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MEASURES RELATED TO OPERATOR IDEALS AND SPARR'S INTERPOLATION METHODS

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We establish an estimate for the outer measure $\gamma_I(T)$ and the inner measure $\beta_I(T)$ of an operator T acting between some intermediate spaces constructed for n-tuples of Banach spaces. We also show that many operator ideals have the strong interpolation property for Sparr's interpolation methods.

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1. INTRODUCTION

The behaviour of compact linear operators under real interpolation for Banach couples or Banach n-tuples has been extensively studied by many authors. The class of compact operators between Banach spaces is an injective surjective closed operator ideal in the sense of Pietsch [11]. It is therefore natural to investigate whether the similar results are valid for such ideals. There are two measures, $\gamma_I(T)$ (outer measure) and $\beta_I(T)$ (inner measure) of an operator $T \in \mathcal{L}(A, B)$, introduced, respectively, by Astala [1] and by Tylli [14], which show the deviation of T from the ideal I.

It is known that

if I is surjective and closed, then $\gamma_I(T_{A,B}) = 0$ if and only if $T \in I(A, B)$ (see [1])

and, analogously that

if I is injective and closed, then $\beta_I(T_{A,B}) = 0$ if and only if $T \in I(A,B)$ (see [14]).

Particularizing the operator ideal I, measures γ_I and β_I coincide with well-known notion. For example, when I is the ideal of compact operator k, $\gamma_k(T)$ is equal to the measure of non-compactness of T and $\beta_k(T)$ turns out to be the limit of the Gelfand numbers of T.

The behaviour under real interpolation of measures γ_I and β_I in the case of Banach couples has been pointed out by Cobos, Manzano and A. Martinez [3] and Cobos, Cwikel and Matos [2]. They have derived estimates for the measures γ_I and β_I provided that one of the couples degenerated into a single Banach space, or that the ideal I satisfied the so-called Σp-condition (see [8]), without assuming any condition on the Banach couples.

In this paper we establish an estimate for the measures $\gamma_I(T)$ and $\beta_I(T)$ of an operator T acting between some intermediate spaces constructed for n-tuples of Banach spaces. We consider here 4-tuples of Banach spaces and Sparr's interpolation method. We obtain similar results for $n \geq 5$.

We also show that weakly compact operators, Rosenthal operators, Banach-Saks operators and Radon-Nikodym operators have the strong interpolation property for Sparr's interpolation methods.

2. PRELIMINARIES

Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, that is to say, a family of 4 Banach spaces A_j all of them continuously embedded in a common linear Hausdorff space. If $\bar{t} = (t_1, t_2, t_3)$ and $\bar{s} = (s_1, s_2, s_3)$ are triples of positive numbers, we set

$$\bar{t}\bar{s} = (t_1 s_1, t_2 s_2, t_3 s_3), \quad 2^{\bar{t}} = (2^{t_1}, 2^{t_2}, 2^{t_3}), \quad |\bar{t}| = t_1 t_2 t_3$$

$$K(\bar{t}, a, \overline{A}) = \inf \left\{ \|a_0\|_{A_0} + \sum_{i=1}^{3} t_i \|a_i\|_{A_i}, \ a = \sum_{i=0}^{3} a_i, \ a_i \in A_i \right\},$$

$$a \in \Sigma(\overline{A}) := A_0 + A_1 + A_2 + A_3$$

and

$$J(\bar{t}, a, \overline{A}) = \max_{1 \leq i \leq 3} \left\{ \|a\|_{A_0}, \ t_i \|a\|_{A_i} \right\}, \ a \in \Delta(\overline{A}) := A_0 \cap A_1 \cap A_2 \cap A_3$$

If A and B are Banach spaces, we denote by $\mathcal{L}(A, B)$ the space of all bounded linear operators between A an B with the usual norm. Given two Banach 4-tuples $\overline{A} = (A_0, A_1, A_2, A_3)$, $\overline{B} = (B_0, B_1, B_2, B_3)$, we write $T \in \mathcal{L}(\overline{A}, \overline{B})$ or $T : \overline{A} \to \overline{B}$, meaning that T is linear operators from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j $(j = 0, 1, 2, 3)$. For each $T \in \mathcal{L}(\overline{A}, \overline{B})$ we consider the norm:

$$\|T\|_{\overline{A}, \overline{B}} := \max_{0 \leq i \leq 3} \left\{ \|T\|_{A_i, B_i} \right\}.$$

If one of the 4-tuples $\overline{A}$ or $\overline{B}$ reduces to single Banach spaces, i.e. if $A_0 = A_1 = A_2 = A_3 = A$, or if $B_0 = B_1 = B_2 = B_3 = B$, then we write $T \in \mathcal{L}(A, \overline{B})$ or, respectively $T \in \mathcal{L}(\overline{A}, B)$.

