Y \in Po, \forall Z \in Po)(\forall \alpha \in Pl, \forall \beta \in Pl)((\forall l \in Li)((\neg X \in l) \vee (\neg Y \in l) \vee (\neg Z \in l)) \wedge X\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\alpha \wedge Z\varepsilon_{1,3}\alpha \wedge X\varepsilon_{1,3}\beta \wedge Y\varepsilon_{1,3}\beta \wedge Z\varepsilon_{1,3}\beta \Rightarrow \alpha = \beta)$. *For each 3 points that is not incident with the same line, there is maximum one plane that is incident with all 3 points.*

8. $(\forall \alpha \in Pl)(\exists X \in Po)(X\varepsilon_{1,3}\alpha)$. *For each plane there is a point that is incident with the plane.*

9. $(\forall \alpha \in Pl)(\exists X \in Po)(\neg X\varepsilon_{1,3}\alpha)$. *For each plane there is a point that is not incident with the plane.*

10. $(\forall l \in Li)(\forall \alpha \in Pl)(\forall X \in Po, \forall Y \in Po)(X \neq Y \wedge X\varepsilon_{1,2}l \wedge Y\varepsilon_{1,2}l \wedge X\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\alpha \Rightarrow (\forall Z \in Po)(Z\varepsilon_{1,2}l \Rightarrow Z\varepsilon_{1,3}\alpha))$. *If 2 different points are incident with a line and with a plane at the same time, then each point which is incident with the line is incident with the plane too. So if 2 different points from a line are incident with the plane then the whole line lays onto the plane.*

11. $(\forall\alpha \in Pl)(\forall\beta \in Pl)(\exists X \in Po)(X\varepsilon_{1,3}\alpha \wedge X\varepsilon_{1,3}\beta \Rightarrow (\exists Y \in Po)(X \neq Y \wedge Y\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\beta))$ *If 2 planes have a point which is incident with both. then the planes have another point different from the first one which is also incident with the 2 planes.*

These relations $\varepsilon_{1,2}$ and $\varepsilon_{1,3}$ we call the incidence relations.

The relation $\varepsilon_{2,3} \subseteq Li \times Pl$ which says that *a line lays onto a plane*, is expressible with the incidence relations $\varepsilon_{1,2}$ and $\varepsilon_{1,3}$.

Definition 2.2. The line lays onto a plane *if any point that is incident with the line is incident with the plane. We can express that relation* $\varepsilon_{2,3} \subseteq Li \times Pl$ *with the equivalence:*

$l\varepsilon_{2,3}\alpha \Leftrightarrow (\forall X \in Po)(X\varepsilon_{1,2}l \Rightarrow X\varepsilon_{1,3}\alpha)$, *where l is a line and α is a plane.*

This way the relation if a line is incident with a plane (lays onto a plane) is expressible with the other 2 incidence relations.

The incidence spaces we call them just spaces. And we turn them into categories with defining the notion of homomorphism between spaces.

Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ and $S' = (Po', Li', Pl', \varepsilon'_{1,2}, \varepsilon'_{1,3})$ be 2 incidence spaces.

Definition 2.3. Homomorphism between incidence spaces. *We says that the 2 incidence spaces has a homomorphism f from S into S' if the f is a function with domain $Po \cup Li \cup Pl$ and range $Po' \cup Li' \cup Pl'$. $f : Po \cup Li \cup Pl \to Po' \cup Li' \cup Pl'$. which follows the conditions:*

1. $(\forall X \in Po)(\forall y \in Li)(\forall\alpha \in Pl)(f(X) \in Po' \wedge f(y) \in Li' \wedge f(\alpha) \in Pl')$
2. $(\forall X \in Po)(\forall y \in Li)(X\varepsilon_{1,2}y \Rightarrow f(X)\varepsilon'_{1,2}f(y))$
3. $(\forall X \in Po)(\forall\alpha \in Pl)(X\varepsilon_{1,3}\alpha \Rightarrow f(X)\varepsilon'_{1,3}f(\alpha))$

   f is an isomorphism if it follows the additional conditions:

4. $f : Po \to Po'$. $f : Li \to Li'$. $f : Pl \to Pl'$ *all these are bijective.*
5. $(\forall X \in Po)(\forall y \in Li)(f(X)\varepsilon'_{1,2}f(y) \Rightarrow X\varepsilon_{1,2}y)$
6. $(\forall X \in Po)(\forall\alpha \in Pl)(f(X)\varepsilon'_{1,3}f(\alpha) \Rightarrow X\varepsilon_{1,3}\alpha)$

2.2. STRUCTURE OF INCIDENCE

The aim is to introduce a new kind of structures which contains only one sort objects, and it is suitable for frames of modal languages. The new structures are equivalent to the incidence spaces and the properties of the incidence spaces are translated as properties of the structures.

So we introduce a construction with which from an incidence space we can create a new kind of one-sort structure. That structures are the structure of incidence.

Definition 2.4. *Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ be an incidence space and we call the structure $\underline{W}(S) = (W(S), \equiv_1, \equiv_2, \equiv_3)$* structure of incidence over an incidence space S. *if:*

1. *The set W(S) is defined as:*

$$W(S) = \{(X, y, \alpha) | (X \in Po) \wedge (y \in Li) \wedge (\alpha \in Pl) \wedge (X\varepsilon_{1,2}y) \wedge (y\varepsilon_{2,3}\alpha)\}$$

2. *The relations $\equiv_1, \equiv_2, \equiv_3$ are defined as:*

$$(X_1, y_1, \alpha_1) \equiv_1 (X_2, y_2, \alpha_2) \Leftrightarrow X_1 = X_2$$

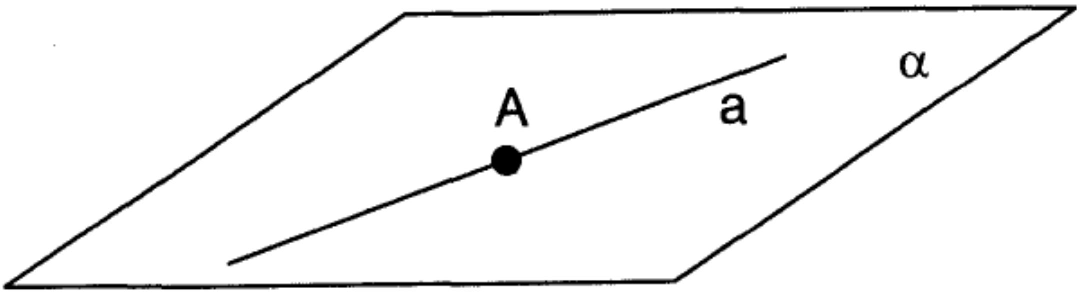
$$(X_1, y_1, \alpha_1) \equiv_2 (X_2, y_2, \alpha_2) \Leftrightarrow y_1 = y_2$$

$$(X_1, y_1, \alpha_1) \equiv_3 (X_2, y_2, \alpha_2) \Leftrightarrow \alpha_1 = \alpha_2$$

where $(X_1, y_1, \alpha_1) \in W(S)$ and $(X_2, y_2, \alpha_2) \in W(S)$.

The relations $\equiv_1, \equiv_2, \equiv_3$ are equivalence relations.

Elements of $W(S)$ are triples of a point, a line and a plane, such that the point is incident with the line and the line is incident with the plane. Each triple of $W(S)$ plays as a point, a line and a plane at the same time. See the figure:



As subsequence of that, the point is incident with the plane too - if $(X, y, \alpha) \in W(S)$ then $X\varepsilon_{1,3}\alpha$. We call $W(S)$ only W for shortly.

Definition 2.5. *The relations $\in_{1,2}, \in_{1,3}, \in_{2,3}$ into $W(S)$ defined with the equations are called* structure incidence relations*:*

$$(X_1, y_1, \alpha_1) \in_{1,2} (X_2, y_2, \alpha_2) \Leftrightarrow X_1\varepsilon_{1,2}y_2$$

$$(X_1, y_1, \alpha_1) \in_{1,3} (X_2, y_2, \alpha_2) \Leftrightarrow X_1\varepsilon_{1,3}\alpha_2$$

$$(X_1, y_1, \alpha_1) \in_{2,3} (X_2, y_2, \alpha_2) \Leftrightarrow y_1\varepsilon_{2,3}\alpha_2$$

where $(X_1, y_1, \alpha_1) \in W(S)$ and $(X_2, y_2, \alpha_2) \in W(S)$.

These relations $\in_{1,2}, \in_{1,3}, \in_{2,3}$ which corresponds to the incidence relations are expressible as compositions of the equivalence $S5$ relations $\equiv_1, \equiv_2, \equiv_3$

Lemma 2.1. *The equations are valid:*

$$\in_{1,2} = \equiv_1 \circ \equiv_2$$

$$\in_{1,3} = \equiv_1 \circ \equiv_3$$

$$\in_{2,3} = \equiv_2 \circ \equiv_3$$

Proof. Using the definitions 2.1, 2.4, 2.5 it is easy to prove that:

$$\overline{x} \in_{1,2} \overline{y} \Leftrightarrow (\exists \overline{z} \in W)(\overline{x} \equiv_1 \overline{z} \wedge \overline{z} \equiv_2 \overline{y})$$

$$x \in_{1,3} y \Leftrightarrow (\exists z \in W)(x \equiv_1 z \wedge z \equiv_3 y)$$

$$x \in_{2,3} y \Leftrightarrow (\exists z \in W)(x \equiv_2 z \wedge z \equiv_3 y)$$

The proofs of the implications above in the direction "$\Leftarrow$" are very simple. We show the proof for $(\exists \overline{z} \in W)(\overline{x} \equiv_1 \overline{z} \wedge \overline{z} \equiv_2 \overline{y}) \Rightarrow \overline{x} \in_{1,2} \overline{y}$:

Let $\overline{x} = (X, x, \alpha)$, $\overline{y} = (Y, y, \beta)$, $\overline{z} = (Z, z, \gamma)$.

If for some $\overline{z} \in W$ is true that $\overline{x} \equiv_1 \overline{z}$ and $\overline{z} \equiv_2 \overline{y}$ then from the definition of $\equiv_1$ and $\equiv_2$ it follows that $X = Z$ and $z = y$. And from $Z\varepsilon_{1,3}z$ we conclude that $X\varepsilon_{1,3}y$ which is $(X, x, \alpha) \in_{1,2} (Y, y, \beta)$ and that's the definition of $\overline{x} \in_{1,2} \overline{y}$. The other 2 implications are proved in the same way.

Proofs of the implications of the direction "$\Rightarrow$" are also simple. We show the proof for $\overline{x} \in_{1,3} \overline{y} \Rightarrow (\exists \overline{z} \in W)(\overline{x} \equiv_1 \overline{z} \wedge \overline{z} \equiv_3 \overline{y})$:

Let $\overline{x} = (X, x, \alpha)$, $\overline{y} = (Y, y, \beta)$. We must find a suitable $\overline{z} = (Z, z, \gamma)$. From $\overline{x} \in_{1,3} \overline{y}$ it follows that $X\varepsilon_{1,3}\beta$ and from $\overline{x} \in W(S)$ follows that $X\varepsilon_{1,3}\alpha$. So both planes α and β have a common point X. From the definition 2.1 axiom (12) it follows that there is another point Z different from X, which belongs to both planes. $\exists Z \in Po, X \neq Z \wedge Z\varepsilon_{1,3}\alpha \wedge Z\varepsilon_{1,3}\beta$. For the 2 different points X and Z from the definition 2.1 axiom (3) there is a line z that is incident with the 2 points, $\exists z \in Li, X\varepsilon_{1,2}z \wedge Z\varepsilon_{1,2}z$ We choose $\overline{z} = (X, z, \beta)$ and we have proved so far that $X \neq Z \wedge X\varepsilon_{1,3}\beta \wedge Z\varepsilon_{1,3}\beta \wedge X\varepsilon_{1,2}z \wedge Z\varepsilon_{1,2}z$, then from the definition 2.1 axiom (11) we can conclude that it is true $(\forall T \in Po)(T\varepsilon_{1,2}z \Rightarrow T\varepsilon_{1,3}\beta)$, which is the definition of $z\varepsilon_{2,3}\beta$. So we discovered a triple $\overline{z} = (X, z, \beta)$ such that $X\varepsilon_{1,2}z \wedge z\varepsilon_{2,3}\beta$ so $\overline{z} \in W(S)$, and $(X, x, \alpha) \equiv_1 (X, z, \beta)$, and $(X, z, \beta) \equiv_3 (Y, y, \beta)$, finally the $\overline{z} = (X, z, \beta)$ suffice $x \equiv_1 z \wedge z \equiv_3 y$.

