ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 98

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 98

A REDUCIBILITY IN THE THEORY OF ITERATIVE COMBINATORY SPACES

DIMITER SKORDEV

The notion of iterative combinatory space introduced in the past by the present author gave the framework for an algebraic generalization of a part of Computability Theory. In the present paper a reducibility concerning iterative combinatory spaces is considered, as well as the corresponding equivalence. A statement of J. Zashev about variants of iterative combinatory spaces is shown to fail under certain interpretations of the notion of variant in the terms of the relations in question.

Keywords: combinatory space, iterative combinatory space, computable element, computable mapping

2000 MSC: 03D75, 03D50

1. SOME PRELIMINARIES

According to Definition II.1.1 in [5], a combinatory space is a 9-tuple

$$\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F),$$

where $\mathcal{F}$ is a partially ordered semigroup, I is its identity, $\mathcal{C} \subseteq \mathcal{F}$, $\Pi : \mathcal{F}^2 \to \mathcal{F}$, $\Sigma : \mathcal{F}^3 \to \mathcal{F}$, $L, R, T, F \in \mathcal{F}$, and the following conditions are identically satisfied, when $\varphi, \psi, \theta, \chi$ range over $\mathcal{F}$, and a, b, c range over $\mathcal{C}$:

$$\forall c(\varphi c \geq \psi c) \;\Rightarrow\; \varphi \geq \psi,$$
$$\Pi(a,b) \in \mathcal{C},\ L\,\Pi(a,b) = a,\ R\,\Pi(a,b) = b,$$
$$\Pi(\varphi,\psi)c = \Pi(\varphi c, \psi c),\ \Pi(I,\psi c)\theta = \Pi(\theta,\psi c),\ \Pi(c,I)\theta = \Pi(c,\theta),$$
$$T \neq F,\ Tc \in \mathcal{C},\ Fc \in \mathcal{C},$$

$$\Sigma(T,\varphi,\psi)=\varphi,\ \Sigma(F,\varphi,\psi)=\psi,\ \theta\,\Sigma(\chi,\varphi,\psi)=\Sigma(\chi,\theta\varphi,\theta\psi)$$
$$\Sigma(\chi,\varphi,\psi)c=\Sigma(\chi c,\varphi c,\psi c),\ \Sigma(I,\varphi c,\psi c)\theta=\Sigma(\theta,\varphi c,\psi c),$$
$$\varphi\geq\psi,\ \theta\geq\chi\ \Rightarrow\ \Sigma(I,\varphi,\theta)\geq\Sigma(I,\psi,\chi)$$

(the same notion is named "semicombinatory space" in [3,4]). The definition implies that multiplication, Π and Σ are monotonically increasing operations. If for some given σ,χ in $\mathcal{F}$ the equation $\theta=\Sigma(\chi,\theta\sigma,I)$ has a least solution θ, and this solution has certain additional nice properties, then the solution in question is called *the iteration of σ controlled by χ*, and it is denoted by $[\sigma,\chi]$. [1] In the present paper it will be also called *the $\mathcal{S}$-iteration of σ controlled by χ*, and the notation $[\sigma,\chi]^{\mathcal{S}}$ will be also used for it.

The triple $(\mathcal{F},I,\mathcal{C})$ will be further called *the kernel* of the combinatory space $(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$. We shall often consider pairs of combinatory spaces having one and the same kernel. The following statement concerning such pairs can be obtained as an immediate corollary of the definition of iteration.

Lemma 1.1. *Let $\mathcal{S}_i=(\mathcal{F},I,\mathcal{C},\Pi_i,L_i,R_i,\Sigma_i,T_i,F_i),\ i=0,1,$ be combinatory spaces, let χ_0,χ_1 be elements of $\mathcal{F}$ such that $\Sigma_0(\chi_0,\varphi,\psi)=\Sigma_1(\chi_1,\varphi,\psi)$ for all φ,ψ in $\mathcal{F}$, and let σ,ι be elements of $\mathcal{F}$ such that ι is the $\mathcal{S}_0$-iteration of σ controlled by χ_0. Then ι is also the $\mathcal{S}_1$-iteration of σ controlled by χ_1.*

A combinatory space $\mathcal{S}=(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$ is called *iterative* if the iteration of σ controlled by χ exists for any σ and χ in $\mathcal{F}$. A notion of computability in iterative combinatory spaces was studied, and some versions of the First Recursion Theorem and of the Normal Form Theorem are among the results about it (intuitively, the elements of $\mathcal{F}$ play the role of functions in that theory, ordinary computability in the set of the natural numbers and abstract first order computability in the sense of Moschovakis [1] being particular instances). The considered computability is a relative one, namely for any subset $\mathcal{B}$ of $\mathcal{F}$ some elements of $\mathcal{F}$ and some operations in $\mathcal{F}$ are called $\mathcal{S}$-computable in $\mathcal{B}$ (however, mainly the particular case of an empty $\mathcal{B}$ will matter for the present paper).

