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AFFINE APPLICATIONS IN OPERATIVE SPACES [1]

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We prove a new normal form theorem for a special kind of expressible mappings in operative spaces with iteration. As a consequence, this provides a large class of models for the type-free implicative linear logic and a natural connection between operative spaces and the systems of algebraic recursion theory based on the linear or affine application which were studied previously.

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INTRODUCTION

Operative spaces are a special class of partially ordered algebras developed in [2] for the purposes of the axiomatization of recursion theory. Recently, they were shown ([8]) to give rise to a large class of combinatory algebras. There is, on the other hand, an alternative to combinatory algebras - a kind of algebras called 'type-free models of the linear logic' below, since they can be regarded as models of a type-free version of the proof calculus for a Hilbert-styled system of a suitable fragment of linear logic. The last algebras have a natural connection with recursion theory and some other advantages which make reasonable the question whether we can model them in a way similar to that in which combinatory algebras were modeled in [8]. In the present paper we discuss this question and indicate a large class of operative spaces (including the iterative ones in the sense of [2]) and combinatory spaces in the sense of [4] in which the type-free models of the linear

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logic can be modeled. Some of these results were briefly mentioned in the last section of [8]; in the present paper we give them a detailed exposition. As in the paper [8], the last results will follow from a suitable normal form theorem for some kind of expressible mappings in operative spaces with iteration. The normal form theorem of the present paper can be regarded as a refinement of a normal form theorem of Georgieva [1].

The type-free models of the implicative linear logic and other partially ordered algebras based on linear (otherwise called BCI-) application were studied previously by the present author ([5, 6, 7]) for the same purposes of axiomatization of recursion theory; but until recently no natural connection was known with other systems of the algebraic recursion theory like operative spaces or combinatory spaces. The normal form theorem of the present paper changes this situation since it enables one to define a natural affine (or otherwise BCK-) application in every operative space with iteration. This opportunity is used in Section 3 below, where we show how to model one of the most important application-based systems of the algebraic recursion theory in such spaces, indicating in this way that the last system comprises the majority of the kinds of recursivenesses dealt with in the theory of operative spaces.

1. BASIC DEFINITIONS

An *operative space*, according to [2], up to some notational modifications, is a partially ordered algebra $\mathcal{F}$ with two binary operations called multiplication and pairing and three constants I,T,F, considered as 0-ary operations, which satisfy the conditions (OS1) and (OS2) below. (Note that the condition that $\mathcal{F}$ is a partially ordered algebra includes the requirement that all operations of $\mathcal{F}$ are increasing on each argument.) We use the following notations for the operations in question: the multiplication is denoted by juxtaposition and the result of applying pairing to the arguments $\varphi, \psi \in \mathcal{F}$ will be denoted by $[\varphi, \psi]$. The conditions defining an operative space $\mathcal{F}$ are the following ones:

(OS1) $\mathcal{F}$ is a monoid with unit I with respect to the multiplication;

(OS2) the identities $\chi[\varphi,\psi] = [\chi\varphi,\chi\psi]$, $[\varphi,\psi]T = \varphi$, and $[\varphi,\psi]F = \psi$ are satisfied for all elements φ, ψ, χ of $\mathcal{F}$.

The operations of multiplication and pairing and the constants I,T,F are called *basic* operations of operative spaces or, more briefly, basic OS-operations. We shall denote by m^+ the standard representation $F^mT = F...FT$ of a natural number m in arbitrary operative space $\mathcal{F}$. We employ also the shorthand notation

$$[\varphi_0, \varphi_1, ..., \varphi_{m-1}] = [\varphi_0, [\varphi_1, ...[\varphi_{m-2}, \varphi_{m-1}]...]];$$

so we have the identities

$$[\varphi_0, \varphi_1, ..., \varphi_m]i^+ = \varphi_i$$

for $i \leq m-1$ and

$$[\varphi_0, \varphi_1, ..., \varphi_m]F^m = \varphi_m$$

in every operative space.