A Banach space A is said to be an intermediate space with respect to $\overline{A}$ if

$$\Delta(\overline{A}) \hookrightarrow A \hookrightarrow \Sigma(\overline{A}),$$

where the notation $\hookrightarrow$ means continuous inclusion.

An intermediate space A is said to be interpolation for the 4-tuple $\overline{A}$ if, for all operators $T \in \mathcal{L}(\overline{A}, \overline{B})$, there exists a constant $C = C(A, \overline{A})$ such that

$$\|T\|_{A,A} \leq C\|T\|_{\overline{A},\overline{A}}.$$

If we consider only the one-dimensional operator T, i.e.

$$Tx = f(x)a, \quad a \in \Delta(\overline{A}), \quad f \in (\Sigma(\overline{A}))^*$$

then the space A is called party interpolation, or rank-one interpolation.

Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuples and let A be an intermediate space with respect to $\overline{A}$. For the triple $\overline{t} = (t_1, t_2, t_3)$ of positive numbers set

$$\psi(\overline{t}) = \psi(\overline{t}, A, \overline{A}) = \sup\left\{K(\overline{t}, a, \overline{A}) : \ \|a\|_A = 1\right\}$$

and

$$\rho(\overline{t}) = \rho(\overline{t}, A, \overline{A}) = \inf\left\{J(\overline{t}, a, \overline{A}) : \ a \in \Delta(\overline{A}), \ \|a\|_A = 1\right\}.$$

Proposition 2.1. [6] *Let $\overline{A}$ be a Banach 4-tuple and let A be an intermediate space with respect to $\overline{A}$. Then A is party interpolation space if and only if there exists a constant $C = C(A, \overline{A})$ such that*

$$\psi(\overline{t}) \leq C\rho(\overline{t}), \quad \textit{for all}\, \overline{t} = (t_1, t_2, t_3) \in (0, \infty)^3.$$

Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple. Then $\Delta(\overline{A}) \hookrightarrow A_0$ and $A_0 \hookrightarrow \Sigma(\overline{A})$. We denote by A_0^0 the closure of $\Delta(\overline{A})$ in A_0 and by $\widetilde{A}_0$ the completion of A_0 with respect to $\Sigma(\overline{A})$ (or the Gagliardo completion of A_0 in $\Sigma(\overline{A})$).

Proposition 2.2. [6] *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple and let A be a party interpolation space with respect to $\overline{A}$*

(i) If $\lim\limits_{\overline{t}\to\overline{0}} \psi(\overline{t}) > 0$, then $A_0^0 \hookrightarrow A$

(ii) If $\lim\limits_{\overline{t}\to\overline{\infty}} (1/\rho(\overline{t})) > 0$, then $A \hookrightarrow \widetilde{A}_0$.

Let A be an intermediate space with respect to Banach 4-tuple $\overline{A}$. We say that A is of class $\mathcal{C}_K(\overline{\theta}, \overline{A})$, where $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in [0,1]^3$, $\theta_1 + \theta_2 + \theta_3 \leq 1$ if there is a constant c such that for all $\overline{t} = (t_1, t_2, t_3) \in (o, \infty)^3$ and $a \in A$

$$|\overline{t}^{-\overline{\theta}}|K(\overline{t}, a, \overline{A}) \leq C\|a\|_A$$

and of class $\mathcal{C}_J(\overline{\theta}, \overline{A})$ if there is a constant C such that for all $\overline{t} = (t_1, t_2, t_3) \in (0, \infty)^3$ and $a \in \Delta(\overline{A})$

$$\|a\|_A \leq C|\overline{t}^{-\overline{\theta}}| J(\overline{t}, a, \overline{A}).$$

An important example of spaces of class $\mathcal{C}_K(\overline{\theta}, \overline{A})$ is the real interpolation K-space $\overline{A}_{\overline{\theta},p,K} := (A_0, A_1, A_2, A_3)_{\overline{\theta},p,K}$ (or Sparr's K-space). We remind that for $1 \leq p \leq \infty$ and $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ the space $\overline{A}_{\overline{\theta},p,K}$ consists of all $a \in \Sigma(\overline{A})$, which have a finite norm:

$$\|a\|_{\overline{\theta},p,K} = \begin{cases} \left(\sum_{\overline{n} \in \mathbb{Z}^3} (|2^{-\overline{n}\overline{\theta}}| K(2^{\overline{n}}, a, \overline{A}))^p \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup_{\overline{n} \in \mathbb{Z}^3} \{|2^{-\overline{n}\overline{\theta}}| K(2^{\overline{n}}, a, \overline{A})\} & \text{if } p = \infty. \end{cases}$$