Numerous examples of iterative combinatory spaces are given in the books [2,5]. A class of such examples (actually the simplest ones) is indicated in Example II.1.2 of [5], the iterativeness of the corresponding combinatory spaces being established in Section II.4 of [5]. The construction of these examples looks as follows. We take an infinite set M, an injective mapping J of M^2 into M, partial mappings L and R of M into M such that $L(J(s,t))=s,\ R(J(s,t))=t$ for all s,t in M, as well as two total mappings T and F of M into M and a partial predicate H on M such that $H(T(u))$ is true and $H(F(u))$ is false for any u in M (any 7-tuple (M,J,L,R,T,F,H) with such components is called a *computational structure*). Then we consider the 9-tuple $(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$, where $\mathcal{F}$ consists of all partial mappings of M into M, the multiplication in $\mathcal{F}$ is defined by $\varphi\psi=\lambda u.\,\varphi(\psi(u))$, the inequality $\varphi\geq\psi$ means that φ is an extension of ψ, I is

[1] The precise definition of iteration can be found in Section II.3 of [5], and, up to an exchange of the second and the third arguments of Σ, also in [3,4].

the identity mapping of M onto itself, $\mathcal{C}$ consists of all total constant mappings of M into M, $\Pi(\varphi,\psi) = \lambda u.\, J(\varphi(u),\psi(u))$ for any φ and ψ in $\mathcal{F}$, and we have $\Sigma(\chi,\varphi,\psi)(u) = v$ iff either $H(\chi(u))$ is true and $\varphi(u) = v$, or $H(\chi(u))$ is false and $\psi(u) = v$. It is shown that each 9-tuple constructed in such a way is an iterative combinatory space, and the equality $[\sigma,\chi](u) = v$ holds iff there are a non-negative integer m and a finite sequence $w_0, w_1, \dots, w_m$ of elements of M such that $w_0 = u$, $w_m = v$, $H(\chi(w_j))$ is true and $w_{j+1} = \sigma(w_j)$ for $j = 0, 1, \dots, m-1$, whereas $H(\chi(w_m))$ is false. The combinatory spaces of this kind will be called here *pf-spaces* (*combinatory spaces of partial functions*). A pf-space will be called *ordinary* if its last two components are constant functions. Without naming them so, the ordinary pf-spaces are considered already in [2] – they actually form the content of Example 1 in Section II.1.3 there, and their iterativeness is shown in Section III.3.2 of the book.

The next two examples indicate certain concrete pf-spaces corresponding to computational structures whose first component is the set $\mathbb{N}$ of the non-negative integers.

Example 1.1. Let J be the bijection from $\mathbb{N}^2$ to $\mathbb{N}$ defined by

$$J(s,t) = \frac{(s+t)(s+t+1)}{2} + s\,,$$

L, R, T, F be the functions from $\mathbb{N}$ to $\mathbb{N}$ defined by the equalities

$$L(J(s,t)) = s,\;\; R(J(s,t)) = t,\;\; T(u) = 1,\;\; F(u) = 0,$$

and H be the predicate that is false at 0 and true at all other elements of $\mathbb{N}$. Then $(\mathbb{N}, J, L, R, T, F, H)$ is a computational structure, and we may consider its corresponding pf-space.

Example 1.2. The same as the previous example, except that T is defined by means of the equality $T(u) = u + 1$ (the corresponding pf-space is not an ordinary one).

Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be an iterative combinatory space. The notion of $\mathcal{S}$-computability (coinciding with $\mathcal{S}$-computability in the empty set in the terminology of [5]) is defined as follows. An element of $\mathcal{F}$ will be called *$\mathcal{S}$-computable* if this element can be obtained from the elements L, R, T, F by means of multiplication, the operation Π and $\mathcal{S}$-iteration (if $\mathcal{B}$ is a subset of $\mathcal{F}$ then an element of $\mathcal{F}$ is called *$\mathcal{S}$-computable in $\mathcal{B}$* if this element can be obtained from elements of the set $\{L, R, T, F\} \cup \mathcal{B}$ by means of the three operations in question). A mapping Γ of $\mathcal{F}^n$ into $\mathcal{F}$ will be called *$\mathcal{S}$-computable* if for arbitrary $\theta_1, \dots, \theta_n$ in $\mathcal{F}$ there is an explicit expression for $\Gamma(\theta_1, \dots, \theta_n)$ through $L, R, T, F, \theta_1, \dots, \theta_n$ by means of multiplication, the operation Π and $\mathcal{S}$-iteration, the form of this expression not depending on the choice of $\theta_1, \dots, \theta_n$ ($\mathcal{S}$-computability of Γ in a given subset $\mathcal{B}$ of $\mathcal{F}$ is defined similarly, but the expression for $\Gamma(\theta_1, \dots, \theta_n)$ may contain now also notations for some fixed elements of $\mathcal{B}$).

Remark 1.1. By the equality $[\sigma, F] = I$, the function I is $\mathcal{S}$-computable. The mapping Σ is also $\mathcal{S}$-computable, since (as shown in Section II.5 of [5])

$$\Sigma(\chi, \varphi, \psi) = [R_\star \psi][R_\star^2 \varphi R]\Pi(\chi, L_\star),$$

where $[\sigma] = R[\sigma R, L]$, $L_\star = \Pi(T, I)$, $R_\star = \Pi(F, I)$. Hence adding I to the initial elements and Σ to the used operations in the above definitions would not enlarge the set of the $\mathcal{S}$-computable elements of $\mathcal{F}$ and the set of the $\mathcal{S}$-computable mappings of $\mathcal{F}^n$ into $\mathcal{F}$.

Example 1.1 (continuation). Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be the combinatory space indicated in Example 1.1. Then all $\mathcal{S}$-computable elements of $\mathcal{F}$ are one-argument partial recursive functions. However, the converse statement is not true. For instance the primitive recursive function θ defined by the equality $\theta(u) = |u - 1|$ is not $\mathcal{S}$-computable. To prove this, we consider the family of all pre-images of the sets $\{0\}$ and $\mathbb{N} \setminus \{0\}$ under products of finitely many L's and R's (the function I being also regarded as such a product). Let $\mathcal{T}$ be the topology in $\mathbb{N}$ having as a prebase this family. The functions J, L, R, T, F can be easily shown to be continuous with respect to $\mathcal{T}$. It follows from here by Exercise II.4.21 of [5] that all functions from $\mathcal{F}$ have open domains and are continuous with respect to $\mathcal{T}$. On the other hand, the function θ is not continuous with respect to $\mathcal{T}$ since $\theta^{-1}\{0\} = \{1\}$, and the set $\{1\}$ is not open in $\mathcal{T}$ because any open set containing 1 contains also some number distinct from 1, namely some number of the form $J(0, J(0, \ldots J(0, J(0, 2)) \ldots))$.