For a unary operation $\mathbf{I} : \mathcal{F} \to \mathcal{F}$ in an operative space $\mathcal{F}$ we shall say that it is an *iteration* iff it satisfies the inequality

$$[I, \mathbf{I}(\varphi)]\varphi \leq \mathbf{I}(\varphi)$$

for all $\varphi \in \mathcal{F}$, and for all $\alpha, \xi \in \mathcal{F}$ the inequality $[\alpha, \xi]\varphi \leq \xi$ implies $\alpha\mathbf{I}(\varphi) \leq \xi$. Therefore the iteration $\mathbf{I}$ in $\mathcal{F}$, if it exists, is uniquely determined by the fact that for every $\varphi \in \mathcal{F}$ the element $\mathbf{I}(\varphi)$ is bound to be the least solution of the inequality $[I, \xi]\varphi \leq \xi$ with respect to ξ; in particular, it satisfies the corresponding equality

$$[I, \mathbf{I}(\varphi)]\varphi = \mathbf{I}(\varphi).$$

When the iteration $\mathbf{I}$ exists in the space $\mathcal{F}$, we shall say that $\mathcal{F}$ is an operative space with iteration; this notion is equivalent to the notion of G-space in [4] and to that of an operative space satisfying the axiom $(\mathcal{L}\mathcal{L})$ of Ivanov [2]. Hence every *iterative* operative space in the sense of [2] is an operative space with iteration, but the reverse is not necessarily true. Every operative space $\mathcal{F}$ in which the least upper bound $supB$ exists for every well ordered part $B \subseteq \mathcal{F}$ and commutes with the left multiplication: $\varphi supB = sup\{\varphi x \mid x \in B\}$, is iterative and therefore has iteration. Another important and most commonly apearing class of iterative operative spaces is that of *continuous* ones, i.e. those in which the least upper bounds of countable increasing sequences exist and commute with all basic OS-operations.

In every operative space $\mathcal{F}$ with iteration the element $O = \mathbf{I}(F)$ is the least element of $\mathcal{F}$, and it satisfies also the equality $\alpha O = O$ for all $\alpha \in \mathcal{F}$. This follows from the equality $[\alpha, \xi]F = \xi$ by the definition of iteration. The last definition implies also that the iteration is an increasing operation.

Let $\mathcal{F}$ be arbitrary operative space and $B \subseteq \mathcal{F}$. We shall say for a mapping $f : \mathcal{F}^n \to \mathcal{F}$ that it is OS-expressible in B iff f can be defined by an explicit expression constructed by means of the basic operations of operative spaces and the elements of B as constants. Similarly, when the space $\mathcal{F}$ has an iteration, the mapping f will be called OSI-expressible in B iff it can be defined by similar expression, which may contain also the operation of iteration $\mathbf{I}$. Instead of OS- or OSI-expressible in B mappings of zero arguments we shall speak also of OS- or OSI-expressible in B elements of $\mathcal{F}$, respectively. We shall also drop 'in B' when B is clear from the context or arbitrary.

2. NORMAL FORM OF SINGULAR MAPPINGS

An OSI-expressible (in B) mapping $f : \mathcal{F}^n \to \mathcal{F}$ in an operative space $\mathcal{F}$ with iteration will be called (B-)*singular* iff in the expression defining f all the variables for the arguments of f occur exactly once. More precisely, the B-singular mappings of n arguments are those which belong to the least class $\mathcal{O}$ of operations in $\mathcal{F}$ satisfying the following conditions:

0) $\mathcal{O}$ contains the identity operation in $\mathcal{F}$ of one argument;

1) $\mathcal{O}$ contains the basic constants I, T, F and the elements of B, considered as operations of zero arguments;

2) for all two operations f and g in $\mathcal{O}$ of n and m arguments, respectively, the operation h of $n+m$ arguments defined by

$$h(\xi_0,...,\xi_{n-1},\eta_0,...,\eta_{m-1}) = f(\xi_0,...,\xi_{n-1})g(\eta_0,...,\eta_{m-1})$$

is also in $\mathcal{O}$;