On the other hand, the real interpolation J-space $\overline{A}_{\overline{\theta},p,J} := (A_0, A_1, A_2, A_3)_{\overline{\theta},p,J}$ (or Sparr's J-space) is an important example of space of class $\mathcal{C}_J(\overline{\theta}, \overline{A})$. We remind that for $1 \leq p \leq \infty$ and $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ the space $\overline{A}_{\overline{\theta},p,J}$ consists of all $a \in \Sigma(\overline{A})$ which can be represented in the form:

$$a = \sum_{\overline{n} \in \mathbb{Z}^3} u_{\overline{n}} \qquad \text{(convergence in } \Sigma(\overline{A}))$$

with $(u_{\overline{n}})_{\overline{n} \in \mathbb{Z}^3} \subset \Delta(\overline{A})$ and

$$\sum_{\overline{n} \in \mathbb{Z}^3} (|2^{-\overline{n}\overline{\theta}}| J(2^{\overline{n}}, u_{\overline{n}}, \overline{A}))^p < \infty.$$

(The sum should be replaced by supremum if $p = \infty$.)
Moreover

$$\|a\|_{\overline{\theta},p,J} = \inf \left\{ \left(\sum_{\overline{n} \in \mathbb{Z}^3} (|2^{-\overline{n}\overline{\theta}}| J(2^{\overline{n}}, u_{\overline{n}}, \overline{A}))^p \right)^{1/p} : a = \sum_{\overline{n}} u_{\overline{n}} \right\}$$

defines a norm on $\overline{A}_{\overline{\theta},p,J}$.

An operator ideal I is any subclass of the class $\mathcal{L}$ of all bounded linear operators between arbitrary Banach spaces such that the components $I(A,B) = I \cap \mathcal{L}(A,B)$ satisfy the following conditions:

(i) $I(A,B)$ is a linear subspace of $\mathcal{L}(A,B)$;

(ii) $I(A,B)$ contains the finite rank operators;

(iii) if $R \in \mathcal{L}(X,A)$, $T \in I(A,B)$ and $S \in \mathcal{L}(B,Y)$ then $STR \in I(X,Y)$.

The operator ideal I is injective if for every isomorphic embedding $J \in \mathcal{L}(B,Y)$ one has that $T \in \mathcal{L}(A,B)$ and $JT \in I(A,Y)$ imply $T \in I(A,B)$, it is surjective if for every surjection $Q \in \mathcal{L}(X, A)$ we have $T \in \mathcal{L}(A, B)$ and $TQ \in I(X, B)$ imply $T \in I(A, B)$. The ideal I is closed if the components $I(A, B)$ are closed subspaces of $\mathcal{L}(A, B)$ (see Pietsch [11]).

The outer measure of $T \in \mathcal{L}(A, B)$ is denoted by $\gamma_I(T) = \gamma_I(T_{A,B})$ and is the infimum of all positive numbers σ such that $T(U_A) \subset \sigma U_B + R(U_Z)$ for some Banach space Z and some operators $R \in I(Z, B)$ (where U_X denotes the closed unit ball of X). The inner measure of the same operator is denoted by $\beta_I(T) = \beta_I(T_{A,B})$ and is the infimum of all positive numbers σ such that for some Banach space Z and some operators $R \in I(A, Z)$ the inequality

$$\|Tx\|_B \leq \sigma \|x\|_A + \|Rx\|_Z$$

holds for all $x \in A$.

The ideal I possesses the strong interpolation property for a method $\mathcal{F}$ of interpolation if the interpolated operator $T_{\mathcal{F}} : \mathcal{F}(\overline{A}) \to \mathcal{F}(\overline{B})$ belongs to I when the induced operators $T_{IS} : \Delta(\overline{A}) \to \Sigma(\overline{B})$ is in I.

3. ESTIMATES FOR THE OUTER MEASURE $\gamma_I(T)$

In this section we establish an estimate for the measure $\gamma_I(T)$ when one of the Banach 4-tuples reduces to a single Banach space.