Example 1.2 (continuation). Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be the combinatory space from Example 1.2. Then again all $\mathcal{S}$-computable elements of $\mathcal{F}$ are one-argument partial recursive functions, but now the converse statement is also true. In view of Theorem I.3.1 of [5] it is sufficient to show the $\mathcal{S}$-computability of the function $\lambda u.\, u \dot{-} 1$, where $u \dot{-} 1$ is $u - 1$ for $u \in \mathbb{N} \setminus \{0\}$ and 0 for $u = 0$. Its $\mathcal{S}$-computability is seen from the fact that $R(J(u, u) + 1) = u - 1$ for any positive integer u, and therefore $\lambda u.\, u \dot{-} 1 = \Sigma(I, RT\Pi(I, I), F)$. [2]

2. REDUCIBILITY OF AN ITERATIVE COMBINATORY SPACE TO A GIVEN ONE

We shall again be interested in pairs of combinatory spaces having one and the same kernel (as in Lemma 1.1).

Lemma 2.1. *Let $(\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be combinatory spaces, and τ be such an element of $\mathcal{F}$ that $\Pi_0(a, b) = \tau\Pi_1(a, b)$ for all $a, b \in \mathcal{C}$. Then $\Pi_0(\varphi, \psi) = \tau\Pi_1(\varphi, \psi)$ for all $\varphi, \psi \in \mathcal{F}$.*

[2] Actually a slight generalization of Theorem I.3.1 of [5] holds that allows an arbitrary function from $\mathcal{F}$ coinciding with the function $\lambda u.\, u \dot{-} 1$ on the positive integers to be used instead of it. $RT\Pi(I, I)$ is such a function already (the functions $RT^2\Pi(\theta, T)$ with $\theta \in \{L, R, T, F\}$ are also such ones).

Proof. Let φ, ψ be arbitrary elements of $\mathcal{F}$. Since $\Pi_i(a,b) = \Pi_i(a,I)b$, $i = 0,1$, we see that $\Pi_0(a,I)b = \tau\Pi_1(a,I)b$ for all $a,b \in \mathcal{C}$, hence $\Pi_0(a,I) = \tau\Pi_1(a,I)$ for all $a \in \mathcal{C}$. Therefore

$$\Pi_0(I,\psi c)a = \Pi_0(a,\psi c) = \Pi_0(a,I)\psi c = \tau\Pi_1(a,I)\psi c = \tau\Pi_1(a,\psi c) = \tau\Pi_1(I,\psi c)a$$

for all $a, c \in \mathcal{C}$, hence $\Pi_0(I,\psi c) = \tau\Pi_1(I,\psi c)$ for all $c \in \mathcal{C}$. It follows from here that

$$\Pi_0(\varphi,\psi)c = \Pi_0(\varphi c,\psi c) = \Pi_0(I,\psi c)\varphi c = \tau\Pi_1(I,\psi c)\varphi c = \tau\Pi_1(\varphi c,\psi c) = \tau\Pi_1(\varphi,\psi)c$$

for all $c \in \mathcal{C}$, and this proves the equality $\Pi_0(\varphi,\psi) = \tau\Pi_1(\varphi,\psi)$. □

Whenever $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0,1$, are combinatory spaces with one and the same kernel, we set

$$P_{\mathcal{S}_0}^{\mathcal{S}_1} = \Pi_1(L_0,R_0),\ Q_{\mathcal{S}_0}^{\mathcal{S}_1} = \Sigma_1(L_0, T_0R_0, F_0R_0),\ \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1} = \Sigma_1(I, T_0, F_0).$$

Lemma 2.2. *Let* $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0,1$, *be combinatory spaces. Then*

$$\Pi_1(\varphi,\psi) = P_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\varphi,\psi), \quad \Sigma_1(\chi,\varphi,\psi) = \Sigma_0(Q_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\chi,I),\varphi,\psi)$$

for all φ, ψ, χ *in* $\mathcal{F}$. *If* T_0 *and* F_0 *belong to* $\mathcal{C}$ *then* $Q_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\chi,I) = \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}\chi$, *thus* $\Sigma_1(\chi,\varphi,\psi) = \Sigma_0(\dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}\chi,\varphi,\psi)$ *in that case.*

Proof. The first of the equalities follows by Lemma 2.1 from the fact that

$$\Pi_1(a,b) = P_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(a,b)$$

for all $a, b \in \mathcal{C}$. For the general case in the rest of the proof we first observe that

$$Q_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\chi,I) = \Sigma_1(\chi,T_0,F_0)$$

for all $\chi \in \mathcal{F}$ (we get this equality by applying Proposition II.1.8 of [5] to the "mixed" combinatory space $(\mathcal{F}, I, \mathcal{C}, \Pi_0, L_0, R_0, \Sigma_1, T_1, F_1)$). In the case when T_0 and F_0 belong to $\mathcal{C}$, we also have the equality

$$\dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}\chi = \Sigma_1(\chi,T_0,F_0),$$

because then, by Proposition II.1.2 of [5], we have $T_0 = T_0c$, $F_0 = F_0c$ for any $c \in \mathcal{C}$. On the other hand, $\Sigma_0(\Sigma_1(\chi,T_0,F_0),\varphi,\psi) = \Sigma_1(\chi,\varphi,\psi)$, since