The remaining 2 implications are simple. □

The reverted relations $\in_{1,2}^{-1}, \in_{1,3}^{-1}, \in_{2,3}^{-1}$ of $\in_{1,2}, \in_{1,3}, \in_{2,3}$ are also expressed with a composition of the equivalence relations.

$$\in_{1,2}^{-1} = \equiv_2 \circ \equiv_1$$

$$\in_{1,3}^{-1} = \equiv_3 \circ \equiv_1$$

$$\in_{2,3}^{-1} = \equiv_3 \circ \equiv_1$$

Lemma 2.2. *Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ be an incidence space and $W(S)$ is the structure of incidence over S. Then the following conditions are true:*

1. *If X is a point then exist a line x and a plane α such that the triple $(X, x, \alpha) \in W(S)$.*
2. *If x is a line then exist a point X and a plane α such that the triple $(X, x, \alpha) \in W(S)$.*
3. *If α is a plane then exist a point X and a line x such that the triple $(X, x, \alpha) \in W(S)$.*

Proof. For the first one:

Let $X \in Po$ is a point, then from the definition 2.1 axiom (1), there is 2 different points $Y_1 \in Po$, and $Y_2 \in Po$ and $Y_1 \neq Y_2$. So because $Y_1 \neq Y_2$ then if $X = Y_1$ then $X \neq Y_2$. Let's note with Y the point of Y_1 and Y_2 which is different from X.

Apply the definition 2.1 axiom (3) and let y be the line incident with X and Y. $X\varepsilon_{1,2}y \wedge Y\varepsilon_{1,2}y$.

Apply the definition 2.1 axiom (7) for the 3 points X, Y, Y and let γ be the plane that is incident with X and Y.

From the definition 2.1 axiom (11) and from the definition 2.2 for
$X \neq Y \wedge X\varepsilon_{1,2}y \wedge Y\varepsilon_{1,2}y \wedge X\varepsilon_{1,3}\gamma \wedge X\varepsilon_{1,3}\gamma$ we conclude that $y\varepsilon_{2,3}\gamma$.

Similar reasoning proves 2 and 3. □

The meaning of this lemma is that each point, line or plane can be completed with redundant points, lines and planes to produce a triple that belong to the structures of incidences. This way the working with multi-sort points, lines and planes can be replaced with one-sort objects which are points, lines and planes at the same time. All geometrical properties can be translated as properties of these one sort objects.

Lemma 2.3. *Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ be an incidence space and $\underline{W}(S)$ $(W(S), \equiv_1, \equiv_2, \equiv_3)$ is the structure of incidence over S. Then for the structure of incidence $W(S)$ the following conditions are true.*

- *The relations $\equiv_1, \equiv_2, \equiv_3$ are equivalence relations and they follow the conditions below:*

* $(\forall x \in W(S))(\forall y \in W(S))((x \equiv_1 y) \wedge (x \equiv_2 y) \wedge (x \equiv_3 y) \Rightarrow x = y)$

** $(\forall x \in W(S))(\forall y \in W(S))(\forall z \in W(S))((x \in_{1,2} y) \wedge (y \in_{2,3} z) \Rightarrow (\exists t \in W(S))((x \equiv_1 t) \wedge (y \equiv_2 t) \wedge (z \equiv_3 t)))$

*** $(\forall x \in W(S))(\forall y \in W(S))((\forall z \in W(S))(z \in_{1,2} x \wedge z \in_{1,3} y) \Rightarrow x \in_{2,3} y)$

The next conditions correspond to the geometrical axioms of the incidence space:

1. $(\exists x \exists y \in W(S))(\neg x \equiv_1 y)$

2. $(\forall x \forall y \in W(S))(\exists z \in W(S))((x \in_{1,2} z) \wedge (y \in_{1,2} z))$

3. $(\forall x \forall y \forall z \forall t \in W(S))((\neg x \equiv_1 y) \wedge (x \in_{1,2} z) \wedge (y \in_{1,2} z) \wedge (x \in_{1,2} t) \wedge (y \in_{1,2} t) \Rightarrow (z \equiv_2 t))$

4. $(\forall x \exists y \exists z \in W(S))((\neg y \equiv_1 z) \wedge (y \in_{1,2} x) \wedge (z \in_{1,2} x))$

5. $(\forall x \exists y \in W(S))(\neg y \in_{1,2} x)$

6. $(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(x \in_{1,3} t \wedge y \in_{1,3} t \wedge z \in_{1,3} t)$

7. $(\forall x \forall y \forall z \in W(S))(\forall u \forall v \in W(S))((x \in_{1,3} u) \wedge (y \in_{1,3} u) \wedge (z \in_{1,3} u) \wedge (x \in_{1,3} v) \wedge (y \in_{1,3} v) \wedge (z \in_{1,3} v) \wedge (\forall l \in W(S))((\neg x \in_{1,2} l) \vee (\neg y \in_{1,2} l) \vee (\neg z \in_{1,2} l)) \Rightarrow (u \equiv_3 v))$

8. $(\forall x \exists y \in W(S))(y \in_{1,3} x)$

9. $(\forall x \exists y \in W(S))(\neg y \in_{1,3} x)$

10. $(\forall x \forall y \forall z \forall t \in W(S))((\neg x \equiv_1 y) \wedge (x \in_{1,2} z) \wedge (y \in_{1,2} z) \wedge (x \in_{1,3} t) \wedge (y \in_{1,3} t) \Rightarrow (z \in_{2,3} t))$

11. $(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))((z \in_{1,3} x) \wedge (z \in_{1,3} y) \Rightarrow (\neg t \equiv_1 z) \wedge (t \in_{1,3} x) \wedge (t \in_{1,3} y))$

Proof. It is a simple check with applying the definitions and use the lemma 2.2. □

Thus the lemma 2.3 gives us the confidence to introduce the next abstract definition od the notion of *structure of incidence*.

Definition 2.6. *We say that the structure* $\underline{\underline{W}} = (W, \equiv_1, \equiv_2, \equiv_3)$ *is an incidence structure if the set is a non empty set* $W \neq \emptyset$ *and the relations* $\equiv_1, \equiv_2, \equiv_3$ *are relations of equivalence, and they follow all the conditions from the* lemma 2.3, *where* $\in_{1,2} = \equiv_1 \circ \equiv_2, \in_{1,3} \equiv_1 \circ \equiv_3, \in_{2,3} = \equiv_2 \circ \equiv_3$.

Now we turn the class of the structures of incidences into a category with introducing the notion of the homomorphism.

Definition 2.7. *Let* $\underline{\underline{W}} = (W, \equiv_1, \equiv_2, \equiv_3)$ *and* $\underline{\underline{W'}} = (W', \equiv'_1, \equiv'_2, \equiv'_3)$ *be two structures of incidence, and* $f : W \to W'$ *is a function. We says that* f *is a* homomorphism *if it follows the next condition:*

1. $(\forall x \in W)(\forall y \in W)(x \equiv_i y \Rightarrow f(x) \equiv'_i f(y))$ *for each* $i = 1, 2, 3$

   We says that f *is a* isomorphism *if it follows the next conditions:*

2. f *is a bijection.*

3. $(\forall x \in W)(\forall y \in W)(f(x) \equiv_i' f(y) \Rightarrow x \equiv_i y)$ *for each* $i = 1, 2, 3$

The category of the structures of incidences we note with Φ_i. And the category of incidence spaces we note with Σ_i.

2.3. EQUIVALENCE BETWEEN THE CATEGORIES OF THE INCIDENCE SPACES AND THE STRUCTURES OF INCIDENCES

Similarly to the functional correspondence from definition 2.4 which for each incidence space finds a structure of incidence, we make another functional correspondence which for each structure of incidence finds an incidence space.

Let $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3)$ be a structure of incidence then we can split the set W into equivalence classes with each equivalence relations.

Definition 2.8. *For each* $x \in W$ *we define the classes:*

$$|x|_1 = \{y \in W | x \equiv_1 y\}, |x|_2 = \{y \in W | x \equiv_2 y\}, |x|_3 = \{y \in W | x \equiv_3 y\}$$

These classes are equivalence classes.

Definition 2.9. *Let* $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3)$ *be a structure of incidence then we define* $S(\underline{W})$ *to be the structure* $S(\underline{W}) = (Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W}))$ *where:*

$$Po(\underline{W}) = W/\equiv_1 = \{|x|_1 | x \in W\}$$

$$Li(\underline{W}) = W/\equiv_2 = \{|x|_2 | x \in W\}$$

$$Pl(\underline{W}) = W/\equiv_3 = \{|x|_3 | x \in W\}$$

$$\varepsilon_{1,2}(\underline{W}) = \{(|x|_1, |y|_2) | x\epsilon_{1,2}y\}$$

$$\varepsilon_{1,3}(\underline{W}) = \{(|x|_1, |y|_3) | x\epsilon_{1,3}y\}$$

Lemma 2.4. *The following conditions are true:*

1. *The definition of the relations* $\varepsilon_{1,2}(\underline{W})$ *and* $\varepsilon_{1,3}(\underline{W})$ *is correct.*
2. $Po(\underline{W}) \cap Li(\underline{W}) = \emptyset, Po(\underline{W}) \cap Pl(\underline{W}) = \emptyset, Li(\underline{W}) \cap Pl(\underline{W}) = \emptyset$

Proof. For 1 we have to proof that the relations $\varepsilon_{1,2}(\underline{W})$ and $\varepsilon_{1,3}(\underline{W})$ are independent from the concrete representatives of the equivalence classes.

From the definition 2.6 it follows that $\in_{1,2} = \equiv_1 \circ \equiv_2, \in_{1,3} \equiv_1 \circ \equiv_3$ and the relations $\equiv_1, \equiv_2, \equiv_3$ are relations of equivalence.

So let $x\epsilon_{1,2}y \wedge x \equiv_1 x' \wedge y \equiv_2 y'$. From $\in_{1,2} \equiv_1 \circ \equiv_2$ it follows that there is $z \in W$ such that $x \equiv_1 z \wedge z \equiv_2 y$. Because the $\equiv_1$ and $\equiv_2$ are equivalence relations it follows that $x' \equiv_1 z \wedge z \equiv_2 y'$, thus we conclude that $x'\epsilon_{1,2}y'$. The same way we can see that for any x, x', y, y' it is true $x\epsilon_{1,3}y \wedge x \equiv_1 x' \wedge y \equiv_3 y' \Rightarrow x'\epsilon_{1,3}y'$

We can prove 2 with accepting the opposite and proof the contradiction. Let us assume that there are $|x|_1, |x|_2, |y|_2, |y|_3$ equivalence classes such that $|x|_1|y|_2$ or $|x|_1 = |y|_3$, or $|x|_2 = |y|_3$, and from that it follows — there is $x \in W$ such that $|x|_1|x|_2$ or $|x|_1 = |x|_3$, or $|x|_2 = |x|_3$.

Let us assume that there is $x \in W$ such that $|x|_1|x|_2$. From the definition 2.6 of the incidence structure the axiom (4) we know that $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_1 z) \wedge y \in_{1,2} x \wedge z \in_{1,2} x)$. Applying that "axiom" for the x we find $y \in W, z \in W$ and $y \in_{1,2} x$ and $z \in_{1,2} x$ and $(\neg y \equiv_1 z)$. From the definition of $\in_{1,2}$ and $\in_{1,3}$, it follows that there are $u \in W$ and $v \in W$ such that $y \equiv_1 u \wedge u \equiv_2 x$ and $z \equiv_1 v \wedge v \equiv_2 x$. So $u \in |x|_2, v \in |x|_2$. Form $(\neg y \equiv_1 z) \wedge z \equiv_1 v \wedge y \equiv_1 u$ we conclude that $(\neg u \equiv_1 v)$. But if $|x|_1 = |x|_2$ and $u \in |x|_2, v \in |x|_2$ then $u \in |x|_1, v \in |x|_1$ and $u \equiv_1 v$ contradiction with $(\neg u \equiv_1 v)$. Finally we proved that for any $x \in W$ it is true that $|x|_1 \neq |x|_2$.

From the definition 2.6 axioms (8) and (11) it follows that:

$((\forall x \exists z \in W)(z \in_{1,3} x)) \wedge ((\forall x \forall x \forall z \in W)(\exists t \in W)(z \in_{1,3} x \wedge z \in_{1,3} x \Rightarrow (\neg t \equiv_1 z) \wedge t \in_{1,3} x \wedge t \in_{1,3} x))$ and from that we can conclude that for any "plane" there are 2 different "points" that belong to the "plane" $(\forall x \in W)(\exists z \in W)(\exists t \in W)((\neg t \equiv_1 z) \wedge z \in_{1,3} x \wedge t \in_{1,3} x)$. From that statement we can proof that $|x|_1 \neq |x|_3$ in the same way as for $|x|_1 \neq |x|_2$.