Theorem 3.1. *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, let A be an intermediate space with respect to $\overline{A}$, and let B be an arbitrary Banach space. Let I be an operator ideal. Then for each $T \in \mathcal{L}(\overline{A}, B)$ we have*

$$\gamma_I(T) \leq \psi(t_1, t_2, t_3) \left[\gamma_I(T_{A_0,B}) + \frac{1}{t_1}\gamma_I(T_{A_1,B}) + \frac{1}{t_2}\gamma_I(T_{A_2,B}) + \frac{1}{t_3}\gamma_I(T_{A_3,B}) \right] \tag{3.1}$$

for all $t_1, t_2, t_3 > 0$.

Proof. In view of the definition of $\gamma_I(T_{A_i,B})$, for $i = 0, 1, 2, 3$ and for each $\varepsilon > 0$ there exist Banach spaces Z_i and operators $S_i \in I(Z_i, B)$ such that

$$T(U_{A_i}) \subset (\varepsilon + \gamma_I(T_{A_i,B}))U_B + S_i(U_{Z_i}) \tag{3.2}$$

Now, consider an arbitrary element $a \in U_A$ and fixed positive numbers t_1, t_2, t_3 and δ. Since

$$K(t_1, t_2, t_3, a, \overline{A}) < \delta + \psi(t_1, t_2, t_3) \tag{3.3}$$

there exists a decomposition $a = a_0 + a_1 + a_2 + a_3$, with $a_i \in A_i$ $(i = 0, 1, 2, 3)$ such that

$$\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + t_2\|a_2\|_{A_2} + t_3\|a_3\|_{A_3} < \delta + \psi(t_1, t_2, t_3).$$

Thus $a_0 \in (\delta + \psi(t_1, t_2, t_3))U_{A_0}$, $a_1 \in t_1^{-1}(\delta + \psi(t_1, t_2, t_3))U_{A_1}$, $a_2 \in t_2^{-1}(\delta + \psi(t_1, t_2, t_3))U_{A_2}$ and $a_3 \in t_3^{-1}(\delta + \psi(t_1, t_2, t_3))U_{A_3}$.

From this and (3.2) it follows that

$$\begin{aligned} T(U_A) &\subset (\delta+\psi(t_1,t_2,t_3))\left[T(U_{A_0})+\frac{1}{t_1}T(U_{A_1})+\frac{1}{t_2}T(U_{A_2})+\frac{1}{t_3}T(U_{A_3})\right] \\ &\subset (\delta+\psi(t_1,t_2,t_3))\left[\varepsilon+\gamma_I(T_{A_0,B})+\frac{\varepsilon}{t_1}+\frac{1}{t_1}\gamma_I(T_{A_1,B})+\frac{\varepsilon}{t_2}+\right. \\ &+ \left.\frac{1}{t_2}\gamma_I(T_{A_2,B})+\frac{\varepsilon}{t_3}+\frac{1}{t_3}\gamma_I(T_{A_3,B})\right]U_B+S_0'(U_{Z_0})+S_1'(U_{Z_1})+ \\ &+ S_2'(U_{Z_2})+S_3'(U_{Z_3}) \end{aligned}$$

where $S_0'=(\delta+\psi(t_1,t_2,t_3))S_0$, $S_1'=\frac{1}{t_1}(\delta+\psi(t_1,t_2,t_3))S_1$, $S_2'=\frac{1}{t_2}(\delta+\psi(t_1,t_2,t_3))S_2$ and $S_3'=\frac{1}{t_3}(\delta+\psi(t_1,t_2,t_3))S_3$ are operators belonging to $I(Z_0,B)$, $I(Z_1,B)$, $I(Z_2,B)$ and $I(Z_3,B)$, respectively. Let Z be the Banach space $Z=Z_0\oplus Z_1\oplus Z_2\oplus Z_3$ with norm $\|(x,y,z,w)\|=\max(\|x\|_{Z_0},\|y\|_{Z_1},\|z\|_{Z_2},\|w\|_{Z_3})$ and define $S:Z\to B$ by $S(x,y,z,w)=S_0'x+S_1'y+S_2'z+S_3'w$. Then $S(U_Z)=S_0'(U_{Z_0})+S_1'(U_{Z_1})+S_2'(U_{Z_2})+S_3'(U_{Z_3})$ and using the ideal properties of I and the projection operators from Z onto Z_i, $i=0,1,2,3$, we have $S\in I(Z,B)$. Consequently

$$\gamma_I(T_{A,B})\le\psi(t_1,t_2,t_3)\left[\gamma_I(T_{A_0,B})+\frac{1}{t_1}\gamma_I(T_{A_1,B})+\frac{1}{t_2}\gamma_I(T_{A_1,B})+\frac{1}{t_3}\gamma_I(T_{A_3,B})\right].\square$$

