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A CLASSIFICATION OF THE UNIFORM COVERINGS [1]

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A classification of the uniform coverings over a given uniformly locally path-wise connected and semi-one connected space is made, by the set of the classes conjugated by themselves subgroups of the fundamental group of the base. We use some well-known theorems in the topological case, proving that they are available in the category of the uniform spaces. We supply the covering space with a suitable uniformity, very closely connected with the uniformity of the base and use it for our investigations.

1. CONNECTION BETWEEN THE U-COVERINGS AND THE FUNDAMENTAL GROUP OF THEIR BASE. AUTOMORFISMS OF THE U-COVERINGS

Definition 1. By an uniform covering we mean a covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$ over an uniform space (X, U), which is trivial over every element U_α of an uniform cover $\delta = \{U_\alpha\}_{\alpha \in A}$ of X, but the family $\{p/\tilde{U}_{\alpha\lambda}\}_{\lambda \in A, \alpha \in A}$ of the uniform isomorfisms is equicontinuous [2].

As a particular case of the topological coverings, the uniform coverings satisfy some well-known theorems. For example, the homomorphism $p_\sharp\colon \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is a monomorphism. If we replace the point $\tilde{x}_0$ by x_0' and connect the two points by a path $\tilde{\omega}$, the monomorphism $p_\sharp$ commutes with the isomorphism of conjugatness $h_{[\omega]}$.

That is: $p_\sharp \pi(\tilde{X}, \tilde{x}_0) = h_{[\omega]} p_\sharp \pi(\tilde{X}, x_0')$. $\quad (\omega = p\tilde{\omega})$

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Theorem. [4] *Let* $p\colon (\tilde{X},\tilde{U}) \to (X,U)$ *be a U-covering and* $x_0 \in X$. *Then the set* $\{p_\sharp\pi(\tilde{X},\tilde{x}_0) \mid \tilde{x}_0 \in p^{-1}(x_0)\}$ *is a class of conjugated by themselves subgroups of* $\pi(X,x_0)$. *The isomorphism* $h_{[\omega]}$ *maps the class of conjugated subgroups of* $\pi(X,x_1)$ *onto the corresponding class in the group* $\pi(X,x_0)$.

Furthermore, the group $\pi(X,p(\tilde{x}_0))$ acts as a group of the right transformations on the set $p^{-1}(p(\tilde{x}_0))$ as follows: For every $\alpha \in \pi(X,p(\tilde{x}_0))$ let $\tilde{\alpha}$ be the unique lifting of α with $\tilde{\alpha}(0) = \tilde{x}_0$. Then by definition $\tilde{x}_0.\alpha = \tilde{\alpha}(1)$. If $\tilde{X}$ and X are linear connected spaces, this action is transitive. Obviously, the isotropy subgroup of the point $\tilde{x}_0$ is just $p_\sharp\pi(\tilde{X},\tilde{x}_0)$. It turns out (from the algebraic considerations) that there exists one to one correspondence between the set of the right classes.

$$\pi(X,p(\tilde{x}_0))/p_\sharp\pi(\tilde{X},\tilde{x}_0) \tag{1.1}$$

and the fibre over point $p(\tilde{x}_0) - p^{-1}(p(\tilde{x}_0))$ [1].

Let us denote by $G_u(p)$ the group of the uniform automorphisms f of the U-covering $(\tilde{X},\tilde{U}) \xrightarrow{p} (X,U)$ (such, that $pf \equiv p$). Then for every $\varphi \in G_u(p)$ the multiplication $\tilde{x}.\alpha$ satisfies the equality $f(\tilde{x}.\alpha) = (f(\tilde{x}))\alpha$. That is, $f/p^{-1}(x)$ is an automorphism of the set $p^{-1}(x)$, treating as a right $\pi(X,x)$ space. We shall prove that if (X,U) is a uniformly locally connected space (ULC), the converse also is true.

Theorem 1. *Every automorphism* $f \in G_u(p)$ *is quite defined by its restriction on* $p^{-1}(x)$.