The proof of the4 statement $|x|_2 \neq |x|_3$ uses the proof of the statement $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_3 z) \wedge x \in_{2,3} y \wedge x \in_{2,3} z)$ which speaks that for any "line" there are 2 different "planes" that contain the "line", next using the definition of $\in_{2,3}$ it follows that there are $u \in W$ and $v \in W$ such that $x \equiv_2 u \wedge u \equiv_3 y$ and $x \equiv_2 v \wedge v \equiv_3 z$, because $(\neg y \equiv_3 z)$ then $(\neg u \equiv_3 v)$, but because $u \equiv_2 x \equiv_2 v$ then $u \in |x|_2$ and $v \in |x|_2$ and if we assume that $|x|_2 = |x|_3$ then $u \in |x|_3$ and $v \in |x|_3$ and it follows the contradiction $u \equiv_3 v$ with $\neg u \equiv_3 v$, so $|x|_3 \neq |x|_2$.

The proof of the fact $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_3 z) \wedge x \in_{2,3} y \wedge x \in_{2,3} z)$.

Let $x \in W$ is the "line". Using the definition 2.6 axiom (4) there are "points" $x_1 \in W$ and $x_2 \in W$ such that $(\neg x_1 \equiv_1 x_2) \wedge x_1 \in_{1,2} x \wedge x_2 \in_{1,2} x$. From axiom (6) there is $u \in W$ such that $x_1 \in_{1,3} u \wedge x_2 \in_{1,3} u$. For the "plane" u using axiom (9) we find a "point" $x_3 \in W$ such that $\neg x_3 \in_{1,3} u$. Using again the axiom (6) for the points x_1, x_2, x_3 there is a "plane" $v \in W$ which contains that points $x_1 \in_{1,3} v \wedge x_2 \in_{1,3} v \wedge x_3 \in_{1,3} v$. If we assume that $u \equiv_3 v$ then from $x_3 \in_{1,3} v$ it follows that there is $t \in W$ and $x_3 \equiv_1 t \wedge t \equiv_3 v$ and $v \equiv_3 u$, so there is $t \in W$ and $x_3 \equiv_1 t \wedge t \equiv_3 u$ which is $x_3 \in_{1,3} v$ contradiction with $\neg x_3 \in_{1,3} u$. So it is true that $\neg u \equiv_3 v$. And because both "planes" contains the "points" x_1, x_2 and $(\neg x_1 \equiv_1 x_2)$, and the "line" contains x_1, x_2 too, using axiom (10) we conclude that $x \in_{2,3} u \wedge x \in_{2,3} v$. Thus we proved that $(\forall x \in W)(\exists u \in W)(\exists v \in W)((\neg u \equiv_3 v) \wedge x \in_{2,3} u \wedge x \in_{2,3} v)$.

Note! The statement $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_2 z) \wedge y \in_{2,3} x \wedge z \in_{2,3} x)$ which says that for any "plane" there is 2 different "lines" that lays onto that "plane" has more complex proof.

Finally we proved that for any $x \in W$ it is true that: $|x|_1 \neq |x|_2$ and $|x|_1 \neq |x|_3$, and $|x|_3 \neq |x|_2$. Which proof that the sets $W/\equiv_1, W/\equiv_2, W/\equiv_3$ have no common elements. ☐

Lemma 2.5. *If $\underline{W}$ is a structure of incidence then $S(\underline{W})$ defined with the definition 2.9 is an incidence space.*

Proof. To prove that, we need to check each of the statements from definition of incidence space 2.1 for $Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W})$.

From the previous lemma 2.4 it follows that the defined sets and relations at the definition 2.9 are correct. Also no "point" is a "line", no "line" is a "plane" and no "plane" is a "point". So the statement $Po(\underline{W}) \cap Li(\underline{W}) = \emptyset, Po(\underline{W}) \cap Pl(\underline{W}) = \emptyset, Li(\underline{W}) \cap Pl(\underline{W}) = \emptyset$ is exactly (1) from the definition 2.1 of the incidence space. The rest of the statements from definition 2.1 - the statements from (2) until (12), are checked easily with using the corresponding "axiom" from (1) until (11) from the definition of the structure of incidence - the definition 2.6. ☐

Lemma 2.6. $|x|_2 \varepsilon_{2,3}(\underline{W}) |y|_3 \Leftrightarrow x \in_{2,3} y$

Proof. According to the definitions 2.2 and 2.9 the following equivalences are true: $|x|_2 \varepsilon_{2,3}(\underline{W}) |y|_3 \Leftrightarrow (\forall Z \in Po(\underline{W}))(Z \varepsilon_{1,2}(\underline{W}) |x|_2 \rightarrow Z \varepsilon_{1,3}(\underline{W}) |y|_3) \Leftrightarrow (\forall z \in W)(|z|_1 \varepsilon_{1,2}(\underline{W}) |x|_2 \rightarrow |z|_1 \varepsilon_{1,3}(\underline{W}) |y|_3) \Leftrightarrow (\forall z \in W)(z \in_{1,2} x \rightarrow z \in_{1,3} y)$

From the incidence structure property (***) it follows that $(\forall z \in W)(z \in_{1,2} x \rightarrow z \in_{1,3} y) \Rightarrow x \in_{2,3} y$

Let for any $z \in W$ be true that $z \in_{1,2} x$ and from $x \in_{2,3} y$ according to the incidence structure property (**) it follows $(\exists t \in W)(z \equiv_1 t \wedge x \equiv_2 t \wedge y \equiv_3 t)$. From lemma 2.1 from $z \equiv_1 t \wedge y \equiv_3 t$ it follows $z \in_{1,3} y$. So we proof $x \in_{2,3} y \Rightarrow (\forall z \in W)(z \in_{1,2} x \rightarrow z \in_{1,3} y)$. ☐

Theorem 2.1. Representing structures of incidences as spaces of incidences.

Let $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3)$ be a structure of incidence.

Let $S(\underline{W}) = (Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W}))$ be an incidence space over $S(\underline{W})$.

Let $\underline{W}(S(\underline{W})) = (W', \equiv_1', \equiv_2', \equiv_3)'$ be a structure of incidence over $S(\underline{W})$.

Then there is an isomorphism from the structure of incidence $\underline{W}$ to the structure of incidence $\underline{W}(S(\underline{W}))$. The structures of incidences $\underline{W}$ and $\underline{W}(S(\underline{W}))$ are isomorphic.

Proof. We define function $f : W \rightarrow W'$ this way $f(x)(|x|_1, |x|_2, |x|_3)$ for every $x \in W$. First we must check that the definition is correct. For every $x \in W$ we must check that $(|x|_1, |x|_2, |x|_3) \in W'$. So let $x \in W$, and from $\equiv_{1,2,3}$ equivalence relations then: $x \equiv_1 x \wedge x \equiv_2 x$ and $x \equiv_2 x \wedge x \equiv_3 x$. From the definition of 2.6 $x \in_{1,2} x$ and $x \in_{2,3} x$. From the definition 2.9 we conclude that $|x| \varepsilon_{1,2}(\underline{W}) |x|$ and from lemma 2.6 we conclude that $|x|\varepsilon_{2,3}(\underline{W})|x|$, so the triple $(|x|_1, |x|_2, |x|_3) \in W'$ according to the definition 2.9.

Check for that f is a homomorphism is also easy:

$x \equiv_{1,2,3} y \leftrightarrow |x|_{1,2,3} = |y|_{1,2,3} \leftrightarrow (|x|_1, |x|_2, |x|_3) \equiv_{1,2,3} (|y|_1, |y|_2, |y|_3) \leftrightarrow f(x) \equiv_{1,2,3} f(y)$, so f is a homomorphism.

If we finally proof that f is a bijection then we can conclude f is an isomorphism.

First we proof that f is a surjective. For any element of $(|x|_1, |y|_2, |z|_3) \in W'$ there is $t \in W$ such that $f(t) = (|x|_1, |y|_2, |z|_3)$. Let $(|x|_1, |y|_2, |z|_3) \in W'$. From the definition 2.4 we know that $|x|_1\varepsilon_{1,2}(\underline{W})|y|_2$ and $|y|_2\varepsilon_{2,3}(\underline{W})|z|_3$. From the definition 2.9 and from lemma 2.6 we conclude that $x \in 1,2y \wedge y \in 2,3z$. Now we apply the statement (**) from the definition of the structure of incidence 2.6, and we find $t \in W$ such that $x \equiv_1 t \wedge y \equiv_2 t \wedge z \equiv_3 t$, and for that t it is true that $(|t|_1, |t|_2, |t|_3) = (|x|_1, |y|_2, |z|_3)$. So f is a surjective.

Let's now proof that f is an injective function. So let $x, y \in W$ and $f(x) = f(y)$. From the definition of f we have that $(|x|_1, |x|_2, |x|_3) = (|y|_1, |y|_2, |y|_3)$ which means that the equivalence classes of x and y for each relation are the same: $|x|_1 = |y|_1$, $|x|_2 = |y|_2$ and $|x|_3 = |y|_3$. And we conclude that the elements x and y are equivalent by each relation: $x \equiv_1 y \wedge x \equiv_2 y \wedge x \equiv_3 y$. Now we use the statement (*) from the definition 2.6 and conclude that x and y are equal, $x = y$, which proofs that f is an injective function.

The defined here function f preserves the relations, and also is a bijection from W to W' So f is an isomorphism. □

Theorem 2.2. Representing spaces of incidences as structures of incidences.

Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ *is an incidence space.*

Let $\underline{W}(S) = (W, \equiv_1, \equiv_2, \equiv_3)$ *be the structure of incidence over the space* S.

Let $S(\underline{W}(S)) = (Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W}))$ *be the incidence space over the structure of incidence* $\underline{W}(S)$.

Then there is an isomorphism from the incidence space S *to the incidence space* $S(\underline{W}(S))$. *The incidence spaces incidences* S *and* $S(\underline{W}(S))$ *are isomorphic.*

Proof. We know from the definition 2.9 that:

$$Po(\underline{W}(S)) = W(S)/\equiv_1 = \{|(X, y, \alpha)|_1 | (X, y, \alpha) \in W(S)\}$$

$$Li(\underline{W}(S)) = W(S)/\equiv_2 = \{|(X, y, \alpha)|_2 | (X, y, \alpha) \in W(S)\}$$

$$Pl(\underline{W}(S)) = W(S)/\equiv_3 = \{|(X, y, \alpha)|_3 | (X, y, \alpha) \in W(S)\}$$

$$\varepsilon_{1,2}(\underline{W}(S)) = \{(|(X', y', \alpha')|_1, |(X'', y'', \alpha'')|_2) | (X', y', \alpha')\epsilon_{1,2}(X'', y'', \alpha'')\}$$

$$\varepsilon_{1,3}(\underline{W}(S)) = \{(|(X', y', \alpha')|_1, |(X'', y'', \alpha'')|_3) | (X', y', \alpha')\epsilon_{1,3}(X'', y'', \alpha'')\}$$

And from definition 2.4 we know the equivalences:

$(X',y',\alpha')\epsilon_{1,2}(X'',y'',\alpha'') \leftrightarrow X'\varepsilon_{1,2}y''$ and $(X',y',\alpha')\epsilon_{1,3}(X'',y'',\alpha'') \leftrightarrow X'\varepsilon_{1,3}\alpha''$. This way we see that the relations $\varepsilon_{1,2}(\underline{W}(S)), \varepsilon_{1,3}(\underline{W}(S))$ follows the next conditions:

$$|(X',y',\alpha')|_1\varepsilon_{1,2}(\underline{W}(S))|(X'',y'',\alpha'')|_2 \leftrightarrow X'\varepsilon_{1,2}y''$$

$$|(X',y',\alpha')|_1\varepsilon_{1,3}(\underline{W}(S))|(X'',y'',\alpha'')|_2 \leftrightarrow X'\varepsilon_{1,3}\alpha''$$

Now let's define the function $f : Po \cup Li \cup Pl \to Po(\underline{W}(S)) \cup Li(\underline{W}(S)) \cup Pl(\underline{W}(S))$. For each point $X \in Po$ from lemma 2.2 statement (1) we know that there is a line incident with the point and a plane incident with the line, so there are $y \in Li, \alpha \in Pl$ such that $(X,y,\alpha) \in W(S)$. Then we define $f(X)|(X,y,\alpha)|_1$. For each line $y \in Li$ using lemma 2.2 statement (2) we know that there are incident a point and a plane, so the triple $(X,y,\alpha) \in W(S)$. Then we $f(y) = |(X,y,\alpha)|_2$. For each plane $\alpha \in Pl$ using using lemma 2.2 statement (2) there are a point and a line such that $(X,y,\alpha) \in W(S)$. Then we define $f(\alpha)|(X,y,\alpha)|_3$

f is a function. That means the result of f is independent of the exact choice of the fictive points, lines, and planes which were chosen to make a triple from $W(S)$. We shall prove it for points only for lines and planes it is the same. So let $X \in Po$ and let $y', y'' \in Li$ and $\alpha', \alpha'' \in Pl$ such that the triples: $(X,y',\alpha') \in W(S)$ and $(X,y'',\alpha'') \in W(S)$. It doesn't matter which one we choose for $f(X)$: $|(X,y',\alpha')|_1$ or $|(X,y'',\alpha'')|_1$, because $(X,y',\alpha') \equiv_1 (X,y'',\alpha'')$ then $|(X,y',\alpha')|_1|(X,y'',\alpha'')|_1$.