Corollary 3.2. *If A is a Banach space of class $\mathcal{C}_K(\overline{\theta},\overline{A})$ with constant C, for some $\overline{\theta}=(\theta_1,\theta_2,\theta_3)\in(0,1)^3$, $\theta_1+\theta_2+\theta_3<1$ then*

$$\begin{aligned} \gamma_I(T_{A,B}) &\le C(1-\theta_1-\theta_2-\theta_3)^{\theta_1+\theta_2+\theta_3-1}\theta_1^{-\theta_1}\theta_2^{-\theta_2}\theta_3^{-\theta_3}\gamma_I(T_{A_0,B})^{1-\theta_1-\theta_2-\theta_3} \\ &\cdot \gamma_I(T_{A_1,B})^{\theta_1}\gamma_I(T_{A_2,B})^{\theta_2}\gamma_I(T_{A_3,B})^{\theta_3} \end{aligned} \tag{3.4}$$

Proof. Let $\sigma_i>\gamma_I(T_{A_i,B})$, $i=0,1,2,3$. By the definition of $\gamma_I(T_{A_i,B})$ for $i=0,1,2,3$ and for each $\varepsilon>0$ there exist Banach spaces Z_i and operators $S_i\in I(Z_i,B)$ such that

$$T(U_{A_i})\subset\sigma_iU_B+S_i(U_{Z_i}).$$

Since $A\in\mathcal{C}_K(\overline{\theta},\overline{A})$, given any $\varepsilon>0$, $t_1>0$, $t_2>0$, $t_3>0$ and $a\in A$ with $\|a\|_A\le1$ we can find $a_i\in A_i$, $i=0,1,2,3$, so that $a=a_0+a_1+a_2+a_3$ and $\|a_0\|_{A_0}\le(1+\varepsilon)Ct_1^{\theta_1}t_2^{\theta_2}t_3^{\theta_3}$, $\|a_1\|_{A_1}\le(1+\varepsilon)Ct_1^{\theta_1-1}t_2^{\theta_2}t_3^{\theta_3}$, $\|a_2\|_{A_2}\le(1+\varepsilon)Ct_1^{\theta_1}t_2^{\theta_2-1}t_3^{\theta_3}$, $\|a_3\|_{A_3}\le(1+\varepsilon)Ct_1^{\theta_1}t_2^{\theta_2}t_3^{\theta_3-1}$. The proof proceeds now in the same way as in Theorem 3.1 to obtain the inequality

$$\gamma_I(T_{A,B}) \le C\inf_{t_1>o,t_2>0,t_3>0}t_1^{\theta_1}t_2^{\theta_2}t_3^{\theta_3}\left[\gamma_I(T_{A_0,B})+\frac{1}{t_1}\gamma_I(T_{A_1,B})+\frac{1}{t_2}\gamma_I(T_{A_2,B})\right.$$

$$+ \quad \frac{1}{t_3}\gamma_I(T_{A_3,B})\Bigg].$$

This inequality implies the result. □

Corollary 3.3. *If I is a surjective closed operator ideal and $T \in \mathcal{L}(\overline{A}, B)$ is such that for some i, say $i = 0$, $T \in I(A_0, B)$ then $T \in I(\overline{A}_{\overline{\theta},p,K}, B)$.*

Proof. Since $\gamma_I(T) = 0$ if and only if $T \in I$ and $\overline{A}_{\overline{\theta},p,K}$ is of class $\mathcal{C}_K(\overline{\theta}, \overline{A})$, form (3.4) the result follows. □

Remark 3.4. When $I = K$ the ideal of compact operators, $\gamma_K(T)$ coincides with the measures of non-compactness of T, so we recover well-known Lions' and Peetre's compactness results [see [9], [4], [6], [10]).

Theorem 3.5. *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, let A be a party interpolation respect to $\overline{A}$, and let B be another Banach space. Let I be a surjective closed operator ideal and $T \in \mathcal{L}(\overline{A}, B)$ such that $T \in I(A_i, B)$, $i = 1, 2, 3$. Then at least one of the following conditions must hold*

(i) $T \in I(A, B)$;

(ii) $A_0^0 \hookrightarrow A$.