$$\begin{gathered}\Sigma_0(\Sigma_1(\chi,T_0,F_0),\varphi,\psi)c = \Sigma_0(\Sigma_1(\chi c,T_0c,F_0c),\varphi c,\psi c) = \\ \Sigma_0(I,\varphi c,\psi c)\Sigma_1(\chi c,T_0c,F_0c) = \Sigma_1(\chi c,\Sigma_0(I,\varphi c,\psi c)T_0c,\Sigma_0(I,\varphi c,\psi c)F_0c) = \\ \Sigma_1(\chi c,\Sigma_0(T_0c,\varphi c,\psi c),\Sigma_0(F_0c,\varphi c,\psi c)) = \Sigma_1(\chi,\Sigma_0(T_0,\varphi,\psi),\Sigma_0(F_0,\varphi,\psi))c = \\ \Sigma_1(\chi,\varphi,\psi)c.\end{gathered}$$

for all $c \in \mathcal{C}$. □

Corollary 2.1. *Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be two combinatory spaces with one and the same kernel $(\mathcal{F}, I, \mathcal{C})$, and let $\mathcal{S}_0$ be iterative. Then $\mathcal{S}_1$ is also iterative, and for any $\sigma, \chi \in \mathcal{F}$ the equality*

$$[\sigma, \chi]^{\mathcal{S}_1} = [\sigma, Q^{\mathcal{S}_1}_{\mathcal{S}_0} \Pi_0(\chi, I)]^{\mathcal{S}_0}$$

holds. Thus if T_0 and F_0 belong to $\mathcal{C}$, then $[\sigma, \chi]^{\mathcal{S}_1} = [\sigma, \dot{Q}^{\mathcal{S}_1}_{\mathcal{S}_0} \chi]^{\mathcal{S}_0}$.

Proof. By Lemma 1.1 and the above lemma. □

If $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ and $\mathcal{S}' = (\mathcal{F}, I, \mathcal{C}, \Pi', L', R', \Sigma', T', F')$ are two iterative combinatory spaces with the same kernel, then $\mathcal{S}'$ will be called *reducible to* S if the elements L', R', T', F' and the mappings Π', Σ' are $\mathcal{S}$-computable. Clearly the space $\mathcal{S}$ is reducible to itself (thanks to the $\mathcal{S}$-computability of Σ). Making use of Corollary 2.1, we see that the iteration operation in any iterative combinatory space reducible to $\mathcal{S}$ is a $\mathcal{S}$-computable mapping of $\mathcal{F}^2$ into $\mathcal{F}$, and therefore the introduced reducibility of iterative combinatory spaces is transitive. The iterative combinatory space $\mathcal{S}'$ will be called *equipowerful with* $\mathcal{S}$ if $\mathcal{S}'$ is reducible to $\mathcal{S}$ and $\mathcal{S}$ is reducible to $\mathcal{S}'$.

The space $\mathcal{S}'$ will be said to be *quasi-reducible* to the space $\mathcal{S}$ if all $\mathcal{S}'$-computable elements of $\mathcal{F}$ are $\mathcal{S}$-computable. Of course, if $\mathcal{S}'$ is reducible to $\mathcal{S}$ then $\mathcal{S}'$ is quasi-reducible to $\mathcal{S}$ (thanks to the $\mathcal{S}$-computability of the $\mathcal{S}'$-iteration). We do not know whether the converse implication holds, however the equipowerfulness of $\mathcal{S}$ and $\mathcal{S}'$ turns out to be equivalent to their mutual quasi-reducibility (i.e. to the equality of the set of the $\mathcal{S}$-computable elements of $\mathcal{F}$ and the set of the $\mathcal{S}'$-computable ones).

Theorem 2.1. *Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces. Then the next three conditions are equivalent:*

(i) *$\mathcal{S}_0$ is equipowerful with $\mathcal{S}_1$;*

(ii) *the set of the $\mathcal{S}_0$-computable elements of $\mathcal{F}$ coincides with the set of the $\mathcal{S}_1$-computable ones;*

(iii) *the elements $P^{\mathcal{S}_1}_{\mathcal{S}_0}$, $Q^{\mathcal{S}_1}_{\mathcal{S}_0}$, L_1, R_1, T_1, F_1 of $\mathcal{F}$ are $\mathcal{S}_0$-computable, and its elements $P^{\mathcal{S}_0}_{\mathcal{S}_1}$, $Q^{\mathcal{S}_0}_{\mathcal{S}_1}$, L_0, R_0, T_0, F_0 are $\mathcal{S}_1$-computable.*

In the case when T_0, F_0, T_1, F_1 belong to $\mathcal{C}$, the condition (iii) *can be replaced by*

(iii˙) *the elements $P^{\mathcal{S}_1}_{\mathcal{S}_0}$, $\dot{Q}^{\mathcal{S}_1}_{\mathcal{S}_0}$, L_1, R_1, T_1, F_1 of $\mathcal{F}$ are $\mathcal{S}_0$-computable, and its elements $P^{\mathcal{S}_0}_{\mathcal{S}_1}$, $\dot{Q}^{\mathcal{S}_0}_{\mathcal{S}_1}$, L_0, R_0, T_0, F_0 are $\mathcal{S}_1$-computable.*

Proof. The implication (i)⇒(ii) is clear from what was said in the paragraph before the theorem. The implications (ii)⇒(iii) and (ii)⇒(iii˙) follow from the fact that multiplication, Π_i and Σ_i preserve $\mathcal{S}_i$-computability for $i = 0, 1$. The validity of the implication (iii)⇒(i) in the general case and of the implication (iii˙)⇒(i) in the case when T, F, T', F' belong to $\mathcal{C}$ are seen from Lemma 2.2. □

Now several examples concerning the notions introduced follow (Corollary 2.1 is used in some of them for showing the iterativeness of the constructed new combinatory spaces).

Example 2.1. The pf-space considered in Example 1.1 is reducible to the one considered in Example 1.2, but these two spaces are not equipowerful.