We have to check that the incidence relations are preserved by the function f.

So let $X \in Po, y \in Li, \alpha \in Pl$ and $f(X)|(X,y',\alpha')|_1$, and $f(y)|(X'',y,\alpha'')|_2$, and $f(\alpha) = |(X''',y''',\alpha)|_3$, where $X'', X''' \in Po$, $y', y''' \in Li$, $\alpha', \alpha'' \in Pl$. Next we use the already proven statements about the relations $\varepsilon_{1,2}(\underline{W}(S))$ and $\varepsilon_{1,3}(\underline{W}(S))$:

$f(X)\varepsilon_{1,2}(\underline{W}(S))f(y) \leftrightarrow |(X,y',\alpha')|_1\varepsilon_{1,2}(\underline{W}(S))|(X'',y,\alpha'')|_2 \leftrightarrow X\varepsilon_{1,2}y$.

$f(X)\varepsilon_{1,3}(\underline{W}(S))f(\alpha) \leftrightarrow |(X,y',\alpha')|_1\varepsilon_{1,3}(\underline{W}(S))|(X''',y''',\alpha)|_3 \leftrightarrow X\varepsilon_{1,3}\alpha$.

It is easy to check that $f \restriction Po$ is a bijection from Po to $Po(\underline{W}(S))$.

$f \restriction Po$ is an injective function. Let $X_1, X_2 \in Po$, and $f(X_1) = f(X_2)$. Let $f(X_1)|(X_1,y',\alpha')|_1$ and $f(X_2) = |(X_2,y'',\alpha'')|_1$, then $|(X_1,y',\alpha')|_1|(X_2,y'',\alpha'')|_1$, and from that we conclude that the triples are equivalent with $\equiv_1$, $(X_2,y'',\alpha'') \equiv_1 (X_2,y'',\alpha'')$, from the definition 2.4 we know that $X_1 = X_2$. So $f \restriction Po$ is an injection.

$f \restriction Po$ is an surjective function. Let $\widetilde{X} \in Po(\underline{W}(S))$, so $\widetilde{X}$ is an equivalence class generated by the triple: $\widetilde{X}|(X,y,\alpha)|_1$, then $f(X) = |(X,y,\alpha)|_1\widetilde{X}$. So $f \restriction Po$ is surjection.

The same way we proof that $f \restriction Li$ and $f \restriction Pl$ are surjective and injective functions. And this way we proof that f is bijection, which with the preserving the relations turns f into isomorphism. □

If we consider $S(\underline{W})$ and $\underline{W}(S)$ as functionals that convert an incidence spaces into structures of incidences, and structures of incidences into incidence spaces, then the conclusion of these 2 theorems is that the 2 functionals behaves as the opposite ones — if S is an incidence space then $S(\underline{W}(S))$ is isomorphic to S, and if $\underline{W}$ is a structure of incidence then $\underline{W}(S(\underline{W}))$ is isomorphic to $\underline{W}$. To proof the equivalence of the category of the incidence spaces and the category of the structures of incidences, we need to turn that functionals into *functors* . To do that we have to extend the functional $S(\underline{W})$ over the class of homomorphisms between structures of incidences and the functional $\underline{W}(S)$ over the class of homomorphisms between incidence spaces.

Definition 2.10. *Let S and S' are incidence spaces and let $\underline{W}(S) = (W, \equiv_1, \equiv_2, \equiv_3)$ and $\underline{W}(S') = (W', \equiv'_1, \equiv'_2, \equiv'_3)$ are the corresponding structures of incidences. For any homomorphism f from S to S', we define the function $\underline{W}(f) : W \to W'$ this way:*

For any triple $(X, x, \alpha) \in W$ we define $\underline{W}(f)((X, x, \alpha)) = (f(X), f(x), f(\alpha))$.

Definition 2.11. *Let $\underline{W}(S) = (W, \equiv_1, \equiv_2, \equiv_3)$ and $\underline{W}(S') = (W', \equiv'_1, \equiv'_2, \equiv'_3)$ are structures of incidences and let $S(\underline{W}) = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ and $S(\underline{W}') = (Po', Li', Pl', \varepsilon'_{1,2}, \varepsilon'_{1,3})$ are the corresponding incidence spaces. For any homomorphism f from $\underline{W}$ to $\underline{W}'$, we define the function $S(f) : Po \cup Li \cup Pl \to Po' \cup Li' \cup Pl'$ this way:*

For any $x \in W$ we define $S(f)(|x|_1) = |f(x)|_1$ and $S(f)(|x|_2) = |f(x)|_2$, and $S(f)(|x|_3) = |f(x)|_3$.

Lemma 2.7. *The definitions above 2.10 and 2.11 are correct.*

Proof. For the definition 2.10 it uses the properties of the homomorphism and the definition of $\underline{W}(S)$, see definition 2.4. For the definition 2.11 we use the definition 2.9 that $Po = W/\equiv_1$, $LiW/\equiv_2$, $Pl = W/\equiv_3$ and $Po'W'/\equiv'_1$, $Li'W'/\equiv'_2$, $Pl'W'/\equiv'_3$ and the properties of homomorphisms. The properties of the homomorphism f is used to proof that the definition $S(f)(|x|_i) = |f(x)|_i$ is independent from the concrete example $x \in W$. □

Theorem 2.3. $\underline{W}$ *is a functor.*

Proof. First we shall proof that the defined above $W(f)$ is a homomorphism from $\underline{W}(S)$ to $\underline{W}(S')$. Let $(X', x', \alpha') \in W$ and $(X'', x'', \alpha'') \in W'$, and $(X', x', \alpha') \equiv_1 (X'', x'', \alpha'')$, then from the definition 2.4 we know that the points are the same $X'X''$, and $f(X') = f(X'')$, again with the definition 2.4 we conclude that $(f(X'), f(x'), f(\alpha')) \equiv_1 (f(X''), f(x''), f(\alpha''))$. In the same way we proof that the function f preserves the relations $\equiv_2$ and $\equiv_3$.

The next question is about the composition of homomorphisms. Let f is a homomorphisms from S to S' and g is a homomorphism from S' to S''. Then $g \circ f$ is a homomorphism from S to S''. We must proof that $W(g \circ f) = W(g) \circ W(f)$.

$W(g \circ f)((X, x, \alpha)) = (W(g \circ f)(X), W(g \circ f)(x), W(g \circ f)(\alpha))$
$= (W(g)(W(f)(X)), W(g)(W(f)(x)), W(g)(W(f)(\alpha)))$
$= W(g)((W(f)(X), W(f)(x), W(f)(\alpha)))$
$= W(g)(W(f)((X, x, \alpha))) = (W(g) \circ W(f))((X, x, \alpha))$. □

Theorem 2.4. *S is a functor.*

Proof. We must check that $S(f)$ is a homomorphism from $S(\underline{W})$ to $S(\underline{W}')$. Let $|x|_1 \in Po(\underline{W})$, $|y|_2 \in Li(\underline{W})$, and $|x|_1 \varepsilon_{1,2}(\underline{W})|y|_2$. From definition 2.9 we know the equivalence $|x|_1 \varepsilon_{1,2}(\underline{W})|y|_2 \leftrightarrow x \in_{1,2} y$. From f homomorphism and the definition of $\in_{1,2}$ we conclude that $f(x) \in'_{1,2} f(y)$. Again applying the definition 2.9 we proof that $|f(x)|_1 \varepsilon'_{1,2}(\underline{W}')|f(y)|_2$, which is $S(f)(|x|_1)\varepsilon'_{1,2}(\underline{W}')S(f)(|y|_2)$. In the same way from $|x|_1\varepsilon_{1,3}(\underline{W})|z|_3$ we proof that $S(f)(|x|_1)\varepsilon'_{1,3}(\underline{W}')S(f)(|z|_3)$. So $S(f)$ is a homomorphism.

Let f is a homomorphism from $\underline{W}$ to $\underline{W}'$, and let g is a homomorphism from $\underline{W}'$ to $\underline{W}''$. Then Then $g \circ f$ is a homomorphism from $\underline{W}$ to $\underline{W}''$. Let's check that $S(g \circ f)S(g) \circ S(f)$. Let $|x|_{1,2,3} \in Po(\underline{W}), Li(\underline{W}), Pl(\underline{W})$, then $S(g \circ f)(|x|_{1,2,3}) = |(g \circ f)(x))|_{1,2,3} = |g(f(x))|_{1,2,3} S(g)(|f(x)|_{1,2,3}) = S(g)(S(f)(|x|_{1,2,3}))$. □

With these theorems 2.1, 2.2, 2.3 and 2.4 we conclude that the category of incidence spaces is equivalent to the category of the structures of incidence.

3. MODAL LOGIC FOR INCIDENCE GEOMETRY

3.1. MODAL LANGUAGE

The language is a modal language with 4 different modal operators. 3 of the operators $[\equiv_1], [\equiv_2], [\equiv_3]$ are interpreted with equivalence relations $\equiv_1, \equiv_2, \equiv_3$, and the 4th one $[\neq]$ with the relation difference $\neq$. The language is consisted also of the set of propositional variables $\{p_1, p_2, p_3, ...\}$ and logical operators $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

3.2. SEMANTICS

The semantics is the Kripke semantics over the frames $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3, \neq)$ where $W \neq \emptyset$ and $\equiv_1, \equiv_2, \equiv_3, \neq$ are binary relations over W. The relation $\neq$ has a special meaning *difference* between 2 elements of W. The elements of W are called *worlds* .

If we assign binary values to the propositional variables, called valuation, we may assign a truth values to all modal formulas. If v is a valuation of propositional variables and $\underline{W}$ is a frame then $\underline{M} = (\underline{W}, v)$ is called a model for that modal logic.

If we have a model $\underline{M} = (\underline{W}, v)$ we can extend the truth value over all formulas using next definition.

Definition 3.1. *The truth value of a modal formula.*

$x \vDash_v A$ means the formula A is true *in the world x according to the valuation v.*

$x \nvDash_v A$ means the formula A is false *in the world x according to the valuation v.*

1. $x \vDash_v p \leftrightarrow v(x,p) = true$*. for any* $p \in \{p_1, p_2, ...\}$ *propositional variable.*

2. $x \vDash_v \neg A \leftrightarrow x \nvDash_v A$.

3. $x \vDash_v (A \wedge B) \leftrightarrow x \vDash_v A$ *and* $x \vDash_v B$.

4. $x \vDash_v [R]A \leftrightarrow (\forall y \in W)(xRy \rightarrow y \vDash_v A)$*. where* $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

Definition 3.2. *We say that A is true into the frame* $\underline{W}$*.* $\underline{W} \vDash A$ *if for each valuation v and for each world* $x \in W$ *it is satisfied* $x \vDash_v A$.

We say that A is true into a class Σ *of frames if A is true into each of frames from that class.*

If Σ *is a class of frames then the set of all formulas A which are true at that class* Σ *forms the logic over that class, and it is noted with* $L(\Sigma)$.

Frames which are interesting for us are frames with $\equiv_1, \equiv_2, \equiv_3$ being equivalence relations and the relation $\neq$ the difference relation. So the class of such frames we note with Σ_0. And the logic over that class $L(\Sigma_0)$ is the minimal logic L_0. First we make the axiomatization of that logic L_0. Next we proceed with axiomatization of the logic over the class of structure of incidences 2.6. That is an extension of the minimal one L_0 with adding axioms for base geometrical properties of the structure of incidences, see 2.6, and next we proof that each of those geometrical properties is a *canonical* property. We note the class of frames which are structures of incidences with Σ_{gsi}. And the geometrical logic is noted with $L(\Sigma_{gsi})$.

3.3. DEFINABLE MODALITIES IN Σ_0

Some relations has modal operators that are definable with these modal operators : $\equiv_1, \equiv_2, \equiv_3, \neq$.

1. Incidence Relations, Like $\in_{1,2}, \in_{1,3}, \in_{2,3}$ and $\in^{-1}_{1,2}, \in^{-1}_{1,3}, \in^{-1}_{2,3}$. The modal operators associated with these relations are $[\in_{1,2}], [\in_{1,3}], [\in_{2,3}], [\in^{-1}_{1,2}], [\in^{-1}_{1,3}], [\in^{-1}_{2,3}]$.

2. Universal Relation. Universal relation U means that every two world are in relation, it connects any with any. So $U = \{< x, y > | x \in W \wedge y \in W\}$. The modality for it is $\blacksquare$.

   The relations below uses the semantics attached with the structures of incidences.