Proof. Since $\gamma_I(T_{A_i,B}) = 0$, $(i = 1, 2, 3)$ from (3.1) we have

$$\gamma_I(T_{A,B}) \leq \gamma_I(T_{A_0,B}) \lim_{\overline{t}\to\overline{0}} \psi(t_1, t_2, t_3, A, \overline{A}).$$

Consequently, either $T \in I(A, B)$ (i.e. $\gamma_I(T_{A,B}) = 0$), or, alternatively, $\lim_{\overline{t}\to\overline{0}} \psi(t_1, t_2, t_3, A, \overline{A}) > 0$, which, by Proposition 2.3, implies that $A_0^0 \hookrightarrow A$. □

4. ESTIMATES FOR THE INNER MEASURE $\beta_I(T)$

In this section we establish an estimate for the measure $\beta_I(T)$ when one of the Banach 4-tuples reduces to a single Banach space.

Theorem 4.1. *Let $\overline{B} = (B_0, B_1, B_2, B_3)$ be a Banach 4-tuple, let B be an intermediate space with respect to $\overline{B}$ and let A be an arbitrary Banach space. Let I be an operator ideal. Then, for each $T \in \mathcal{L}(A, \overline{B})$ we have*

$$\beta_I(T) \leq \frac{1}{\rho(t_1, t_2, t_3)} \max\{\beta_I(T_{A,B_0}), t_1\beta_I(T_{A,B_1}), t_2\beta_I(T_{A,B_2}), t_3\beta_I(T_{A,B_3})\} \quad (4.1)$$

for all $t_1 > 0$, $t_2 > 0$, $t_3 > 0$.

Proof. By the definition of $\beta_I(T_{A,B_i})$, for each $\varepsilon > 0$ there exist Banach spaces Z_i and operators $S_i \in I(A, Z_i)$ such that

$$\|Ta\|_{B_i} \leq (\varepsilon + \beta_I(T_{A,B_i}))\|a\|_A + \|S_i a\|_{Z_i}, \quad \text{for all } a \in A,\ i = 0,1,2,3. \tag{4.2}$$

Since $\|b\|_B \leq \dfrac{J(t_1, t_2, t_3, b, \overline{B})}{\rho(t_1, t_2, t_3)}$ for all $b \in \Delta(\overline{B})$ and all $t_1 > 0$, $t_2 > 0$, $t_3 > 0$ and using (4.2) we obtain:

$$\begin{aligned}
\|Ta\|_B &\leq \frac{1}{\rho(t_1,t_2,t_3)} \max(\|Ta\|_{B_0}, t_1\|Ta\|_{B_1}, t_2\|Ta\|_{B_2}, t_3\|Ta\|_{B_3}) \\
&\leq \frac{1}{\rho(t_1,t_2,t_3)} \max[\varepsilon + \beta_I(T_{A,B_0}), t_1(\varepsilon + \beta_I(T_{A,B_1})), t_2(\varepsilon + \beta_I(T_{A,B_2})), \\
&\qquad t_3(\varepsilon + \beta_I(T_{A,B_3}))]\|a\|_A + \|S_0'a\|_{Z_0} + \|S_1'a\|_{Z_1} + \|S_2'a\|_{Z_2} + \|S_3'a\|_{Z_3}
\end{aligned}$$

where $S_0' = \dfrac{S_0}{\rho(t_1,t_2,t_3)}$, $S_i' = \dfrac{t_i S_i}{\rho(t_1,t_2,t_3)}$, $(i = 1,2,3)$

Let Z be the Banach space $Z = Z_0 \oplus Z_1 \oplus Z_2 \oplus Z_3$ with the norm $\|(x,y,z,w)\| = \|x\|_{Z_0} + \|y\|_{Z_1} + \|z\|_{Z_2} + \|w\|_{Z_3}$ and let the operator $S : A \to Z$ defined by $Sa = (S_0'a, S_1'a, S_2'a, S_3'a)$. Using the ideal properties of I and the canonical embeddings of Z_i into Z $(i = 0,1,2,3)$ we have $S \in I(A,Z)$ and

$$\begin{aligned}
\|Ta\|_B &\leq \frac{1}{\rho(t_1,t_2,t_3)} \max[\varepsilon + \beta_I(T_{A,B_0}), t_1(\varepsilon + \beta_I(T_{A,B_1})), t_2(\varepsilon + \beta_I(T_{A,B_2})), \\
&\qquad t_3(\varepsilon + \beta_I(T_{A,B_3}))]\|a\|_A + \|Sa\|_Z, \text{ for all } a \in A.
\end{aligned}$$