3) for all two operations f and g in $\mathcal{O}$ of n and m arguments, respectively, the operation h of $n+m$ arguments defined by

$$h(\xi_0,...,\xi_{n-1},\eta_0,...,\eta_{m-1}) = [f(\xi_0,...,\xi_{n-1}), g(\eta_0,...,\eta_{m-1})]$$

is also in $\mathcal{O}$;

4) for all operations f in $\mathcal{O}$ of n arguments the operation h of n arguments defined by

$$h(\xi_0,...,\xi_{n-1}) = \mathbf{I}(f(\xi_0,...,\xi_{n-1}))$$

is also in $\mathcal{O}$;

5) for all operations f in $\mathcal{O}$ of n arguments and every bijection

$$p : \{0,...,n-1\} \to \{0,...,n-1\}$$

the operation f_p of n arguments defined by

$$f_p(\xi_0,...,\xi_{n-1}) = f(\xi_{p(0)},...,\xi_{p(n-1)}))$$

is also in $\mathcal{O}$.

The main purpose of the present section is to prove the following normal form theorem for singular mappings.

Theorem 1. *For every B-singular mapping $f:\mathcal{F}^n \to \mathcal{F}$ in an operative space $\mathcal{F}$ with iteration there is an element $\varphi \in \mathcal{F}$, OS-expressible in $B \cup \{O\}$, such that*

$$f(\xi_0,...,\xi_{n-1}) = \mathbf{I}([I, 2^+\xi_0,...,(n+1)^+\xi_{n-1}]\varphi)T$$

for all $\xi_0,...,\xi_{n-1} \in \mathcal{F}$.

For that purpose we shall employ the technique of homogeneous systems introduced by Skordev [3], [4]. Let $\mathcal{F}$ be an arbitrary operative space. Following Skordev, we shall call a mapping $f : \mathcal{F}^n \to \mathcal{F}$ *left homogeneous* iff it satisfies the equality

$$f(\varphi\vartheta_0,...,\varphi\vartheta_{n-1}) = \varphi f(\vartheta_0,...,\vartheta_{n-1})$$

for all $\varphi, \vartheta_0, ..., \vartheta_{n-1} \in \mathcal{F}$. Left homogeneous mappings f are easily seen to be increasing on each argument since they satisfy the equalities

$$\begin{aligned} f(\vartheta_0, ..., \vartheta_{n-1}) &= f([\vartheta_0, ..., \vartheta_{n-1}]0^+, ..., [\vartheta_0, ..., \vartheta_{n-1}]F^n) \\ &= [\vartheta_0, ..., \vartheta_{n-1}]f(0^+, ..., F^{n-1}). \end{aligned}$$

By a homogeneous system we shall mean a system of inequalities of the form

$$\Phi_i(\alpha, x_0, ..., x_{n-1}) \leq x_i \tag{2.1}$$

where i ranges over natural numbers less than n, $\Phi_i : \mathcal{F}^{n+1} \to \mathcal{F}$ are left homogeneous mappings of $n+1$ arguments, $\alpha \in \mathcal{F}$ is a parameter, and $x_0, ..., x_{n-1}$ are unknowns. The following fundamental result for such systems, up to nonessential modifications, belongs to Skordev [3], [4].

Theorem 2. *Suppose the space $\mathcal{F}$ has an iteration. Then the elements μ_i defined by $\mu_i = \alpha\mathbf{I}(\varphi)i^+$, where $i = 0, ..., n-1$,*

$$\varphi = [\varphi_0, ..., \varphi_{n-1}, O],$$

and $\varphi_i = \Phi_i(T, 1^+, ..., n^+)$, form the least solution of the system 2.1 in $\mathcal{F}$ with respect to x_0,...,x_{n-1}, respectively, for all $\alpha \in \mathcal{F}$.