Example 2.2. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be an iterative combinatory space that is symmetric in the sense of [5], i.e. the equality $\Pi(\varphi c, I)\theta = \Pi(\varphi c, \theta)$ holds for all $\varphi, \theta \in \mathcal{F}$ and all $c \in \mathcal{C}$ (in particular, $\mathcal{S}$ can be any pf-space). Let $\mathcal{S}_1 = (\mathcal{F}, I, \mathcal{C}, \Pi_1, R, L, \Sigma, T, F)$, where Π_1 is the mapping of $\mathcal{F}^2$ into $\mathcal{F}$ obtained from Π by exchanging its arguments, i.e. $\Pi_1(\varphi, \psi) = \Pi(\psi, \varphi)$ for all $\varphi, \psi \in \mathcal{F}$. Then $\mathcal{S}_1$ is an iterative combinatory space that is equipowerful with $\mathcal{S}$ (as indicated in Exercise II.1.2 of [5], the combinatory space $\mathcal{S}_1$ is also symmetric).

Example 2.2 (continuation). The assumption in Example 2.2 about the symmetry of $\mathcal{S}$ cannot be omitted without making other changes in the example. However, the definition of Π_1 is equivalent to another one that makes the symmetry assumption superfluous. In fact, an application of Lemma 2.2 in the situation from the example shows that $\Pi_1(\varphi, \psi) = \Pi_1(L, R)\Pi(\varphi, \psi)$, hence the equality

$$\Pi_1(\varphi, \psi) = \Pi(R, L)\Pi(\varphi, \psi) \tag{2.1}$$

holds for all $\varphi, \psi \in \mathcal{F}$. Now it is clear that we would get the same combinatory space $\mathcal{S}_1 = (\mathcal{F}, I, \mathcal{C}, \Pi_1, R, L, \Sigma, T, F)$ in the considered situation if we would define Π_1 by means of the equality (2.1). However, such a definition of $\mathcal{S}_1$ has the advantage that $\mathcal{S}_1$ turns out to be always an iterative combinatory space equipowerful with $\mathcal{S}$ (no symmetry of $\mathcal{S}$ is already needed). Checking everything in this statement is straightforward except for the fact that $\mathcal{S}$ is reducible to $\mathcal{S}_1$. The reducibility of $\mathcal{S}$ to $\mathcal{S}_1$ can be shown by proving the equality

$$\Pi(\varphi, \psi) = \Pi_1(L, R)\Pi_1(\varphi, \psi), \tag{2.2}$$

and this equality follows by Lemma 2.1 from the fact that, as it is easy to be verified, $\Pi(a, b) = \Pi_1(L, R)\Pi_1(a, b)$ for all $a, b \in \mathcal{C}$. [3]

Example 2.3. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be any iterative combinatory space, and let

$$\begin{aligned} \mathcal{S}_0 &= (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma_0, F, T), \\ \mathcal{S}_1 &= (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma_1, \Pi(T, I), \Pi(F, I)), \\ \mathcal{S}_2 &= (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma_2, \Pi(I, T), \Pi(I, F)), \end{aligned}$$

where Σ_0, Σ_1 and Σ_2 are defined by means of the equalities

[3] A proof of the equality (2.2) by using Lemma 2.2 is also possible, namely

$$\Pi_1(L, R)\Pi_1(\varphi, \psi) = \Pi(R, L)\Pi(L, R)\Pi(R, L)\Pi(\varphi, \psi) =$$
$$\Pi(R, L)\Pi(R, L)\Pi(\varphi, \psi) = \Pi(R, L)\Pi_1(\varphi, \psi) = \Pi(\varphi, \psi).$$

$$\Sigma_0(\chi,\varphi,\psi) = \Sigma(\chi,\psi,\varphi),\ \ \Sigma_1(\chi,\varphi,\psi) = \Sigma(L\chi,\varphi,\psi),\ \ \Sigma_2(\chi,\varphi,\psi) = \Sigma(R\chi,\varphi,\psi)$$

(cf. Exercise II.1.1 in [5]). Then $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are iterative combinatory spaces that are equipowerful with $\mathcal{S}$ (the equalities

$$\Sigma(\chi,\psi,\varphi) = \Sigma_1(\Pi(\chi,I),\varphi,\psi) = \Sigma_2(\Pi(I,\chi),\varphi,\psi),$$
$$T = L\Pi(T,I) = R\Pi(I,T),\ F = L\Pi(F,I) = R\Pi(I,F)$$

are used in the proof of the reducibility of $\mathcal{S}_1$ and $\mathcal{S}_2$ to $\mathcal{S}$).

Example 2.4. Let M be an infinite set, m_0 and m_1 be two distinct elements of M, and J be a bijection from M^2 to M such that $J(m_0,m_0) = m_0$, $J(m_0,m_1) = m_1$ (we may for instance set $M = \mathbb{N}$, $m_0 = 0$, $m_1 = 1$, and take J as in Example 1.1). Let L and R be the mappings of M into M defined by means of the equalities $L(J(s,t)) = s$, $R(J(s,t)) = t$. Then clearly $L(m_0) = R(m_0) = L(m_1) = m_0$, $R(m_1) = m_1$. We define a new mapping J' of M^2 into M by means of the equality

$$J'(s,t) = J(L(s),J(R(s),t)).$$

It is easily seen that the equality $J'(s,t) = u$ is equivalent to the pair of equalities $s = J(L(u),L(R(u)))$, $t = R(R(u))$. Therefore J' is also a bijection from M^2 to M, and after setting