This implies (4.1). □

Corollary 4.2. *If B is a Banach space of class $\mathcal{C}_J(\overline{\theta}, \overline{B})$ with constant C, for some $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ and $\beta_I(T_{A,B_i}) > 0$, $(i = 0,1,2,3)$ then*

$$\beta_I(T_{A,B}) \leq C\beta_I(T_{A,B_0})^{1-\theta_1-\theta_2-\theta_3}\beta_I(T_{A,B_1})^{\theta_1}\beta_I(T_{A,B_2})^{\theta_2}\beta_I(T_{A,B_3})^{\theta_3}. \tag{4.3}$$

Proof. Since $B \in \mathcal{C}_J(\overline{\theta}, \overline{B})$ and using (4.2) we obtain the inequality

$$\begin{aligned}
\|Ta\|_B &\leq Ct_1^{-\theta_1}t_2^{-\theta_2}t_3^{-\theta_3} \max\{\beta_I(T_{A,B_0}), t_1\beta_I(T_{A,B_1}), t_2\beta_I(T_{A,B_2}), \\
&\qquad t_3\beta_I(T_{A,B_3})\}\|a\|_A + \|Sa\|_Z, \text{ for all } a \in A \text{ and } t_1 > 0, t_2 > 0, t_3 > 0.
\end{aligned}$$

Taking $t_i = \dfrac{\beta_I(T_{A,B_0})}{\beta_I(T_{A,B_i})}$ $(i = 1, 2, 3)$, we get (4.3). □

Corollary 4.3. *If i is an injective closed operator ideal and $T \in \mathcal{L}(A, \overline{B})$ is such that for some i, say $T \in I(A, B_0)$, $T \in I(A, \overline{B}_{\overline{\theta},p,J})$.*

Proof. Since $\beta_I(T) = 0$ if and only if $T \in I$ and $\overline{B}_{\overline{\theta},p,J}$ is of class $\mathcal{C}_J(\overline{\theta}, \overline{A})$, the result follows from (4.3). □

Theorem 4.4. *Let $\overline{B} = (B_0, B_1, B_2, B_3)$ be a Banach 4-tuple, let B be a party interpolation space with respect to $\overline{B}$ and let A be another Banach space. Let I be an injective closed operator ideal and $T \in \mathcal{L}(A, \overline{B})$ such that $T \in I(A, B_i)$, $(i = 1, 2, 3)$. Then at least one of the following conditions must hold:*

(i) $T \in I(A, B)$,

(ii) $B \hookrightarrow \widetilde{B}_0$.

Proof. Since $\beta_I(T_{A,B_i}) = 0$, $(i = 1, 2, 3)$ from (4.1) we have

$$\beta_I(T_{A,B}) \leq \beta_I(T_{A,B_0}) \lim_{\overline{t}\to\overline{\infty}} \frac{1}{\rho(t_1, t_2, t_3)}.$$

Consequently, either $T \in I(A, B)$ (i.e. $\beta_I(T_{A,B}) = 0$), or, alternatively $\lim_{\overline{t}\to\overline{\infty}} \dfrac{1}{\rho(t_1, t_2, t_3)} > 0$, which, by Proposition 2.2, implies $B \hookrightarrow \widetilde{B}_0$. □

5. THE STRONG INTERPOLATION PROPERTY

In this section we show that many classes of operators ideals possess the strong interpolation property with respect to Sparr's interpolation method. To obtain the strong interpolation property for the ideal I (without assuming any condition on the Banach 4-tuples), we require the operator ideal I to satisfy the so-called $\sum p$-condition (which was introduced by Heinrich [8]).

Given any sequence of Banach spaces $(E_{\overline{m}})_{\overline{m}\in\mathbb{Z}^3}$ we denote by $l_p(E_{\overline{m}})$ the vector-valued l_p space defined by

$$l_p(E_{\overline{m}}) = \left\{ x = (x_{\overline{m}}) : x_{\overline{m}} \in E_{\overline{m}} \text{ and } \|x\|_{l_p(E_{\overline{m}})} = \Big(\sum_{\overline{m}\in\mathbb{Z}^3} (\|x_{\overline{m}}\|_{E_{\overline{m}}})^p \Big)^{1/p} < \infty \right\}.$$

Denote by $Q_{\overline{k}} : l_p(E_{\overline{m}}) \to E_{\overline{k}}$ the projection $Q_{\overline{k}}(x_{\overline{m}}) = x_{\overline{k}}$ and by $I_{\overline{n}} : E_{\overline{n}} \to l_p(E_{\overline{m}})$ the natural (isometric) embedding $I_{\overline{n}} y = \left(\delta_{\overline{m}}^{\overline{n}} y\right)$, where $\delta_{\overline{m}}^{\overline{n}}$ is the Kronecker symbol.