Proof. $(\mu_0, ..., \mu_{n-1})$ is a solution of (2.1) since

$$\begin{aligned} \Phi_i(\alpha, \mu_0, ..., \mu_{n-1}) &= \Phi_i(\alpha, \alpha\mathbf{I}(\varphi)0^+, ..., \alpha\mathbf{I}(\varphi)(n-1)^+) \\ &= \alpha\Phi_i(I, \mathbf{I}(\varphi)0^+, ..., \mathbf{I}(\varphi)(n-1)^+) \\ &= \alpha[I, \mathbf{I}(\varphi)]\Phi_i(T, 1^+, ..., n^+) = \alpha[I, \mathbf{I}(\varphi)]\varphi_i \\ &= \alpha[I, \mathbf{I}(\varphi)]\varphi i^+ = \alpha\mathbf{I}(\varphi)i^+ = \mu_i; \end{aligned}$$

and for an arbitrary solution $(\xi_0, ..., \xi_{n-1})$ of (2.1) with respect to $x_0, ..., x_{n-1}$, respectively, define $\xi = [\xi_0, ..., \xi_{n-1}, O]$; then for arbitrary $\alpha \in \mathcal{F}$ we have

$$\begin{aligned} [\alpha, \xi]\varphi &= [\alpha, \xi][\varphi_0, ..., \varphi_{n-1}, O] = [[\alpha, \xi]\varphi_0, ..., [\alpha, \xi]\varphi_{n-1}, O] \\ &= [[\alpha, \xi]\Phi_0(T, 1^+, ..., n^+), ..., [\alpha, \xi]\Phi_{n-1}(T, 1^+, ..., n^+), O] \\ &= [\Phi_0(\alpha, \xi 0^+, ..., \xi(n-1)^+), ..., \Phi_{n-1}(\alpha, \xi 0^+, ..., \xi(n-1)^+), O] \\ &= [\Phi_0(\alpha, \xi_0, ..., \xi_{n-1}), ..., \Phi_{n-1}(\alpha, \xi_0, ..., \xi_{n-1}), O] \\ &\leq [\xi_0, ..., \xi_{n-1}, O] = \xi, \end{aligned}$$

whence by the definition of iteration $\alpha\mathbf{I}(\varphi) \leq \xi$ and

$$\mu_i = \alpha\mathbf{I}(\varphi)i^+ \leq \xi i^+ = \xi_i$$

for all $i \leq n-1$.□

Note that the last theorem implies that the least solutions of homogeneous systems in operative spaces with iteration are left homogeneous in the following sense: If $(\mu_0, ..., \mu_{n-1})$ is the least solution of a homogeneous system of the form (2.1), then $(\beta\mu_0, ..., \beta\mu_{n-1})$ is the least solution of the system

$$\Phi_i(\beta\alpha, x_0, ..., x_{n-1}) \leq x_i$$

for all $\beta \in \mathcal{F}$.

Henceforth in this section we shall suppose that $\mathcal{F}$ is an operative space with iteration.

Next we shall specify some kind of homogeneous systems, which we shall call canonical. Namely, let $\xi = (\xi_0, ..., \xi_{n-1}) \in \mathcal{F}^n$ and $B \subseteq \mathcal{F}$; then by a $(B;\xi)$-*canonical* system of inequalities we shall mean a homogeneous system of the form

$$\Gamma_i(I, x_0, x_1\xi_0, ..., x_n\xi_{n-1}, x_{n+1}, ..., x_{n+m}) \leq x_i, \tag{2.2}$$

where $i \leq n+m$, $x_0, ..., x_{n+m}$ are unknowns and $\Gamma_0, ..., \Gamma_{n+m} : \mathcal{F}^{n+m+2} \to \mathcal{F}$ are OS-expressible in B left homogeneous mappings of $n+m+2$ arguments. A mapping $f : \mathcal{F}^n \to \mathcal{F}$ will be called *canonically definable* in B iff there is a $(B;\xi)$-canonical system of the form (2.2) such that the first member (corresponding to x_0) of the least solution of (2.2) equals $f(\xi)$ for all $\xi \in \mathcal{F}^n$. The systems of the form (2.2) being homogeneous, the Theorem 2 applies to them, whence we obtain the following

Corollary 1. *Every $(B;\xi)-$canonical system (2.2) has a least solution whose components μ_i ($i \leq n-1$) have the form*

$$\mu_i = \mathbf{I}([I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma)i^+$$

for a suitable element $\gamma \in \mathcal{F}$ which is OS-expressible in $B \cup \{O\}$.