We need some preparations before establishing that the group $G_u(p)$ is isomorphic to a subgroup $\dfrac{N(p_\sharp\pi(\tilde{X},\tilde{x}_0))}{p_\sharp\pi(\tilde{X},\tilde{x}_0)}$ of (1) (Theorem 2). The homomorphism

$$\varphi\colon G_u(p) \to N(p_\sharp\pi(\tilde{X},\tilde{x}_0))/p_\sharp\pi(\tilde{x},\tilde{x}_0) \tag{1.2}$$

is defined as follows: Let $f \in G_u(p)$ and $\tilde{\omega}$ be a curve in $\tilde{X}$ connecting $\tilde{x}_0$ and $f(\tilde{x}_0)$.

Then $\psi(f) = \varphi([p.\tilde{\omega}])$, where φ is the factor map

$$N(p_\sharp\pi(\tilde{X},\tilde{x}_0)) \to N(p_\sharp\pi(\tilde{X},\tilde{x}_0))/p_\sharp\pi(\tilde{X},\tilde{x}_0).$$

Of course, we need the lemma:

Lemma 1. *Let* $p\colon (\tilde{X},\tilde{U}) \to (X,U)$ *be a U-covering,* $p(\tilde{x}_0) = x_0$ *and the map* $f \in G_u(p)$. *If the path* $\tilde{\omega}$ *connects* $\tilde{x}_0$ *and* $f(\tilde{x}_0)$, *then* $[p\tilde{\omega}] \in N(p_\sharp\pi(\tilde{X},\tilde{x}_0))$.

Proof. (see [3]). Since f is a homeomorphism, we can write the following equalities:

$$\begin{aligned}[p\tilde{\omega}]^{-1}p_\sharp\pi(\tilde{X},\tilde{x}_0)[p\tilde{\omega}] &= h_{[p\tilde{\omega}]}(p_\sharp\pi(\tilde{X},\tilde{x}_0)) = p_\sharp(h_{[\tilde{\omega}]}\pi(\tilde{X},\tilde{x}_0)) = \\ &= p_\sharp\pi(\tilde{X},f(\tilde{x}_0)) = p_\sharp(f_\sharp\pi(\tilde{X},\tilde{x}_0)) = p_\sharp\pi(\tilde{X},\tilde{x}_0). \quad \square\end{aligned}$$

Theorem 2. *Let* $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$ *be a uniform covering, its base is being a connected and ULC-space. Then the map (2) is a group isomorphism.*

Proof. Analogous theorem is known about the topologycal case ([3]). We shall prove only that ψ is an epimorphism, which is new. Let the class of the loop $\omega - [\omega]$ belong to $N(p_\sharp \pi(\tilde{X}, \tilde{x}_0))$:

$[\omega]^{-1} p_\sharp \pi(\tilde{X}, \tilde{x}_0)[\omega] = p_\sharp \pi(\tilde{X}, \tilde{x}_0)$. We lift ω to $\tilde{\omega}$ such, that $\tilde{\omega}(0) = \tilde{x}_0$ and put $\tilde{x}'_0 = \tilde{\omega}\,(1)$. Writing the equalities

$$p_\sharp \pi(\tilde{X}, \tilde{x}_0) = p_\sharp h_{[\tilde{\omega}]} \pi(\tilde{X}, \tilde{x}'_0) = h_{[\omega]} p_\sharp \pi(\tilde{X}, \tilde{x}'_0) = [\omega^{-1}] p_\sharp \pi(\tilde{X}, \tilde{x}'_0)[\omega] = p_\sharp \pi(\tilde{X}, \tilde{x}'_0),$$

we get $p_\sharp \pi(\tilde{X}, \tilde{x}'_0)[\omega] = p_\sharp \pi(\tilde{X}, \tilde{x}_0)$. Now we make use of the theorem of the uniformly continuous lifting of the map p ([2]). Denoting the corresponding liftings by f and g, we obtain the diagrams

$$\begin{array}{ccc} (\tilde{X}, \tilde{x}_0) & \overset{f}{\dashrightarrow} & (\tilde{X}, \tilde{x}'_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (X, x_0) & = & (X, x_0) \end{array} \qquad \begin{array}{ccc} (\tilde{X}, \tilde{x}'_0) & \overset{g}{\dashrightarrow} & (\tilde{x}, \tilde{x}_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (X, x_0) & = & (X, x_0) \end{array} \tag{1.3}$$

It follows (by the uniqueness of f and g) that f and g are multually reverse uniform isomorphisms, i.e. $f \in G_u(p)$. □

Remark. If we combine the isomorphism ψ and the diagrams (3), we see that every map $f \in G_u(p)$ is well defined by its restriction on $p^{-1}(x)$.