The operator ideal I satisfies the $\sum p$-condition if for any two sequences $(E_{\overline{m}})_{\overline{m}\in\mathbb{Z}^3}$ and $(F_{\overline{m}})_{\overline{m}\in\mathbb{Z}^3}$ of Banach spaces the following holds:

if $T \in \mathcal{L}(l_p(E_{\overline{m}}), l_p(F_{\overline{m}}))$ and $Q_{\overline{k}} T I_{\overline{n}} \in I(E_{\overline{n}}, F_{\overline{k}})$ for any $\overline{n}, \overline{k} \in \mathbb{Z}^3$, then $T \in I(l_p(E_{\overline{m}}), l_p(F_{\overline{m}}))$.

Theorem 5.1. *Let $1 < p < \infty$, $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ and let I be a closed injective and surjective operator ideal which satisfies the $\sum p$-condition. Suppose that $\overline{A}, \overline{B}$ are Banach 4-tuples and $T \in \mathcal{L}(\overline{A}, \overline{B})$. Let T_{IS} be the induced operator from $\Delta(\overline{A})$ into $\sum(\overline{B})$. If $T_{IS} \in I(\Delta(\overline{A}), \sum(\overline{B}))$ then $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$.*

Proof. Define on $\Delta(\overline{A})$ and $\sum(\overline{B})$ the following equivalent norms

$$\|a\|_{\overline{m}} = 2^{-\theta_1 m_1 - \theta_2 m_2 - \theta_3 m_3} J(2^{m_1}, 2^{m_2}, 2^{m_3}, a, \overline{A}), a \in \Delta(\overline{A}), \overline{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$$

$$\|b\|_{\overline{m}} = 2^{-\theta_1 m_1 - \theta_2 m_2 - \theta_3 m_3} K(2^{m_1}, 2^{m_2}, 2^{m_3}, b, \overline{B}), b \in \Sigma(\overline{B}), \overline{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$$

Denote by $A_{\overline{m}}$ the space $(\Delta(\overline{A}), \|\cdot\|_{\overline{m}})$ and by $B_{\overline{m}}$ the space $(\sum(\overline{B}), \|\cdot\|_{\overline{m}})$. In view of the definition of the space $\overline{A}_{\overline{\theta},p,J}$ there is a surjection Q from $l_p(A_{\overline{m}})$ onto $A_{\overline{\theta},p,J}$ defined by

$$Q((x_{\overline{m}})_{\overline{m}}) = \sum_{\overline{m} \in \mathbb{Z}^3} x_{\overline{m}} \quad \text{(convergence in } \textstyle\sum(\overline{A})).$$

By the definition of the space $\overline{B}_{\overline{\theta},p,K}$ there is an (isomorphic) embedding J from $\overline{B}_{\overline{\theta},p,K}$ into $l_p(B_{\overline{m}})$ defined by

$$J(y) = (\ldots y, y, y, \ldots)$$

Then the operator $Q_{\overline{k}} J T Q J_{\overline{n}}$ is the operator T_{IS} from $A_{\overline{n}} = \Delta(\overline{A})$ into $B_{\overline{k}} = \sum(\overline{B})$. So, it is an operator of the class I. Since I satisfies the $\sum p$-condition the operator JTQ belongs to $I(l_p(A_{\overline{m}}), l_p(B_{\overline{m}}))$. Now, the injectivity and surjectivity of I implies $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$. □

Corollary 5.2. *Let $T \in \mathcal{L}(\overline{A}, \overline{B})$ such that for some i, say $i = 0$, $T \in I(A_0, B_0)$, then $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$.*

Proof. Using the commutative diagram

$$\begin{array}{ccc} \Delta(\overline{A}) & \xrightarrow{T_{IS}} & \sum(\overline{B}) \\ \downarrow & & \uparrow \\ A_0 & \xrightarrow{T} & B_0 \end{array}$$

and the ideal properties of I, we obtain $T_{IS} \in I(\Delta(\overline{A}), \sum(\overline{B}))$. Consequently $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$. □

Remark 5.3. For the case of Banach couples Heinrich [8] has proved rezults like those from theorem 5.1. He has also shown that weakly compact operators, Rosenthal operators, Banach-Saks operators and dual Radon-Nikodym operator satisfy the $\sum p$-condition, for $1 < p < \infty$ (these operators ideals are also injective, surjective and closed) but the above condition is not satisfied by the compact operators. So, Theorem 5.1 does not apply to compact operators, though we have a similar result as in Corollary 5.2 for compact operators (see [4], [6]).

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