Proof. By Theorem 2 the least solution of (2.2) is given by

$$\mu_i = \mathbf{I}(\varphi)i^+,$$

where $i \leq n+m$, $\varphi = [\varphi_0, ..., \varphi_{n+m}, O]$, and

$$\varphi_i = \Gamma_i(T, 1^+, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}, (n+2)^+, ..., (n+m+1)^+).$$

Thence

$$\varphi_i = [I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma_i,$$

where

$$\gamma_i = \Gamma_i(T^2, T1^+, 1^+, ..., (n-1)^+, F^n, T(n+2)^+, ..., T(n+m+1)^+).$$

Then defining $\gamma = [\gamma_0, ..., \gamma_{n+m}, O]$, we obtain

$$\varphi = [I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma$$

and the required representation of μ_i. □

Corollary 2. *Every canonically definable in $B \subseteq \mathcal{F}$ mapping $f : \mathcal{F}^n \to \mathcal{F}$ of n arguments is representable of the form*

$$f(\xi_0, ..., \xi_{n-1}) = \mathbf{I}([I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma)T$$

for a certain element $\gamma \in \mathcal{F}$ which is OS-*expressible in $B \cup \{O\}$.*□

The last corollary shows that to establish Theorem 1 it is enough to prove that all singular mappings are canonically definable. For that purpose it will be convenient to introduce some more terminology about homogeneous systems. Consider two homogeneous systems

$$\Phi_i(I, x_0, ..., x_{n-1}) \le x_i \tag{2.3}$$

and

$$\Psi_j(I, y_0, ..., y_{m-1}) \le y_j, \tag{2.4}$$

where $i \le n-1$ and $j \le m-1$ and the variables x_i,y_j are supposed pairwise different. Then the *product* of the systems (2.3) and (2.4) is defined as the homogeneous system of $n+m$ inequalities, consisting of the inequalities of (2.3) and the m inequalities

$$\Psi_j(x_0, y_0, ..., y_{m-1}) \le y_j. \tag{2.5}$$

Similarly, the homogeneous system of $n+m+1$ inequalities, consisting of the inequalities of (2.3) and (2.4) and the inequality $[x_0, y_0] \le z$, where z is a new variable, will be called *pairing* of the systems (2.3) and (2.4); and the homogeneous system of n inequalities

$$\Phi_i([I, x_0], x_0, ..., x_{n-1}) \le x_i \tag{2.6}$$

will be called *iteration* of the system (2.3).

Lemma 1. *Suppose $(\mu_0, ..., \mu_{n-1})$ and $(\nu_0, ..., \nu_{m-1})$ are the least solutions in $\mathcal{F}$ of the systems* (2.3) *and* (2.4) *with respect to $x_0, ..., x_{n-1}$ and $y_0, ..., y_{m-1}$, respectively. Then:*

(a) *the $(n+m)$-tuple $(\mu_0, ..., \mu_{n-1}, \mu_0\nu_0, ..., \mu_0\nu_{m-1})$ is the least solution in $\mathcal{F}$ of the product of the systems* (2.3) *and* (2.4) *with respect to $x_0, ..., x_{n-1}, y_0, ..., y_{m-1}$, respectively;*

(b) *the $(n+m+1)$-tuple $(\mu_0, ..., \mu_{n-1}, \nu_0, ..., \nu_{m-1}, [\mu_0, \nu_0])$ is the least solution in $\mathcal{F}$ of the pairing of the systems* (2.3) *and* (2.4) *with respect to $x_0, ..., x_{n-1}$, $y_0, ..., y_{m-1}, z$, respectively; and*

(c) *the n-tuple $(\mathbf{I}(\mu_0), [I, \mathbf{I}(\mu_0)]\mu_1, ..., [I, \mathbf{I}(\mu_0)]\mu_{n-1})$ is the least solution in $\mathcal{F}$ of the iteration of the system* (2.3) *with respect to $x_0, ..., x_{n-1}$, respectively.*