Of course, the space (X, U) must be connected and uniformly locally linear connected.

2. REGULAR UNIFORM COVERINGS

In this point we shall assume, that (X, U) is a connected, uniformly locally connected space.

Definition 2. The uniform covering $(\tilde{X}, \tilde{U}) \to (X, U)$ is called regular at the point $x_0 \in X$, iff for every $\tilde{x}_0 \in p^{-1}(x_0)$ the group $p_\sharp \pi(\tilde{X}, \tilde{x}_0)$ is a normal dividor of $\pi(X, x_0)$, i.e. it coincides with all its conjugated subgroup of $\pi(X, x_0)$

As it is known, this definition does not depend on the choise of the point x_0. For regular uniform coverings the isomorphism ψ looks as follows:

$$\psi\colon G_u(p) \to \pi(X, x_0)/p_\sharp \pi(\tilde{X}, \tilde{x}_0)\,. \tag{2.1}$$

Given two points $\tilde{x}_0$ and $\tilde{x}'_0$ of $\tilde{x}$ with $p(\tilde{x}_0) = p(\tilde{x}'_0)$, we can write the diagrams (3) and get that there exists an isomorphism $f \in G_u(p)$ such that $f \in (\tilde{x}_0) = \tilde{x}'_0$. This fact often is accepted as a definition of regularity.

Now, let us connect $\tilde{x}_0$ and $\tilde{x}'_0 = f(\tilde{x}_0)$ by a path and recall the action of the group $\pi(X, x_0)$ on the layer $p^{-1}(x_0)$. We obtain that the group $\pi(X, x_0)$ acts transitively on the layer $p^{-1}(x_0)$.

Let us recall the important particular case of the regular coverings ∐ the universal coverings.

Definition 3. The uniform covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$ is called universal uniform covering if $\pi(\tilde{X}, \tilde{x}_0) = 0$.

In this case $p_\sharp\pi(\tilde{X}, \tilde{x}_0) = 0$ and hence the group $\pi(X, x_0)$ acts on $p^{-1}(x_0)$ without fixed points. We immediately obtain

Theorem 3. *If the uniform covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$ is an universal covering, the groups $G_u(p)$ and $\pi(X, x_0)$ are isomorphic. The order of the group $\pi(X, x_0)$ is equal to the number of the leafs of p.*

Now we go into details in the action of the group $G_u(p)$ over an regular U-covering p. The regular U-covering p is defined by a uniform cover $\{U_\alpha\}_{\alpha\in A}$ of (X, U) that consists of the fundamental neighborhoods of the points, i.e. for each $\alpha \in A$ $\; p^{-1}(U_\alpha) = \bigcup\limits_{\lambda\in A} \tilde{U}_{\alpha\lambda}$ and all isomorphisms $p/\tilde{U}_{\alpha\lambda}\colon \tilde{U}_{\alpha\lambda} \to \tilde{U}_\alpha$ are equicontinuous.

Let the points $\tilde{x}_{\alpha\lambda} \in \tilde{U}_{\alpha\lambda}$ and $\tilde{x}_{\alpha\lambda'} \in \tilde{U}_{\alpha\lambda'}$ satisfy $p(\tilde{x}_{\alpha\lambda}) = p(\tilde{x}_{\alpha\lambda'})$.

Then the automorphism $f_\lambda^{\lambda'}$ maps $\tilde{x}_{\alpha\lambda}$ into $\tilde{x}_{\alpha\lambda'}$ also maps a connected neighborhood $\tilde{U}_{\alpha\lambda}$ of $\tilde{x}_{\alpha\lambda}$ into $\tilde{U}_{\alpha\lambda'}$ uniformly isomorphic. We obtain the next theorem.