3. Two lines has an incident point.

4. Two planes has an incident point.

5. Two planes has an incident line.

Lemma 3.1. *The modalities with* incidence relations $\in_{1,2}, \in_{1,3}, \in_{2,3}$ *and* $\in^{-1}_{1,2}, \in^{-1}_{1,3}, \in^{-1}_{2,3}$ *are definable as:*

$$[\in_{1,2}] = [\equiv_1][\equiv_2], [\in_{1,3}] = [\equiv_1][\equiv_3], [\in_{2,3}] = [\equiv_2][\equiv_3]$$

$$[\in^{-1}_{1,2}] = [\equiv_2][\equiv_1], [\in^{-1}_{1,3}] = [\equiv_3][\equiv_1], [\in^{-1}_{2,3}] = [\equiv_3][\equiv_2]$$

Lemma 3.2. *The modality with* universal relation *is expressible with:*

$$\blacksquare A = A \wedge [\neq] A$$

Lemma 3.3. *If $\underline{W}$ is a structure of incidence then the relations are expressed:*

$$\cap_2 = \equiv_2 \circ \equiv_1 \circ \equiv_2$$

$$\cap_{3,1} = \equiv_3 \circ \equiv_1 \circ \equiv_3, \cap_{3,2} = \equiv_3 \circ \equiv_2 \circ \equiv_3, \cap_{3,2} = \cap_{3,1}$$

Lemma 3.4. *The relations about two lines has a common point and two planes has a common point, line are definable:*

$$[\cap_2] = [\equiv_2][\equiv_1][\equiv_2]$$

$$[\cap_{3,1}] = [\equiv_3][\equiv_1][\equiv_3], [\cap_{3,2}] = [\equiv_3][\equiv_2][\equiv_3]$$

3.4. AXIOMATIZATION FOR THE MINIMAL LOGIC $L(\Sigma_0)$

Axioms

1. All Propositional Tautologies.
2. $[R](A \Rightarrow B) \Rightarrow ([R]A \Rightarrow [R]B)$, where $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.
3. $S5$ axioms for modalities $[\equiv_1], [\equiv_2], [\equiv_3]$.
   $[\equiv_i]A \Rightarrow A$ noted with (T_i).
   $<\equiv_i> [\equiv_i]A \Rightarrow A$ noted with (B_i).
   $[\equiv_i]A \Rightarrow [\equiv_i][\equiv_i]A$ noted with (4_i).
4. $<\neq> [\neq]A \Rightarrow A$ noted with $(B_{\neq})$. The relation $\neq$ is a symmetric.
5. $<\neq><\neq> A \Rightarrow (A \vee <\neq> A)$.
6. $\blacksquare A \Rightarrow [\equiv_i]A$.

Deductive Rules

1. Modus Ponens (MP) $A, A \Rightarrow B \vdash B$.

2. Normality (N_R) $A \vdash [R]B$, where $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

3. Irreflexivity (Irr) $(p \wedge [\neq]\neg p) \Rightarrow A$, the variable p does not enter $A \vdash A$.

Definition 3.3. *Formal proof with the deductive system we call any sequence of formulas each of them could be a variant of one of the axiom schemas or produced by some rule from the previous formulas.*

$\vdash A$ There is a formal proof of A without using the rule (Irr).
$\vdash_{Irr} A$ There is a formal proof of A with using the rule (Irr).

Lemma 3.5. $\vdash \blacksquare A \Rightarrow A, \vdash \blacklozenge\blacksquare A \Rightarrow A, \vdash \blacksquare A \Rightarrow \blacksquare\blacksquare A$

To proof the completeness we will use another rule for irreflexivity which will offer simpler completeness proof, and next the deductive equivalence of the two rules will proof completeness of the logic $L(\Sigma_0)$.

3.5. DIFFERENT IRREFLEXIVITY RULES AND THEIR DEDUCTIVE EQUIVALENCE

The all mentioned deductive systems will contain the axioms 1 - 6 and the rules MP and $N_{\equiv_1, \equiv_2, \equiv_3, \neq}$, they will only differ with the irreflexivity rule.

The new infinite irreflexivity rule is:

(Irr^*) $(p \wedge [\neq]\neg p) \Rightarrow A$, for each variable $p \vdash^* A$.

This rule makes the ordinary definition of formal proof not appropriate and requires a new definition. The set of axioms 1 - 6 we note with A_0.

Definition 3.4. *Infinite formal proof of the formula ϕ is the ordered pair (Γ, ρ), where Γ is a tree with a* finite *path from the root to any leaf. and ρ is the correspondence between each tree node and a formula from the modal language. For Γ and ρ it is true that:*

1. *if v is a leaf from the tree then $\rho(v) \in A_0$*

2. *if v is not a leaf then $\rho(v)$ is a formula which is a conclusion of some of the rules.*

3. *if v is the root of the tree then $\rho(v) = \phi$.*

We can note such infinite proof of ϕ with $\triangleright \phi$. The "triangle" sign symbolizes the infinite tree with finite path to the leaves. if the rule (Irr) is used then we use $\triangleright \phi$, else if the rule (Irr^*) is used we use $\triangleright^* \phi$. So the formula ϕ is proved with (Irr), $\vdash_{Irr} \phi$ if and only if there is $\triangleright \phi$. Also ϕ is proved with (Irr^*), $\vdash_{Irr^*} \phi$ if and only if $\triangleright^* \phi$.

Lemma 3.6. $\vdash_{Irr} \phi$ *if and only if* $\vdash_{Irr^*} \phi$.

Proof. If $\vdash_{Irr} \phi$ then there is a infinite proof $\rhd\ \phi$. Next with induction over the max tree path length we can create infinite proof $\rhd^*\ \phi$. The only interesting case is when ϕ is a conclusion of irreflexivity rule. Let the proof looks like $\rhd\ ((p\wedge[\neq]\neg p) \Rightarrow \phi) \vdash \phi$, where the variable p does not enter ϕ. Then everywhere inside the infinite proof $\rhd\ ((p\wedge[\neq]\neg p) \Rightarrow \phi)$ we can replace the variable p with any other variable q, so the result will be the infinite proof $\rhd\ ((q\wedge[\neq]\neg q) \Rightarrow \phi)$, ϕ remains unchanged because p does not enter ϕ. So for each infinite proof by induction assumption we have $\rhd^*\ ((q\wedge[\neq]\neg q) \Rightarrow \phi)$, for each variable q. Now applying of the infinite rule Irr^* we construct the infinite proof $\rhd^*\ \phi$.

In the other direction if we have the infinite proof $\rhd^*\ \phi$, again induction over the height of the tree, also the interesting case is with ϕ is being conclusion of the infinite rule Irr^*. The proof for ϕ looks like: $\rhd_1^*\ ((q_1\wedge[\neq]\neg q_1) \Rightarrow \phi), ..., \rhd_n^*\ ((q_n\wedge[\neq]\neg q_n) \Rightarrow \phi), ... \vdash \phi$. Because ϕ has a finite number of variables we can choose one variable p which does not enter ϕ and for which there is a proof $\rhd_n^*\ ((p\wedge[\neq]\neg p) \Rightarrow \phi)$, now applying the inductional assumption and applying the finite Irr rule we receive the proof $\rhd^*\ \phi$.

The obvious observation that if (Γ, ρ) is an infinite proof but if it uses only finite rules then it is equivalent to the ordinary finite proof. □

The conclusion of this lemma 3.6 is that the formal systems with the infinite rule $Irr*$ is deductive equivalent with the formal system with the finite rule Irr.

Because all our relations are symmetric for all the modalities $[\equiv_1], [\equiv_2], [\equiv_3], [\neq]$, there are the axioms for symmetric relation: $< R > [R]\phi \Rightarrow \phi$.

Lemma 3.7. *(Rijke [5]) If for one modal operator we have the symmetric axiom* $< R > [R]\phi \Rightarrow \phi$*. then:* $\vdash \phi \Rightarrow [R]\psi$ *if and only if* $\vdash< R > \phi \Rightarrow \psi$

Proof. It is used contraposition, the normality rule N_R, the axiom 2 $[R](\phi \Rightarrow \psi) \Rightarrow ([R]\phi \Rightarrow [R]\psi)$, and the above axiom $B_R < R > [R]\phi \Rightarrow \phi$, and propositional tautologies.

Let $\vdash< R > \phi \Rightarrow \psi$, then using N_R, we have $\vdash [R](< R > \phi \Rightarrow \psi)$, using axiom 2 and MP, we have $\vdash [R] < R > \phi \Rightarrow [R]\psi$. Now with contraposition we have $\vdash \neg[R]\psi \Rightarrow \neg[R] < R > \phi$, which is the same as $\vdash< R > \neg\psi \Rightarrow< R > [R]\neg\phi$. Now using the B_R axiom for $\neg\phi$, $< R > [R]\neg\phi \Rightarrow \neg\phi$ and tautology we have that $\vdash< R > \neg\psi \Rightarrow \neg\phi$, contraposition, $\vdash \phi \Rightarrow [R]\psi$. □

This lemma gives us the opportunity to use an infinite many rules instead of a single irreflexivity rule:

Definition 3.5. *Long Irreflexive Rules. For each natural number* $n \geqq 0$ *:*
$(Adm_0 Irr^*)$ *is* (Irr^*).

$(Adm_n Irr^*)$ $A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow A)...))$*. for each variable* $p \vdash A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 \Rightarrow ... \Rightarrow [R_n](A)...))$*, where* $n > 0$*,* $\{R_1, R_2, ...R_n\} \subseteq \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

Lemma 3.8. *If into the proof of the formula ϕ the rule Adm_nIrr^* is used for some n, then that proof can be reworked into a proof of ϕ with the rule Adm_nIrr^* eliminated and replaced with Irr^*.*

Proof. Induction over n can show the elimination and the replacement of the rule Adm_nIrr^* with the rule Irr^*, also we choose to eliminate the Adm_nIrr^* rule closest to the leaves of the infinite proof.

If $n = 0$ then Adm_nIrr^* is Irr^*.

If $n = 1$ then Adm_1Irr^* is : $A_1 \Rightarrow [R_1]((p\wedge[\neq]\neg p) \Rightarrow A)$, for each $p \vdash A_1 \Rightarrow [R_1]A$. Because $\vdash_{Irr^*} A_1 \Rightarrow [R_1]((p\wedge[\neq]\neg p) \Rightarrow A)$, for each p, and using the lemma 3.7 $\vdash_{Irr^*} < R_1 > A_1 \Rightarrow ((p\wedge[\neq]\neg p) \Rightarrow A)$, according to the tautology : $\vdash_{Irr^*} ((< R_1 > A_1\wedge(p\wedge[\neq]\neg p)) \Rightarrow A)$, and $\vdash_{Irr^*} (p\wedge[\neq]\neg p) \Rightarrow (< R_1 > A_1 \Rightarrow A))$, for each p, then applying the rule Irr^*, we get $\vdash_{Irr^*} < R_1 > A_1 \Rightarrow A$, again using the lemma 3.7 we proof that $\vdash_{Irr^*} A_1 \Rightarrow [R_1]A$.

Let's for some n the assumption is true.

Let's have the occurrence $Adm_{n+1}Irr^*$, the closest to the tree-proof leaves, and it is look like:

$(A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}]((p\wedge[\neq]\neg p) \Rightarrow A)...)))$, for each variable $p \vdash (A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}](A)...)))$.

Let's note with $\psi(p) = [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}]((p\wedge[\neq]\neg p) \Rightarrow A)...))$, for each p. Let's note with $\chi = [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}](A)...))$. Now the rule is written as : $A_1 \Rightarrow [R_1](A_2 \Rightarrow \psi(p))$, for each $p \vdash A_1 \Rightarrow [R_1](A_2 \Rightarrow \chi)$.

Because $\vdash_{Irr^*} A_1 \Rightarrow [R_1](A_2 \Rightarrow \psi(p))$ then using the lemma 3.7 we get that $\vdash_{Irr^*} < R_1 > A_1 \Rightarrow (A_2 \Rightarrow \psi(p))$, propositional tautology, $\vdash_{Irr^*} (< R_1 > A_1\wedge A_2) \Rightarrow \psi(p)$, for each p. Now we can apply the rule Adm_nIrr^* and also the inductive assumption, and the result is : $\vdash_{Irr^*} (< R_1 > A_1\wedge A_2) \Rightarrow \chi$, that formula is proved with Irr^* only. Using propositional tautology : $\vdash_{Irr^*} (< R_1 > A_1 \Rightarrow (A_2 \Rightarrow \chi))$. Again from the lemma 3.7 we receive $\vdash_{Irr^*} A_1 \Rightarrow [R_1](A_2 \Rightarrow \chi)$. This shows how to eliminate the $Adm_{n+1}Irr^*$ rule. □

Theorem 3.1. *The rules Irr, Irr^*, and the set Adm_nIrr^* of rules are deductive equivalent:* $\{\phi| \vdash_{Irr} \phi\} = \{\phi| \vdash_{Irr^*} \phi\} = \{\phi| \vdash_{Adm_nIrr^*} \phi\}$.

Note: The rule is needed only to proof the lemma 3.14, neither deduction lemma 3.11 nor Lindenbaum's lemma 3.13 needs that rule and they can be proofed with the finite irreglexivity rule, but the rules Adm_nIrr^* change the nature of the ω-theories.