Proof. The n-tuple $(\mu_0, ..., \mu_{n-1})$ satisfies the inequalities of (2.3), and by the left homogeneity of Ψ_j the $(m+1)$-tuple

$$(\mu_0, \mu_0\nu_0, ..., \mu_0\nu_{m-1})$$

satisfies those of (2.5). Hence the $(n+m)$-tuple

$$(\mu_0, ..., \mu_{n-1}, \mu_0\nu_0, ..., \mu_0\nu_{m-1})$$

is a solution of the product of the systems (2.3) and (2.4). Consider an arbitrary solution $(\xi_0, ..., \xi_{n-1}, \eta_0, ..., \eta_{m-1})$ of the last product with respect to $x_0, ..., x_{n-1}$, $y_0, ..., y_{m-1}$, respectively. Since $(\mu_0, ..., \mu_{n-1})$ is the least solution of (2.3), we have $\mu_i \leq \xi_i$ for all natural numbers i less than n. On the other hand, by Theorem 2 it follows that $(\mu_0\nu_0, ..., \mu_0\nu_{m-1})$ is the least solution of the system

$$\Psi_j(\mu_0, y_0, ..., y_{m-1}) \leq y_j$$

with respect to $y_0, ..., y_{m-1}$, respectively; and from the inequality $\mu_0 \leq \xi_0$ we can conclude that $(\eta_0, ..., \eta_{m-1})$ satisfies the last system. Therefore $\mu_0\nu_j \leq \eta_j$ for all $j \leq m-1$, which proves (a). The proof of (b) is similar, but simpler and straightforward, and we leave it to the reader. Finally, to prove (c), denote shortly by λ the iteration $\mathbf{I}(\mu_0)$; then using the equality $[I, \lambda]\mu_0 = \lambda$ and the left homogeneity of Φ_i, we see that the n-tuple $(\lambda, [I, \lambda]\mu_1, ..., [I, \lambda]\mu_{n-1})$ is a solution of (2.6):

$$\Phi_i([I, \lambda], \lambda, [I, \lambda]\mu_1, ..., [I, \lambda]\mu_{n-1}) = [I, \lambda]\Phi_i(I, \mu_0, ..., \mu_{n-1}) \leq [I, \lambda]\mu_i.$$

Let $(\xi_0, ..., \xi_{n-1})$ be an arbitrary solution of (2.6) with respect to $x_0, ..., x_{n-1}$, respectively. Then it is a solution of the system

$$\Phi_i([I, \xi_0], x_0, ..., x_{n-1}) \leq x_i$$

with respect to $x_0, ..., x_{n-1}$, respectively. But Theorem 2 implies that the n-tuple

$$([I, \xi_0]\mu_0, ..., [I, \xi_0]\mu_{n-1})$$

is the least solution of the last system. Therefore $[I, \xi_0]\mu_i \leq \xi_i$ for all $i \leq n-1$, whence $[I, \xi_0]\mu_0 \leq \xi_0$ and $\lambda = \mathbf{I}(\mu_0) \leq \xi_0$, and finally $[I, \lambda]\mu_i \leq \xi_i$. □

Proof of Theorem 1. According to Corollary 2 it suffices to show that all B-singular operations in $\mathcal{F}$ of n arguments are canonically definable in B; and this follows from the fact that the class $\mathcal{O}$ of all canonically definable in B operations in $\mathcal{F}$ satisfies the conditions 0) - 5) in the definition of B-singular mapping above. Indeed, the identity mapping of one argument $e(\xi) = \xi$ is canonically definable by the system of two inequalities $x_1\xi \leq x_0$ and $I \leq x_1$. If b is a basic constant or element of B, then b as an operation in $\mathcal{F}$ of zero arguments is canonically definable by the system of one inequality $Ib \leq x_0$. Thus $\mathcal{O}$ satisfies conditions 0) and 1). To see that it satisfies conditions 2) - 4), it is enough to note, respectively, that the product, the pairing and the iteration of canonical systems are also canonical systems, and to apply Lemma 1. Finally, the class $\mathcal{O}$ satisfies 5), because a canonical definition by a system of the form (2.2) of an operation f in $\mathcal{F}$ of n arguments can be regarded as a canonical definition of the operation f_p after a suitable permutation of the variables $x_1, ..., x_n$. □