$$L'(u) = J(L(u),L(R(u))),\ R'(u) = R(R(u))$$

we have $L'(J'(s,t)) = s$, $R'(J'(s,t)) = t$ for all $s,t \in M$. Moreover, we have also the equalities $J'(m_0,m_0) = m_0$, $J'(m_0,m_1) = m_1$. Now let us consider the computational structures (M,J,L,R,T,F,H) and (M,J',L',R',T,F,H), where $T(u) = m_1$, $F(u) = m_0$ for all $u \in M$, and H is a partial predicate on M such that $H(m_1)$ is true, $H(m_0)$ is false. Let $\mathcal{S} = (\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$ and $\mathcal{S}' = (\mathcal{F},I,\mathcal{C},\Pi',L',R',\Sigma,T,F)$ be the pf-spaces corresponding to these two computational structures. Since $\Pi'(\varphi,\psi) = \Pi(L\varphi,\Pi(R\varphi,\psi))$ for all $\varphi,\psi \in \mathcal{F}$, and the equalities $L' = \Pi(L,LR)$, $R' = R^2$ hold, the pf-space $\mathcal{S}'$ is reducible to $\mathcal{S}$. However, we shall show that $\mathcal{S}'$ is not equipowerful with $\mathcal{S}$, i.e. $\mathcal{S}$ is not reducible to $\mathcal{S}'$. This will be shown by proving that the $\mathcal{S}$-computable element $\Pi(T,F)$ of $\mathcal{F}$ is not $\mathcal{S}'$-computable. For that purpose, let us denote by A the smallest subset of M containing the elements m_0 and m_1 and closed under application of J'. It is easy to show by induction that the image of A under any $\mathcal{S}'$-computable function from $\mathcal{F}$ is a subset of A. On the other hand, $\Pi(T,F)(u) = J(m_1,m_0)$ for all $u \in M$, but $J(m_1,m_0)$ does not belong to A, because $m_0 \neq J(m_1,m_0)$, $m_1 \neq J(m_1,m_0)$, and $J'(s,t) \neq J(m_1,m_0)$ whenever $s \neq J(m_1,m_0)$, due to the equalities $L(m_0) = L(m_1) = m_0$ and $L'(J(m_1,m_0)) = J(m_1,m_0)$.

Remark 2.1. The above example shows how to construct an infinite sequence $\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2,\ldots$ of pf-spaces not differing from one another out of their fourth to sixth components and having the property that $\mathcal{S}_j$ is reducible to $\mathcal{S}_i$ without being equipowerful with it, whenever $j > i$.

Remark 2.2. Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces with one and the same kernel, and let

$$\mathcal{D} = \{P_{\mathcal{S}_0}^{\mathcal{S}_1}, Q_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\}$$

(or $T_0, F_0 \in \mathcal{C}$, $\mathcal{D} = \{P_{\mathcal{S}_0}^{\mathcal{S}_1}, \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\}$). An application of Lemma 2.2 shows that the operations Π_1, Σ_1 and consequently also the iteration in $\mathcal{S}_1$ are $\mathcal{S}_0$-computable in the set $\mathcal{D}$, hence all $\mathcal{S}_1$-computable elements of $\mathcal{F}$ are $\mathcal{S}_0$-computable in $\mathcal{D}$. If $\mathcal{S}_0$ is reducible to $\mathcal{S}_1$ then also the converse is true, hence in this case the $\mathcal{S}_1$-computability of an element of $\mathcal{F}$ is equivalent to its $\mathcal{S}_0$-computability in $\mathcal{D}$.

Intuitively, an iterative combinatory space can be considered as a certain kind of programming system. The intuitive interpretation of the reducibility of the space $\mathcal{S}'$ to the space $\mathcal{S}$ is as emulability of all $\mathcal{S}'$-programs (including the ones that may use oracles) by corresponding $\mathcal{S}$-programs. The quasi-reducibility of $\mathcal{S}'$ to $\mathcal{S}$ can be interpreted similarly, but with having in view only the programs that do not use oracles. Of course, the equipowerfulness will be interpreted as emulability in both directions.

3. ON A STATEMENT OF JORDAN ZASHEV

If (M, J, L, R, T, F, H) is a computational structure whose component J is a bijection from M^2 to M, then the corresponding pf-space $(\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ has the property that $\Pi(L, R) = I$. In a remark on page 78 of [7] Jordan Zashev indicates a way for improving the exposition of the theory for iterative combinatory spaces with this property (assuming that the elements T and F belong to $\mathcal{C}$). According to him the examples of combinatory spaces given in [2,5] do not give reasons to consider the abandonment of the equality $\Pi(L, R) = I$ as essential for the scope of the theory, since, as he writes, "*all of them have more or less obvious variants in which the last equality is true*". No definition is given in [7] for the used notion of variant, and of course no proof or disproof of the quoted statement can be expected without such a definition. We shall present now a refutation of the statement in question for the case when "variant" is interpreted as an iterative combinatory space that is quasi-reducible to the given one. Of course this will also show the failure of the statement for the stronger interpretations as an iterative combinatory space reducible to the given one or as an iterative combinatory space equipowerful with it.

Let us call an iterative combinatory space $(\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ a *Z-space* if the equality $\Pi(L, R) = I$ holds. We shall indicate some ordinary pf-spaces to which no Z-space is quasi-reducible, and this will be the promised refutation, since, as we mentioned in Section 1, the ordinary pf-spaces and all pf-spaces are the subject of some examples in [2] and in [5], respectively. The following lemma will be used.