3.6. COMPLETENESS THEOREM FOR THE MINIMAL LOGIC $L(\Sigma_0)$.

We proof now the completeness of the minimal logic for axioms from 1 to 6 and the rules MP, $N_{\equiv_1,\equiv_2,\equiv_3,\neq}$ and the set of long infinite irreflexive rules Adm_nIrr^* instead of the finite irreflexive rule Irr.

Let Φ be the set of all modal formulas.

Let with $L = \{\phi| \vdash_{Irr} \phi\} = \{\phi| \vdash_{Irr^*} \phi\} = \{\phi| \vdash_{Adm_nIrr^*} \phi\}$ we note the set of all logical theorems.

Definition 3.6. *ω-theory — Any set of modal formulas $X \subseteq \Phi$ such that:*

1. *$L = \{\phi | \vdash_{Adm_n Irr^*} \phi\} \subseteq X$ All logical theorems belongs to X.*
2. *X closed under (MP).*
3. *X closed under $(Adm_n Irr^*)$.*

Note: The ω-theories are not closed under the normality rules $N_{\equiv_1,\equiv_2,\equiv_3,\neq}$.

Definition 3.7. *The ω-theory X is inconsistent if and only if $X = \Phi$.*

Definition 3.8. *The ω-theory X is consistent if and only if $X \neq \Phi$.*

Also X is inconsistent ω-theory if and only if $\perp \in X$, and X is consistent if and only if $\perp \notin X$.

Lemma 3.9. *Intersection of two ω-theories is a ω-theory.*

This lemma 3.9 gives us the opportunity to give the next definition:

Definition 3.9. *For each set of formulas $Y \in \Phi$, the set of formulas $Th(Y)$ is the smallest ω-theory that contains Y.*

Lemma 3.10. $Th(Y) = \bigcap\{X | X$ *is a ω-theory. and* $Y \subseteqq X\}$.

So for each set Y it is true that:

1. $L \subseteq Th(Y)$,
2. $Y \subseteq Th(Y)$,
3. $Th(Y)$ is closed under (MP),
4. $Th(Y)$ is closed under all rules $(Adm_n Irr^*)$.

Lemma 3.11. Deduction Lemma. *Let ϕ formula and X ω-theory, then:*

$$(\phi \Rightarrow \psi) \in X \leftrightarrow \psi \in Th(X \cup \{\phi\})$$

Proof. Let's choose $Y = \{\psi | (\phi \Rightarrow \psi) \in X\}$.

1. Let's proof that $X \subseteq Y$. Let $\psi \in X$. Because $(\psi \Rightarrow (\phi \Rightarrow \psi))$ is a classical axiom, and X closed under (MP) then $(\phi \Rightarrow \psi) \in X$, and according the definition of Y, $\psi \in Y$, so $X \subseteq Y$. Also from $L \subseteq X$ we conclude that $L \subseteq Y$, all logical theorems belong to Y.

2. $(\phi \Rightarrow \phi) \in L$, it is a classical theorem, this way $(\phi \Rightarrow \phi) \in X$, so from the definition of Y, $\phi \in Y$.

3. Let $\psi \in Y$ and $(\psi \Rightarrow \chi)$, so $(\phi \Rightarrow \psi) \in X$ and $(\phi \Rightarrow (\psi \Rightarrow \chi)) \in X$, now using the classical axiom $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$, and X closed

under (MP), then $(\phi \Rightarrow \chi) \in X$, and it follows that $\chi \in Y$, so Y is closed under (MP).

4. Finally we proof that Y is closed under Irr^* the infinite irreflexivity rule. Let $((p \wedge [\neq]\neg p) \Rightarrow \psi) \in Y$, for each variable p. Then $(\phi \Rightarrow ((p \wedge [\neq]\neg p) \Rightarrow \psi)) \in X$, for each variable p. Using propositional tautologies we conclude that for each variable p the formulas $((p \wedge [\neq]\neg p) \Rightarrow (\phi \Rightarrow \psi)) \in X$. X is closed under (Irr^*), so $(\phi \Rightarrow \psi) \in X$, $\psi \in Y$ and we conclude that Y is closed under (Irr^*).

Y is an ω-theory, $(X \cup \{\phi\}) \subseteq Y$, and $Th(X \cup \{\phi\})$ is the minimal ω-theory containing $(X \cup \{\phi\})$. That's why $Th(X \cup \{\phi\}) \subseteq Y$.

Now if $\psi \in Th(X \cup \{\phi\})$, then $\psi \in Y$, and from the definition of Y, $(\phi \Rightarrow \psi) \in X$. □

Lemma 3.12. *If X is an ω-theory, and $\phi \notin X$ then $Th(X \cup \{\neg\phi\})$ is a consistent ω-theory.*

Proof. Let's assume that $Th(X \cup \{\neg\phi\})$ is inconsistent, then $\bot \in Th(X \cup \{\neg\phi\})$, now using deduction lemma 3.11 we have that $(\neg\phi \Rightarrow \bot) \in X$, now apply classical tautology we get that $\phi \in X$, which contradicts with $\phi \notin X$. □

Long infinite rules are hard to write that's why we can specify some short writing form. If the formula ϕ is graphically equal to: $(\psi_0 \Rightarrow [R_0](\psi_1 \Rightarrow [R_1](....(\psi_m \Rightarrow [R_m](\psi))...)))$, then we note with $\Psi_\phi(p,i) = (\psi_0 \Rightarrow [R_0](\psi_1 \Rightarrow [R_1](....(\psi_i \Rightarrow [R_i]((p \wedge [\neq]\neg p) \Rightarrow (\psi_{i+1} \Rightarrow [R_{i+1}](\psi_{i+2}... \Rightarrow [R_m](\psi)...)))...)))$, where $i > 0$, and $\Psi_\phi(p,0) = ((p \wedge [\neq]\neg p) \Rightarrow \phi)$. If the formula ϕ is not of the above form then only $\Psi(p,0)$ makes sense. As a conclusion we can say that for each formula ϕ, if we can specify $\Psi_\phi(p,i)$, then ϕ can be a conclusion of the rule $Adm_i Irr^*$. The rule $Adm_i Irr^*$ looks like: $\Psi_\phi(p,i)$, for each $p \vdash \phi$.

Lemma 3.13. *Lindenbaum's lemma. Let X is an ω-theory and $\phi \notin X$ then there is a maximal consistent ω-theory Y such that $X \subseteq Y$ and $\phi \notin Y$.*

Proof. Let's order all modal formulas into a sequence starting with $\neg\phi$ as the first formula: $\neg\phi = \phi_0, \phi_1, \phi_2, ...\phi_n, ...$ Now we define a sequence of consistent ω-theories : $X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_n \subseteq ...$ inductively. For X_0 we choose $Th(X \cup \{\neg\phi\})$, according lemma 3.12 it is a consistent ω-theory. Also $X \subseteq X_0$. Let's assume for some n the sequence $X_0, ..., X_n$ is created. Now we must define X_{n+1}, and there are two cases:

(i). If $Th(X_n \cup \{\phi_{n+1}\})$ is a consistent ω-theory, then we choose : $X_{n+1} = Th(X_n \cup \{\phi_{n+1}\})$. It is easy seen that $X_n \subseteq X_{n+1}$ and $\phi_{n+1} \in X_{n+1}$.

(ii). If $Th(X_n \cup \{\phi_{n+1}\})$ is a *inconsistent* ω-theory, then $\bot \in Th(X_n \cup \{\phi_{n+1}\})$, and according the deduction lemma 3.11, $(\phi_{n+1} \Rightarrow \bot) \in X_n$, and ω-theories are closed under propositional tautologies, so $\neg\phi_{n+1} \in X_n$. The right choice for X_{n+1} could be X_n , but we must do something more to guarantee that the infinite sequence union would not accumulate all formulas for the infinite rules Irr^* and

$Adm_m Irr^*, m > 1$ that can produce ϕ_{n+1}, and this way the infinite union becomes inconsistent. If the formula ϕ_{n+1} is graphically equal to: $\psi_0 \Rightarrow [R_0](\psi_1 \Rightarrow [R_1](....(\psi_m \Rightarrow [R_m](\psi))...))$, then this formula can be a result of any of the rules $Irr^* Adm_0 Irr^*, Adm_1 Irr^*, ..., Adm_m Irr^*$. If the formula ϕ_{n+1} is not of that form then it can be a result of the rule $Irr^* = Adm_0 Irr^*$ only.

Now we define the sequence $Y_{-1}, Y_0, Y_1, ..., Y_m$, of ω-theories, where $Y_{-1} = X_n$, and the aim is for each of Y_i to prevent ability to produce ϕ_{n+1} from the rule $Adm_i Irr^*$. Inductively. $Y_{-1} = X_n$ is a consistent and $\neg\phi_{n+1} \in Y_{-1}$. Let's we have defined already some Y_{i-1}, now we can define Y_i.

Let's assume that for each variable p, $\Psi_{\phi_{n+1}}(p, i) \in Y_{i-1}$. Then ϕ_{n+1} is a conclusion of the rule $Adm_i Irr^*$ and Y_{i-1} is an ω-theory, thus it is closed under $Adm_i Irr^*$, then the formula $\phi_{n+1} \in Y_i$, but $\neg\phi_{n+1} \in Y_i$, contradiction with Y_{i-1} is a consistent ω-theory. So we conclude that there must be a variable p_i such that $\Psi_{\phi_{n+1}}(p_i, i) \notin Y_{i-1}$. From the lemma 3.12 it follows that the ω-theory $Th(Y_{i-1} \cup \{\neg\Psi_{\phi_{n+1}}(p_i, i)\})$ is a consistent. Thus we choose for $Y_i = Th(Y_{i-1} \cup \{\neg\Psi_{\phi_{n+1}}(p_i, i)\})$. This way the sequence $Y_{-1}, Y_0, Y_1, ..., Y_m$ is defined. It is not hard to see that $X_n \subseteq Y_0 \subseteq Y_1 \subseteq ... \subseteq Y_m$, and $\neg\phi_n \in Y_m$, and for each of the chosen variables $p_0, p_1, ...p_m$ during the inductive definition, it is true that: $(\neg\Psi_{\phi_{n+1}}(p_i, i)) \in Y_m$.

We choose for $X_{n+1} = Y_m$. This way the inductive definition for consistent ω-theories is complete.

Let's $X_\omega = \bigcup_{n=0}^{\infty} X_n$.

1. X_ω is a consistent. Let's assume that X_ω is inconsistent, then $\perp \in X_\omega$ and according definition of X_ω, then there is $m : \perp \in X_m$, and X_m is inconsistent which is a contradiction with the build of X_m.

2. From $L \subseteq X \subseteq X_0 \subseteq X_\omega$, we get that $L \subseteq X_\omega$ and $X \subseteq X_\omega$.

3. Because the sequence is monotonic $X_0 \subseteq X_1 \subseteq ... \subseteq X_n \subseteq ...$ and each set is closed under MP then X_ω is closed under MP.

4. Because of the inductive construction, for each formula ψ, it is true: $\psi \in X_\omega$ or $\neg\psi \in X_\omega$. Also $\neg\phi \in X_0 \subseteq X_\omega$, then $\phi \notin X_\omega$.

5. Let's assume that X_ω is not closed under some $Adm_n Irr^*$ rule, then there is a formula ϕ such that for each variable p, $\Psi_\phi(p, n) \in X_\omega$, and $\phi \notin X_\omega$. According to the previous point, there is an index m: $\phi_m = \phi$, and $\neg\phi \in X_m$. According to the inductive construction, the case (ii) was chosen, and there is a variable p_n^m, such that the formula $\neg\Psi_\phi(p_n^m, n) \in X_m$, and $X_m \subseteq X_\omega$, it follows that $\neg\Psi_\phi(p_n^m, n) \in X_\omega$. This way we conclude that $\perp \in X_\omega$, contradiction with 1. X_ω is closed under any $Adm_n Irr^*$ rule, and it is ω-theory.

6. X_ω is a maximal according the subset relation "$\subseteq$". □

Definition 3.10. $[R]X = \{\phi | [R]\phi \in X\}$, *where* $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

Lemma 3.14. *If* X *is an* ω*-theory then* $[R]X$ *is an* ω*-theory.*

Proof. 1. Let $\phi \in L$, logical theorem, then by normality rule N_R it is a theorem $[R]\phi \in L$, $L \subseteq X$, then $[R]\phi \in X$, and $\phi \in [R]X$, so $L \subseteq [R]X$.

2. Let $\phi \in [R]X$ and $(\phi \Rightarrow \psi) \in [R]X$, then $[R]\phi \in X$ and $[R](\phi \Rightarrow \psi) \in X$. It is an axiom $[R](\phi \Rightarrow \psi) \Rightarrow ([R]\phi \Rightarrow) \in L \subseteq X$. Because X is closed under MP, $[R]\psi \in X$, and finally $\psi \in [R]X$, $[R]X$ is closed under MP.