Theorem 4. *For arbitrary $\alpha \in A$ and a couple of points $\tilde{x}_{\alpha\lambda}$, $\tilde{x}_{\alpha\lambda'}$ with $p(\tilde{x}_{\alpha\lambda}) = p(\tilde{x}_{\alpha\lambda'})$ there exists $f_\lambda^{\lambda'} \in G_u(p)$, $f_\lambda^{\lambda'}\colon (\tilde{X}, \tilde{x}_{\alpha\lambda}) \to (\tilde{X}, \tilde{x}_{\alpha\lambda'})$, that maps a neighborhood $\tilde{U}_{\alpha\lambda}$ uniformly isomorphic. onto $\tilde{U}_{\alpha\lambda'}$. The family $\{f_\lambda^{\lambda'}\}$ is equicontinuous on (out of) $(\lambda, \lambda' \in \Lambda)$, and even on $\alpha \in A$.*

Before we proceed to the construction of a uniform regular covering, we give the following

Definition 4. *Let (Y, V) be a uniform space and G be a group of its equicontinuous uniform isomorphisms. We say that the group G acts uniformly discretely over (Y, V), iff there exists a uniform cover $\{V_\lambda\}_{\lambda\in\Lambda}$ of Y such that: if $gV_\lambda \cap g'V_\lambda \neq \emptyset \to g = g'$ (V_λ is an arbitrary element of $\{V_\lambda\}_{\lambda\in\Lambda}$).*

Theorem 5. *Let (Y, V) be a connected and uniformly locally linear connected space and G is a group of its isomorphisms, which acts uniformly discretely over (Y, V). Then the natural projection p of Y on the space of orbits Y/G is a regular uniform covering with a group of automorphisms $G_u(p)$.*

Proof. First we shall supply Y/G with a factor-uniformity $\overline{V}$. If W belongs to the uniformity V then two orbits yG and y_1G we shall call $\overline{W}$-near if they have representatives yg and y_1g, which are W-near. This definition satisfies the axioms of uniformity as the action of G on (Y, V) is uniformly equicontinuous. V is the strongest uniformity at which p is uniformly continuous.

Now, let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a uniform covering of Y, such that for $g_1 \neq g_2 \in G$ we have $g_1V_\lambda \cap g_2V_\lambda \neq \phi$ and the sets V_λ are linear connected. We put $U_\lambda = p(V_\lambda)$. Obviously $p/V_\lambda \colon V_\lambda \to U_\lambda$ is an uniform isomorphism. If V_μ is another component $p^{-1}(U_\lambda)$, then there exists an automorphism $h \in G$, such that $V_\mu = V_\lambda.h$. Hence $\frac{p}{V_\mu} = \frac{p}{V_\lambda}.h$ is also automorphism. The family $\{p/V_\lambda\}_{\lambda\in\Lambda}$ is equicontinuous at the given condition.

The group of automorphisms $A(Y,p)$ coincides with G and as it acts transitively on $p^{-1}(yG)$, the constructed covering is regular. □

3. CLASSIFICATION OF THE UNIFORM COVERINGS

Let (X, U) be a uniform space and $\langle k\rangle$ be a class of selfconjugated subgroups of the group $\pi(X, x)$. We shall prove that there exists an uniform covering $p\colon \tilde{X} \to X$, such that the group $p_\sharp\pi(\tilde{X}, \tilde{x})$ belongs to the class $\langle k\rangle$. It is known that topologically such unique covering $\tilde{X}$ exists in some additional suppositions about the space X. It is necessary to supply the space $\tilde{X}$ by a suitable uniform structure such that we get a uniform covering. We need to increase the suppositions on (X, U) for this purpose.

In Theorem 6 we solve this task, when there exists a universal covering (Y, q) over (X, U). The construction on this covering (Y, q) is done in Theorem 7.

Theorem 6. *Let the uniform space (X, U) be uniformly locally linear connected and uniformly locally semione-connected. If $\langle k\rangle$ is an arbitrary class of conjugated subgroups of $\pi(X, x)$, there exists a uniform covering $(\tilde{X}, \tilde{U}, p)$: $p_\sharp\pi(\tilde{X}, \tilde{x})$ belongs to the class $\langle k\rangle$.*

Let $(Y, V) \xrightarrow{q} (X, U)$ is the universal uniform covering over (X, U) (see theorem 7). As we know, the group $\pi(X, x)$ acts on $q^{-1}(x)$ transitively and without fixed points. We take $y \in q^{-1}(x)$ and $k \subset \pi(X, x)$. Then the following subgroup $H \subset G_u(q)$ corresponds to K by the isomorphism (4):

$$\varphi \in H \Leftrightarrow \text{ there exist } \alpha \in K\colon \varphi(y) = y\alpha\,.$$