Lemma 3.1. *Let (M, J, L, R, T, F, H) be a computational structure, and let the corresponding pf-space $\mathcal{S}$ be such that some Z-space with the same kernel is* *quasi-reducible to $\mathcal{S}$. Then there is an $\mathcal{S}$-computable bijection from M to the range of J.*

Proof. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$, and let $(\mathcal{F}, I, \mathcal{C}, \Pi', L', R', \Sigma', T', F')$ be a Z-space that is quasi-reducible to $\mathcal{S}$. Then the element $\Pi(L', R')$ of $\mathcal{F}$ is $\mathcal{S}$-computable thanks to the $\mathcal{S}$-computability of L' and R'. By Lemma 2.2, the equality $\Pi'(L', R') = \Pi'(L, R)\Pi(L', R')$ holds, hence $\Pi'(L, R)\Pi(L', R') = I$. Therefore $\Pi(L', R')$ is an injective mapping of M into M. Taking into account the definition of Π, we conclude that in fact $\Pi(L', R')$ is an injective mapping of M into the range of J. To show that any element of the range of J is a value of $\Pi(L', R')$, let us consider such an element u. Then $u = J(s, t)$ for some s and t in M. Denoting by a and b the elements of $\mathcal{C}$ with values s and t, respectively, we consider the element $\Pi'(a, b)$ of $\mathcal{C}$. Let v be the value of this constant function. The equalities $L'\Pi'(a, b) = a$, $R'\Pi'(a, b) = b$ imply that $L'(v) = s$, $R'(v) = t$, hence $\Pi(L', R')(v) = u$. □

Having the above lemma at our disposal, we shall proceed by indicating some computational structures (M, J, L, R, T, F, H) such that T and F are constant mappings of M into M, and, if $\mathcal{S}$ is the corresponding pf-space, then no $\mathcal{S}$-computable bijection from M to the range of J exists.

Example 3.1. We consider a computational structure (M, J, L, R, T, F, H) of the following kind. The set M is the closure of A under formation of ordered pairs, where A is some non-empty set, and none of its elements is an ordered pair, J is the function from M^2 to M defined by the equality $J(s, t) = (s, t)$, L and R are the functions from the range of J to M defined by means of the equalities $L(J(s, t)) = s$, $R(J(s, t)) = t$, T and F are the constant functions from M to M with values (o, o) and o, respectively, where o is some distinguished element of A, H is the predicate on M that is false on A and true everywhere in $M \setminus A$. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be the pf-space corresponding to this computational structure. Under the additional assumption that A is finite and has more than one element, we shall show that no Z-space with the same kernel is quasi-reducible to $\mathcal{S}$. Let A have k elements, where $k > 1$. Suppose there is an $\mathcal{S}$-computable bijection θ_0 from M to the range of J. Then θ_0 is a computable bijection from M to $M \setminus A$. From here a contradiction will be produced as follows. We define inductively a family $\mathcal{M}$ of subsets of M by the clauses that $A \in \mathcal{M}$ and $X \times Y \in \mathcal{M}$ whenever $X, Y \in \mathcal{M}$. One proves by induction that all members of $\mathcal{M}$ are non-empty finite sets, and, whenever $Z \in \mathcal{M}$, then either $Z = A$, or the cardinality of Z is divisible by k^2. Another induction shows that each element of M belongs to exactly one member of $\mathcal{M}$. By means of a third induction we prove that whenever θ is an $\mathcal{S}$-computable element of $\mathcal{F}$, the image by θ of any member of $\mathcal{M}$ is a subset of some member of $\mathcal{M}$. In particular, the mapping θ_0 will have this property. Since θ_0 is a bijection from M to $M \setminus A$, each member of $\mathcal{M}$ different from A will be the union of its subsets that are images by θ_0 of members of $\mathcal{M}$, and these subsets will be pairwise disjoint. Let Z be the member of $\mathcal{M}$ that contains as a subset the image by θ_0 of the set A. Clearly $Z \neq A$, and therefore the cardinality of Z is divisible by k^2. On the other hand, this cardinality must be equal to the sum of k and some numbers divisible by k^2, and this is a contradiction.

Remark 3.1. We could reason in the same way as above if we would make the functions L and R total by additionally setting $L(u) = R(u) = u$ for all $u \in A$. On the other hand, as seen from [6], the situation would become essentially different if we would make them total in the way from [1], namely by setting $L(o) = R(o) = o$, $L(u) = R(u) = (o, o)$ for all $u \in A \setminus \{o\}$. Then, independently of the cardinality of A, there would be a Z-space having the same kernel as $\mathcal{S}$ and reducible to it.

4. AN EXTENSION OF THE CONSIDERED REDUCIBILITY

The application of an iterative combinatory space $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ for the characterization of some concrete computability notion usually makes use of $\mathcal{S}$-computability in certain subset $\mathcal{B}$ of $\mathcal{F}$. The intuitive interpretation of $\mathcal{S}$ as a programming system can be transferred also to pairs $(\mathcal{S}, \mathcal{B})$ by replacing $\mathcal{S}$-computability with $\mathcal{S}$-computability in $\mathcal{B}$. The case of $\mathcal{S}$-computability will then correspond to the pair $(\mathcal{S}, \emptyset)$. It is natural to extend the reducibility notions introduced in Section 2 for the case of two pairs $(\mathcal{S}, \mathcal{B})$ and $(\mathcal{S}', \mathcal{B}')$, where $\mathcal{S}$ and $\mathcal{S}'$ are iterative combinatory spaces with one and the same kernel, and $\mathcal{B}, \mathcal{B}'$ are subsets of their first component. Here are the corresponding definitions.

If $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ and $\mathcal{S}' = (\mathcal{F}, I, \mathcal{C}, \Pi', L', R', \Sigma', T', F')$ are iterative combinatory spaces, and $\mathcal{B}$, $\mathcal{B}'$ are subsets of $\mathcal{F}$, then the pair $(\mathcal{S}', \mathcal{B}')$ will be called *reducible to* the pair $(\mathcal{S}, \mathcal{B})$ if the elements L', R', T', F', all elements of $\mathcal{B}'$ and the mappings Π', Σ' are $\mathcal{S}$-computable in $\mathcal{B}$. The pair $(\mathcal{S}', \mathcal{B}')$ will be said to be *quasi-reducible* to the pair $(\mathcal{S}, \mathcal{B})$ if all elements of $\mathcal{F}$ that are $\mathcal{S}'$-computable in $\mathcal{B}'$ are also $\mathcal{S}$-computable in $\mathcal{B}$. If each of the pairs $(\mathcal{S}, \mathcal{B})$ and $(\mathcal{S}', \mathcal{B}')$ is reducible to the other one then these pairs will be called *equipowerful*.