3. Let $(\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow \phi)...))) \in [R]X$, for each variable p. Then $[R](\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow \phi)...))) \in X$, it can be written into equivalent form $(T \Rightarrow [R](\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow \phi)...)))) \in X$, for each variable p. The ω-theory X is closed under $Adm_{n+1}Irr^*$, then $(T \Rightarrow [R](\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n](\phi)...)))) \in X$, and back using the definition of $[R]X$, we get that $(\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n](\phi)...))) \in [R]X$. So the set $[R]X$ is closed under Adm_nIrr^* rule, and it is closed under the whole set of rules Adm_nIrr^*, $n \geqslant 0$. □

Now we can define *canonical frame* using ω-theories. Canonical frame would not be perfect for difference $\neq$ relation and will be improved to *generic canonical frame* in which the difference is a real difference.

Definition 3.11. *Canonical Frame.* $\underline{W_k} = (W_k, \equiv_{1k}, \equiv_{2k}, \equiv_{3k}, \neq_k)$. *where* $W_k = \{X | X$ *is a maximal consistent* ω *theory* $\}$. $X \equiv_{1k} Y \leftrightarrow [\equiv_1]X \subseteq Y$. $X \equiv_{2k} Y \leftrightarrow [\equiv_2]X \subseteq Y$. $X \equiv_{3k} Y \leftrightarrow [\equiv_3]X \subseteq Y$. $X \neq_k Y \leftrightarrow [\neq]X \subseteq Y$.

Definition 3.12. *Canonical Evaluation.* $V_k(X,p) = true \leftrightarrow p \in X$, *for each variable* p.

Definition 3.13. *Canonical Model.* $M_k = (\underline{W_k}, V_k)$.

Lemma 3.15. *Truth Lemma*

Let $M_k = (\underline{W_k}, V_k)$ *canonical model, then for each formula* ϕ, *and for each maximal* ω*-theory* X *it is true that:* $V_k(X,\phi) = true \leftrightarrow \phi \in X$.

The relations $\equiv_{1k}, \equiv_{2k}, \equiv_{3k}$ are equivalence relations because of $S5$ axioms for each of them. Alas the relation $\neq_{1k}$ is not exactly the difference relation, that's why we will rework this canonical model into a new one.

Definition 3.14. *Let's* X *and* Y *are maximal consistent* ω*-theories. Then we say that* Y *is a* finite reachable *from* X, $X \curvearrowright Y$, *if there is a finite sequence of maximal consistent* ω*-theories* $X = Z_0, Z_1, ... Z_{m-1}, Z_m = Y$, *and relations* $R_{1k}, R_{2k}, ..., R_{mk}$. $R_{ik} \in \{\equiv_{1k}, \equiv_{2k}, \equiv_{3k}, \neq_k\}$,
such that $X = Z_0 R_{1k} Z_1 R_{2k} ... Z_{m-1} R_{mk} Z_m = Y$.

Definition 3.15. *Generic Canonical Frame. Let* X *is a maximal consistent* ω*-theory. The generic canonical frame is* $\underline{W'_k} = (W'_k, \equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}, \neq'_k)$. *where* $W'_k = \{Y | X \curvearrowright Y\}$, *the set of finite reachable from* X. *The relations* $\equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}$, $\neq'_k$ *are restrictions of* $\equiv_{1k}, \equiv_{2k}, \equiv_{3k}, \neq_k$ *over the set* W'_k.

The generic canonical model $M'_k = (\underline{W'_k}, V'_k)$ is defined in the same way, and also the truth lemma 3.15 is true for the generic canonical frames and generic models.

Lemma 3.16. *The relations $\equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}$ are equivalence relations and the relation $\neq'_{1k}$ is a symmetric relation.*

Analogically to the definition 3.10 we can define $\blacksquare X$, where X is an ω-theory. $\blacksquare X = \{\phi | \blacksquare\phi \in X\}$, from the definition of $\blacksquare\phi = (\phi \wedge [\neq]\phi)$, we see that: $\blacksquare X = \{\phi | \phi \wedge [\neq]\phi \in X\} = \{\phi | \phi \in X\} \cap \{\phi | [\neq]\phi \in X\} = X \cap [\neq]X$, intersection of ω-theories is a ω-theory, so $\blacksquare X$ is an ω-theory. Now we can define the relation $Y \blacksquare'_k Z \leftrightarrow \blacksquare Y \subseteq Z$, and we can see that it is the universal relation into $\underline{W'_k}$, any 2 objects are with relation $\blacksquare'_k$.

Lemma 3.17. *For any two maximal ω-theories $X \in \underline{W'_k}$ and $Y \in \underline{W'_k}$, then $\blacksquare X \subseteq Y$.*

Proof. First we proof that if $[R]X \subseteq Y$ then $\blacksquare X \subseteq Y$, and for that we use that it is an axiom: $\blacksquare\phi \Rightarrow [\equiv_i]\phi$. When R is $\neq$ then we use the tautology $(\phi \wedge [\neq]\phi) \Rightarrow [\neq]\phi$. Second from the lemma 3.5 we know that $\vdash \blacksquare\phi \Rightarrow \blacksquare\blacksquare\phi$, and using that theorem we can proof that if $\blacksquare X \subseteq Y$, and $\blacksquare Y \subseteq Z$, then $\blacksquare X \subseteq Z$, so $\blacksquare'_k$ is a transitive. Thus finally if $X, Y \in \underline{W'_k}$, then there is a finite sequence $X = Z_0 R'_{1k} Z_1 R'_{2k} ... R'_{mk} Zm = Y$, now applying what we have proof, leads to $\blacksquare X \subseteq Y$. □

The formulas $p \wedge [\neq]\neg p$, where p is a variable, have special meaning, they "lock" the variable to be true at exactly one world and false in any other. So we can call them constants.

Definition 3.16. *If p is a variable then the formula noted with Op. $Op = (p \wedge [\neq]\neg p)$ is called a* constant.

Lemma 3.18. *If X is a maximal consistent ω-theory then there is a variable p. such that the constant $Op \in X$.*

Proof. Let'a assume that for each variable p, $Op \notin X$, because X is a maximal consistent ω-theory, then $\neg Op \in X$, the equivalent from is $(Op \Rightarrow \bot) \in X$, for each p, X is closed under $Irr^* = Adm_0 Irr^*$, then $\bot \in X$, contradiction with X consistent. □

Lemma 3.19. *For any variable p there is a maximal consistent ω-theory X. such that $Op \in X$. it contains the constant of p.*

Proof. $\neg Op \notin L$. If it were $\neg Op \in L$, then the formula $\neg Op$ must be true in any frame in any evaluation, it is simple to show a frame and an evaluation, and a world x such that $x \nvDash_v \neg Op$, thus $\neg Op \notin L$.

L is a consistent ω-theory, and $\neg Op \notin L$, according to lemma 3.13 there is a maximal consistent ω-theory X such that $L \subseteq X$, and $\neg Op \notin X$, thus $Op \in X$. □

Lemma 3.20. *Canonically defined $\neq_k$ is irreflexive. For any maximal consistent ω-theory X. $[\neq]X \nsubseteq X$.*

Proof. Let's assume that for some X maximal consistent ω-theory, $[\neq]X \subseteq X$. From lemma 3.18, there is p, and the constant $Op \in X$, $(p \wedge [\neq]\neg p) \in X$, so $p \in X$ and $[\neq]\neg p \in X$, $\neg p \in [\neq]X$, and from $[\neq]X \subseteq X$, we conclude that $\neg p \in X$ and from $p \in X$, X is an inconsistent, which is the contradiction. □

Lemma 3.21. *Let $W'_k = (W'_k, \equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}, \neq'_k)$ be a generic canonical frame, then for each $X \in W'_k$ and $Y \in W'_k$, and $X \neq Y$, then $X \neq'_k Y$, $[\neq]X \subseteq Y$.*

Proof. Because $X, Y \in W'_k$ from lemma 3.17, then $\blacksquare X \subseteq Y$. From $X \neq Y$ the there is a formula ϕ: $\phi \in X$ and $\phi \notin Y$. Let's now assume that $[\neq]X \nsubseteq Y$, so there is a formula ψ: $\psi \in [\neq]X$ and $\psi \notin Y$. For Y maximal consistent ω-theory we have $\neg(\phi \vee \psi) \in Y$ or $(\phi \vee \psi) \notin Y$. From $\blacksquare X \subseteq Y$ it follows that $\blacksquare(\phi \vee \psi) \notin X$, next we have $\neg((\phi \vee \psi) \wedge [\neq](\phi \vee \psi)) \in X$.

From the classical axiom $(\phi \Rightarrow (\phi \vee \psi)) \in X$, and $\phi \in X$, we conclude that $(\phi \vee \psi) \in X$. From the classical axiom $\vdash (\psi \Rightarrow (\phi \vee \psi))$, now applying the normality rule $N_{\neq}$, $\vdash [\neq](\psi \Rightarrow (\phi \vee \psi))$, and from the monotonic axiom $\vdash [\neq](\psi \Rightarrow (\phi \vee \psi)) \Rightarrow ([\neq]\psi \Rightarrow [\neq](\phi \vee \psi)))$, we get the theorem $([\neq]\psi \Rightarrow [\neq](\phi \vee \psi)) \in L \subseteq X$, and $[\neq]\psi \in X$, then $[\neq](\phi \vee \psi) \in X$, thus we get that $(\phi \vee \psi) \wedge [\neq](\phi \vee \psi) \in X$. Contradiction with X consistent. □

The conclusion of the last lemma is that the generic canonical frames belongs to the class of frames Σ_0, and it gives us the completeness theorem.

Theorem 3.2. *Completeness theorem for the minimal logic. Each formula ϕ that is true at the class of frames Σ_0 is provable, $\vdash_{Irr} \phi$.*

Proof. Contraposition. Let $\nvdash_{Irr} \phi$, ϕ is not a theorem, then $\phi \notin L$, using the Lindenbaum's lemma 3.13, there is a maximal consistent ω-theory X, such that $\phi \notin X$. Let's get the generic canonical frame and model, in which $W'_k = \{Y | X \curvearrowright Y\}$, generated from X. That frame belongs to the class Σ_0. In that model using the truth lemma 3.15, we get that $V_k(X, \phi) = false$, because $\phi \notin X$. And the deductive equivalence of the rules makes no difference between Irr and $Adm_n Irr^*$.
□

In the end, some properties about constants and maximal consistent ω-theories that are useful and reveals the character of the maximal consistent ω-theories are expressed:

Lemma 3.22. *Let X and Y are maximal consistent ω-theories such that they are finite reachable, $\blacksquare X \subseteq Y$. If there is a variable p such that $Op \in X$ and $p \in Y$ then $X = Y$.*

Proof. Let's assume that $X \neq Y$, because $X \curvearrowright Y$, $\blacksquare X \subseteq Y$, then from lemma 3.21, we conclude the key fact that $[\neq]X \subseteq Y$. From $Op \in X$ it follows: $p \wedge [\neq]\neg p \in X$, so $p \in X$ and $[\neq]\neg p \in X$, $\neg p \in [\neq]X$. From $[\neq]X \subseteq Y$, then $\neg p \in Y$, and $p \in Y$, contradiction with Y consistent. □

If two maximal consistent ω-theories possess the same constant Op, they are the equal ω-theories.

Lemma 3.23. *$[R]X \subseteq Y$ if and only if there is a variable p such that $Op \in Y$ and $< R > Op \in X$. where $R \in \{\equiv_1, \equiv_2, \equiv_3, \blacksquare, \in_{ij}\}$.*

Proof. Let $[R]X \subseteq Y$, then from lemma 3.18, there is a constant $Op \in Y$, then $\neg Op \notin Y$, and from $[R]X \subseteq Y$, $\neg Op \notin [R]X$, then $[R]\neg Op \notin X$, and finally $\neg[R]\neg Op \in X$, which is $< R > Op \in X$.

Let there is a constant $Op \in Y$ and $< R > Op \in X$. From $< R > Op \in X$ it follows that $[R]\neg Op \notin X$, and $\neg Op \notin [R]X$. From lemma 3.14 then $[R]X$ is an ω-theory. Because $\neg Op \notin [R]X$, then $[R]X$ we know that it is a consistent theory. Now applying Lindenbaum's lemma 3.13 we get that there is Z maximal consistent ω-theory such that $[R]X \subseteq Z$, and $\neg Op \notin Z$. Z is a maximal, then $Op \in Z$, but $Op \in Y$, from lemma 3.22, $Y = Z$, and from $[R]X \subseteq Z$, then $[R]X \subseteq Y$. □

Next three lemmas are related with expressible modalities as the incidences: $[\in_{12}]$, $[\in_{13}]$, $[\in_{23}]$, $[\in_{12}^{-1}]$, $[\in_{13}^{-1}]$, $[\in_{23}^{-1}]$, or simply about $[\in_{ij}]$. Actually $\in_{ij} = \equiv_i \circ \equiv_j$.