As H is a subgroup of $G_u(q)$, it acts uniformly discretely on Y and we can introduce a factor uniformity in the space of orbits Y/H. Let $\tilde{X} = Y/H$ and $p\colon Y/H \to X$ is the map defined by q. We got the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{r} & Y/H \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & = & X \end{array}$$

which shows, that p is a uniform covering. The isotropy group of the point $\tilde{x} = p^{-1}(x) = r(y)$ is obviously K. Hence $p_\sharp\pi(\tilde{X}, \tilde{x}) = K$. □

Before we construct the universal uniform covering q, we need the following

Definition 5. The space (X, U) is called uniformly semilocally oneconnected (USL1) if there exist arbitrary little open uniform covers δ with the property: every loop $S^1 \to X$, whose image consists of some element of δ, is contractable.

Theorem 7. *Let (X, U) be a USL1-space. Then it has a unique (precisely to a uniform isomorphism) universal covering.*

Proof. Although the proof reminds the traditional in the topological case, we shall expose it, because it is specific in the ushering an uniform structure $\tilde{U}$ in X. The space $\tilde{X}$ is constructed as a space of the classes $[\alpha_x]$ homotopic curves in X, beginning at x_0. The map $p\colon \tilde{X} \to X$ is $p[\alpha] = \alpha(1)$. To usher an uniform structure in $\tilde{X}$ first of all we choose a basic open uniform cover δ of X, satisfying the USL1 ĻĮ condition. For every $U \in \delta$ and a class $[\alpha]\colon p(\alpha) \in U$, we put $\langle \alpha, U \rangle = \{\beta\colon \beta = \alpha.\alpha'$, where $\alpha'(I) \subset U\}$.

We got a cover $\tilde{\delta}$ of $\tilde{x}$, consisting of the sets $\langle \alpha, U \rangle$ the base of $u \in \delta, \alpha(1) \in U$. If δ varies trough U, then the family $\tilde{\delta}$ defines the base of a uniform structure $\tilde{U}$ on $\tilde{X}$. $\tilde{U}$ is the coarsest uniform structure in $\tilde{X}$ with which p is uniform continuous. Some important, but easily proved properties of sets $\langle \alpha, U \rangle$ are available.

I. The map $p/\langle \alpha, U \rangle$ is an isomorphism of $\langle \alpha, U \rangle$ on U.

II. Let $U \in \delta$, $x_0 \in X$, and $x \in U$ are fixed. Let $\langle \alpha_\lambda \rangle_{\lambda \in \Lambda}$ is the set of all classes of paths, beginning at x_0 and ending at x. If x varies through U we get that $p^{-1}(U) = \bigcup\limits_{\lambda \in \Lambda} \langle [\alpha_\lambda], U \rangle$ and the sets $\langle [\alpha_\lambda], U \rangle$ do not intersect as U is one-connected set.

III. The family of isomorphisms $\{p/\langle [\alpha_\lambda], U \rangle\}_{\lambda \in \Lambda}$ is equicontinuous.

Hence we got a uniform covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$.

We shall not repeat the known fact that $\tilde{x}$ is a linear connected space, but we shall prove that it is one-connected. For this purpose we shall recall how the curves in X can be lifted in $\tilde{x}$.

Let $\alpha\colon I \to X$ be a curve, beginning at $x_0 \in X$. We denote by $x_0 \in X$ the class of the constant curve, i.e. $\tilde{x}_0 = [c_{x_0}]$. For an arbitrary $t \in I$ let α^t be the curve $\alpha^t(s) = \alpha(st)$. Then, for $\tilde{\alpha}$ we have $\tilde{\alpha}(t) = [\alpha^t]$. Obviously $\tilde{\alpha}(0) = [\alpha^0] = [c_{x_0}] = \tilde{x}_0$. As $p_\sharp$ is a mononmorphism, we have to prove that $p_\sharp(\tilde{X}, \tilde{x}_0) = 0$, i.e., if α is a loop in (X, x_0), whose lifting is a loop, then $\alpha \in c_{x_0}$. But this follows from the definition. The equality $\tilde{\alpha}(1) = [\alpha^1] = \tilde{x}_0 = [c_{x_0}]$ holds iff the curves α and c_{x_0} are homotopic. The existence of the universal uniform covering over each uniform L1C-space (X, U) is proved.

We shall not prove the uniqueness of $(\tilde{X}, \tilde{U})$, although it does not follow automatically from those in the topological case. □

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