As in Section 2 the reducibility is seen to be reflexive and transitive, and it implies quasi-reducibility. Also, Theorem 2.1 remains valid after replacing the combinatory spaces with pairs of the considered kind, the proof being quite similar. Here is the result of the replacements.

Theorem 4.1. *Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces. and $\mathcal{B}_0$, $\mathcal{B}_1$ be subsets of $\mathcal{F}$. Then the next three conditions are equivalent:*

(i) *$(\mathcal{S}_0, \mathcal{B}_0)$ is equipowerful with $(\mathcal{S}_1, \mathcal{B}_1)$;*

(ii) *the set of the elements of $\mathcal{F}$ that are $\mathcal{S}_0$-computable in $\mathcal{B}_0$ coincides with the set of the ones that are $\mathcal{S}_1$-computable in $\mathcal{B}_1$;*

(iii) *the elements of the set $\{P_{\mathcal{S}_0}^{\mathcal{S}_1}, Q_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\} \cup \mathcal{B}_1$ are $\mathcal{S}_0$-computable in $\mathcal{B}_0$, and the elements of the set $\{P_{\mathcal{S}_1}^{\mathcal{S}_0}, Q_{\mathcal{S}_1}^{\mathcal{S}_0}, L_0, R_0, T_0, F_0\} \cup \mathcal{B}_0$ are $\mathcal{S}_1$-computable in $\mathcal{B}_1$.*

In the case when T_0, F_0, T_1, F_1 *belong to* $\mathcal{C}$, *the condition* (iii) *can be replaced by*

(iii˙) *the elements of the set* $\{P_{\mathcal{S}_0}^{\mathcal{S}_1}, \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\} \cup \mathcal{B}_1$ *are* $\mathcal{S}_0$*-computable in* $\mathcal{B}_0$, *and the elements of the set* $\{P_{\mathcal{S}_1}^{\mathcal{S}_0}, \dot{Q}_{\mathcal{S}_1}^{\mathcal{S}_0}, L_0, R_0, T_0, F_0\} \cup \mathcal{B}_0$ *are* $\mathcal{S}_1$*-computable in* $\mathcal{B}_1$.

The statements in Remark 2.2 can be strengthened in the following way.

Remark 4.1. Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces with one and the same kernel, and let

$$\mathcal{D} = \{P_{\mathcal{S}_0}^{\mathcal{S}_1}, Q_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\}$$

(or $T_0, F_0 \in \mathcal{C}$, $\mathcal{D} = \{P_{\mathcal{S}_0}^{\mathcal{S}_1}, \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\}$). Then the pair $(\mathcal{S}_1, \emptyset)$ is reducible to the pair $(\mathcal{S}_0, \mathcal{D})$. If $\mathcal{S}_0$ is reducible to $\mathcal{S}_1$ then $(\mathcal{S}_1, \emptyset)$ and $(\mathcal{S}_0, \mathcal{D})$ are equipowerful.

The following obvious monotonicity can also be mentioned: if $\mathcal{S}_0$ and $\mathcal{S}_1$ are iterative combinatory spaces with one and the same kernel $(\mathcal{F}, I, \mathcal{C})$, and $\mathcal{B}_0$, $\mathcal{B}_1$ are subsets of $\mathcal{F}$ such that the pair $(\mathcal{S}_0, \mathcal{B}_0)$ is reducible to the pair $(\mathcal{S}_1, \mathcal{B}_1)$, then for any subset $\mathcal{E}$ of $\mathcal{F}$ the pair $(\mathcal{S}_0, \mathcal{B}_0 \cup \mathcal{E})$ is reducible to the pair $(\mathcal{S}_1, \mathcal{B}_1 \cup \mathcal{E})$.

Acnowledgements. Thanks are due to Jordan Zashev for attracting the author's attention to such a kind of reducibility problems, especially by constructing an iterative combinatory space $(\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ with $\Pi(L, R) = I$ (mentioned in Remark 3.1) that is reducible to the iterative combinatory space straightforwardly connected with Moschovakis' abstract first order computability on a given set.

REFERENCES

1. Moschovakis, Y. N. Abstract first order computability. I. *Trans. Amer. Math. Soc.*, **138**, 1969, 427–464.
2. Skordev, D. Combinatory Spaces and Recursiveness in Them. Publishing House of the Bulgarian Academy of Sciences, Sofia, 1980 (in Russian).
3. Skordev, D. Semicombinatory spaces. *C. R. Acad. Bulgare Sci.*, **33**, 1980, 6, 151–154 (in Russian).
4. Skordev, D. The First Recursion Theorem for iterative semicombinatory spaces. In *Mathematical Logic, Proc. of the Conference on Mathematical Logic Dedicated to the Memory of A. A. Markov (1903–1979), Sofia, Sept. 22–23, 1980*, Publ. House of the Bulg. Acad. Sci., Sofia, 1984, 89–111 (in Russian).
5. Skordev, D. Computability in Combinatory Spaces. Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.
6. Zashev, J. Private communication. March 2004.

7. Zashev, J. Affine applications in operative spaces. *Ann. Univ. Sofia, Fac. Math. Inf.*, **97**, 2005, 63–79.

Received on October 1, 2005
Revised on December 23, 2005

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: skordev@fmi.uni-sofia.bg