Lemma 3.24. *If X is an ω-theory then $[\in_{ij}]X$ is an ω-theory.*

Proof. It uses that $[\in_{ij}][\equiv_i][\equiv_i]$, thus $[\in_{ij}]X = \{\phi | [\equiv_i][\equiv_j]\phi \in X\}$, so $[\in_{ij}]X[\equiv_i][\equiv_j]X$, and now applying lemma 3.14. □

Lemma 3.25. *The expressible relations are compliant with the canonical model.*

$\in_{ij} X \subseteq Y \leftrightarrow X \in_{ijk} Y \leftrightarrow (\exists Z)(X \equiv_{ik} Z \wedge Z \equiv_{jk} Y)$

Lemma 3.26. $X \in_{ijk} Y \leftrightarrow Y \in_{jik}^{-1} X$ *or* $[\in_{ij}]X \subseteq Y \leftrightarrow [\in_{ji}^{-1}]Y \subseteq X$

3.7. AXIOMATIZATION FOR THE STRUCTURES OF INCIDENCES LOGIC $L(\Sigma_{GSI})$.

The axioms of the logic $L(\Sigma_{gsi})$ will contain all axioms of the minimal logic $L(\Sigma_0)$, and the rules are MP and Irr, the finite one, and also several other axioms specific for the geometrically related properties of the structures of incidences. Each axiomatic property of the structures of incidences have a corresponding modal axiom, which modally expresses it, and also it makes that property a property of the generated canonical frame — canonical property. The new axioms are:

A_0^* $(<\equiv_1> Op\wedge <\equiv_2> Op\wedge <\equiv_3> Op) \Rightarrow Op$, axiom for the property:

$(\forall x \in W(S))(\forall y \in W(S))(x \equiv_1 y \wedge x \equiv_2 y \wedge x \equiv_3 y \Rightarrow x = y)$

A_0^{**} $<\in_{12}> (Op\wedge <\in_{12}> Oq) \Rightarrow <\equiv_1> (<\equiv_2> Op\wedge <\equiv_3> Oq)$, axiom for the property:

$(\forall x \in W(S))(\forall y \in W(S))(\forall z \in W(S))(x \in_{1,2} y \wedge y \in_{2,3} z \Rightarrow (\exists t \in W(S))(x \equiv_1 t \wedge y \equiv_2 t \wedge z \equiv_3 t))$

A_0^{***} $\blacklozenge Op \wedge [\in_{12}^{-1}] <\in_{13}> Op \Rightarrow <\in_{23}> Op$, axiom for the property:

$(\forall x \in W(S))(\forall y \in W(S))((\forall z \in W(S))(z \in_{1,2} x \wedge z \in_{1,3} y) \Rightarrow x \in_{2,3} y)$

- The property $(\exists x \exists y \in W(S))(\neg x \equiv_1 y)$ does not need an axiom.

A_2 $\blacklozenge A \Rightarrow <\in_{12}><\in_{12}^{-1}> A$, axiom for the property:

$(\forall x \forall y \in W(S))(\exists z \in W(S))(x \in_{1,2} z \wedge y \in_{1,2} z)$

A_3 $\blacklozenge Op \wedge [\equiv_1]\neg Op \wedge <\in_{12}> (Oq\wedge <\in_{12}^{-1}> Op) \Rightarrow [\in_{12}](<\in_{12}^{-1}> Op \Rightarrow <\equiv_2> Oq)$, axiom for the property:

$(\forall x \forall y \forall z \forall t \in W(S))(\neg x \equiv_1 y \wedge x \in_{1,2} z \wedge y \in_{1,2} z \wedge x \in_{1,2} t \wedge y \in_{1,2} t \Rightarrow z \equiv_2 t)$

A_4 $Op \Rightarrow <\equiv_2> [\equiv_1]\neg Op$, axiom for the property:

$(\forall x \exists y \exists z \in W(S))(\neg y \equiv_1 z \wedge y \in_{1,2} x \wedge z \in_{1,2} x)$

A_5 $Op \Rightarrow \blacklozenge([\in_{12}]\neg Op)$, axiom for the property:

$(\forall x \exists y \in W(S))(\neg y \in_{1,2} x)$

A_6 $\blacklozenge Op \wedge \blacklozenge Oq \Rightarrow <\in_{13}> (<\in_{13}^{-1}> Op\wedge <\in_{13}^{-1}> Oq)$, axiom for the property:

$(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(x \in_{1,3} t \wedge y \in_{1,3} t \wedge z \in_{1,3} t)$

A_7 $\blacklozenge Op \wedge \blacklozenge Oq \wedge [\in_{12}]([\in_{12}^{-1}]\neg Op \vee [\in_{12}^{-1}]\neg Oq) \wedge <\in_{13}> (Or\wedge <\in_{13}^{-1}] > \neg Op\wedge <\in_{13}^{-1}> \neg Oq) \Rightarrow [\in_{13}](<\in_{13}^{-1}> Op\wedge <\in_{13}^{-1}] > Oq \Rightarrow <\equiv_3> Or)$, axiom for the property:

$(\forall x \forall y \forall z \in W(S))(\forall u \forall v \in W(S))(x \in_{1,3} u \wedge y \in_{1,3} u \wedge z \in_{1,3} u \wedge x \in_{1,3} v \wedge y \in_{1,3} v \wedge z \in_{1,3} v \wedge (\forall l \in W(S))(\neg x \in_{1,2} l \vee \neg y \in_{1,2} l \vee \neg z \in_{1,2} l) \Rightarrow u \equiv_3 v)$

- The property $(\forall x \exists y \in W(S))(y \in_{1,3} x)$ does not need an axiom.

A_9 $Op \Rightarrow \blacklozenge[\in_{13}]\neg Op$, axiom for the property:

$(\forall x \exists y \in W(S))(\neg y \in_{1,3} x)$

A_{10} $\blacklozenge Op \wedge [\equiv_1]\neg Op \wedge <\in_{12}> (Or\wedge <\in_{12}^{-1}> Op) \wedge <\in_{13}> (Oq\wedge <\in_{13}^{-1}> Op) \Rightarrow \blacksquare(Or \Rightarrow <\in_{23}> Oq)$, axiom for the property:

$(\forall x \forall y \forall z \forall t \in W(S))(\neg x \equiv_1 y \wedge x \in_{1,2} z \wedge y \in_{1,2} z \wedge x \in_{1,3} t \wedge y \in_{1,3} t \Rightarrow z \in_{2,3} t)$

A_{11} $\blacklozenge Op \wedge <\in_{13}^{-1}> (Oq \wedge <\in_{13}> Op) \Rightarrow <\in_{13}^{-1}> (<\in_{13}> Op \wedge [\equiv_1]\neg Oq)$, axiom for the property:

$(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(z \in_{1,3} x \wedge z \in_{1,3} y \Rightarrow (\neg t \equiv_1 z) \wedge t \in_{1,3} x \wedge t \in_{1,3} y)$

Lemma 3.27. *All that modal formulas are true at the class of structures of incidences* Σ_{gsi}

Proof. Simple check that each modal formula is true at frames with the corresponding property, using contraposition. If the formula is not true then the frame does not have the corresponding property. □

Lemma 3.28. *All the modal formulas above modally expresses their correspondent properties of the structures of incidences.*

Proof. Simple check for each modal formula using contraposition. If the frame does not possess its correspondent property then there is an evaluation in which the the formula is not true. □

Lemma 3.29. *Adding each formula from above list as an axiom, makes the generic canonical frame to posses the same property, which the axiom modally expresses — generic canonical frame is a structure of incidence.*

Proof. Check that adding each modal formula, makes its property a property of the generic canonical frame, using properties of the *constant* formulas and maximal consistent ω-theories — lemmas 3.19 and 3.18. We can demonstrate it for 2 formulas, for A_4 and A_6:

A_4: Let we have added the axiom A_4, so $Op \Rightarrow <\equiv_2> [\equiv_1]\neg Op \in L$, for any variable p. Let X is a maximal consistent ω-theory, according to lemma 3.18, then there is a variable p_1, such that the constant $Op_1 \in X$, and also $Op_1 \Rightarrow <\equiv_2> [\equiv_1]\neg Op_1 \in L \subseteq X$. X closed under MP, then $<\equiv_2> [\equiv_1]\neg Op_1 \in X$, equivalent to $\neg[\equiv_2]\neg[\equiv_1]\neg Op_1 \in X$, X is a maximal, then $[\equiv_2]\neg[\equiv_1]\neg Op_1 \notin X$, and then according to 3.10, $\neg[\equiv_1]\neg Op_1 \notin [\equiv_2]X$, from the lemma 3.14 it follows that $[\equiv_2]X$ is an ω-theory, from Lindenbaum's lemma 3.13, there is a maximal consistent ω-theory Z, such that $[\equiv_2]X \subseteq Z$ and $\neg[\equiv_1]\neg Op_1 \notin Z$, $<\equiv_1> Op_1 \notin Z$. From $X \equiv_{2k} Z$, reachable from Z, then Z is into the domain of the generic canonical model, see definition 3.15. Now if we assume that $X \equiv_{1k} Z$ from lemma 3.23 $<\equiv_1> Op_1 \in Z$, contradiction, so $\neg(X \equiv_{1k} Z)$. From $X \equiv_{2k} Z$, and $\equiv_{2k}$ equivalence relation we have $Z \equiv_{2k} X$, and $\equiv_{1k}$ equivalence relation we get that $Z \equiv_{1k} X \equiv_{2k} X$ and $X \equiv_{1k} X \equiv_{2k} X$, from the lemma 3.25 $Z \in_{12k} X$, and $X \in_{12k} X$. As conclusion we can say that for each maximal consistent ω-theory X we found a maximal consistent ω-theories $Y = X$ and Z such that $\neg(Y \equiv_{1k} Z)$ and $Z \in_{12k} X$ and $Y \in_{12k} X$.

Also it is seen that the property $(\forall x \exists y \exists z \in W(S))(\neg y \equiv_1 z \wedge y \in_{1,2} x \wedge z \in_{1,2} x)$ is equivalent to the property $(\forall x \exists z \in W(S))(\neg x \equiv_1 z \wedge x \equiv_2 z)$.

A_6: Let we have added the axiom A_6. Let X, Y and Z are maximal consistent ω-theories such that they belong to the generic canonical frame, thus $\blacksquare X \subseteq Y$, $\blacksquare X \subseteq Z$. From lemma 3.18 there is a variables p, q such that $Op \in Y$ and $Oq \in Z$, now from lemma 3.23, it follows that $\blacklozenge Op \in X$ and $\blacklozenge Oq \in X$. A_6 axiom is $\blacklozenge Op \wedge \blacklozenge Oq \Rightarrow <\in_{13}> (<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq)$, and X is closed under MP, then $<\in_{13}> (<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \in X$, so $[\in_{13}]\neg(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \notin X$, from lemma 3.25 we get that $\neg(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \notin [\in_{13}]X$, and $[\in_{13}]X$ ω-theory. Lindenbaum's lemma 3.13 found that there is a maximal consistent ω-theory T such that $[\in_{13}]X \subseteq T$ and $\neg(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \notin T$, or $(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \in T$. From the lemma 3.23 we conclude for T that $[\in_{13}^{-1}]T \subseteq Y$ and $[\in_{13}^{-1}]T \subseteq Z$, and finally from the lemma 3.26 we conclude for T that: $Y \in_{13k} T$ and $Y \in_{13k} Z$. And from $[\in_{13}]X \subseteq T$ then $X \in_{13k} T$, which shows that we found T maximal consistent ω-theory from generic canonical frame that suffices the property: $(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(x \in_{1,3} t \wedge y \in_{1,3} t \wedge z \in_{1,3} t)$. □

Theorem 3.3. *The logic with the axiomatization above is complete for the class of structures of incidences.*

This completes the axiomatization of $L(\Sigma_{gsi})$ logic of the class of structures of incidences.

3.8. OPEN QUESTIONS

Besides the axiomatization with finite number of axiom schemas for the logic of the structures of incidences, and ability to proof geometrically related properties with it. There are several open question unsolved up to now.

- Q_1 Is it decidable? It is not clean if there is an algorithm about checking if a formula is a theorem or not.

- Q_2 Is it useful to proof some interesting? Some simple properties that are easy proofed with first order logic are not seen how to be proof with this modal logic. For example the property that: "for each line there is another line that is crossed to the first one, the 2 lines has no common point". That property is modally definable with the formula: $Op \Rightarrow \blacklozenge[\equiv_2][\equiv_1][\equiv_2]\neg Op$ or it's semantically equivalent form $Op \Rightarrow \blacklozenge[\equiv_2][\equiv_1][\equiv_2]\neg p$. Desarque's Theorem is also modally definable. Both modal formulas should be theorems, but the proof is not seen.

- Q_3 Is it has a simpler axiomatization? For example is it possible to eliminate the rule Irr. Also it is not known if the rule Irr is useful in any proofs with the current axiomatization.

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