3. AFFINE APPLICATIONS IN OPERATIVE SPACES

The normal form Theorem 1 proved in the previous section enables us to define in arbitrary operative space $\mathcal{F}$ with iteration a binary operation which has the properties of affine application. Namely, for all $\varphi, \psi \in \mathcal{F}$ define

$$(\varphi \cdot \psi) = \mathbf{I}([I, 2^+\psi]\varphi)T. \tag{3.1}$$

We shall adopt the shorthand notation $(\varphi_0 \cdot \varphi_1 \cdot \ldots \cdot \varphi_{n-1} \cdot \varphi_n)$ for the iterated application $((\ldots(\varphi_0 \cdot \varphi_1) \cdot \ldots \cdot \varphi_{n-1}) \cdot \varphi_n)$ and we shall also omit the external parentheses in such expressions. In the case $n = 0$ the last notation should be interpreted as φ_0. We have the following

Corollary 3. *Let $\mathcal{F}$ be an operative space with iteration and let f be a B-singular operation in $\mathcal{F}$ of n arguments. Then there is an element $\varphi \in \mathcal{F}$, OS-expressible in $B \cup \{O\}$, which represents f in the sense that for all $\xi_0,\ldots,\xi_{n-1} \in \mathcal{F}$ we have*

$$f(\xi_0, \ldots, \xi_{n-1}) = \varphi \cdot \xi_0 \cdot \ldots \cdot \xi_{n-1}.$$

Proof. Induction on n. The case $n = 0$ is trivial; suppose $n \geq 1$ and assume the induction hypothesis for $n - 1$. By Theorem 1 we have an element $\varphi_0 \in \mathcal{F}$ OS-expressible in $B \cup \{O\}$ such that for all $\xi_0, \ldots, \xi_{n-1} \in \mathcal{F}$

$$f(\xi_0, \ldots, \xi_{n-1}) = \mathbf{I}([I, 2^+\xi_{n-1}, 3^+\xi_0, \ldots, (n+1)^+\xi_{n-2}]\varphi_0)T.$$

Thence for the B-singular mapping $f_1 : \mathcal{F}^{n-1} \to \mathcal{F}$ defined by

$$f_1(\xi_0, \ldots, \xi_{n-2}) = [T, F, T3^+\xi_0, \ldots, T(n+1)^+\xi_{n-2}]\varphi_0$$

we have

$$f(\xi_0, \ldots, \xi_{n-1}) = \mathbf{I}([I, 2^+\xi_{n-1}]f_1(\xi_0, \ldots, \xi_{n-2}))T = f_1(\xi_0, \ldots, \xi_{n-2}) \cdot \xi_{n-1},$$

and by the induction hypothesis applied to f_1 we obtain the required representation of f.□

Corollary 4. *In an arbitrary operative space $\mathcal{F}$ with iteration, the binary operation defined by* (3.1) *is an affine application in $\mathcal{F}$ in the sense that there are three elements A, C, $K \in \mathcal{F}$, OS-expressible in $\{O\}$, such that for all $\varphi, \psi, \chi \in \mathcal{F}$ we have*

$$A \cdot \varphi \cdot \psi \cdot \chi = \varphi \cdot (\psi \cdot \chi), \tag{3.2}$$

$$C \cdot \varphi \cdot \psi = \psi \cdot \varphi, \tag{3.3}$$

$$K \cdot \varphi \cdot \psi = \varphi. \tag{3.4}$$

Moreover, there are two elements $C_*, D_* \in \mathcal{F}$, OS-expressible in $\{O\}$, which satisfy for all φ, ψ, $\chi \in \mathcal{F}$ the equalities

$$C_* \cdot \varphi \cdot (D_* \cdot \psi \cdot \chi) = \varphi \cdot \psi \cdot \chi \tag{3.5}$$

and

$$D_* \cdot \psi \cdot \chi = [T\psi, F\chi]. \tag{3.6}$$

Proof. The existence of A, C and D_* satisfying (3.2), (3.3) and (3.6), respectively, follows immediately from Corollary 3 since the right-hand sides of the last three equalities are the values of suitable $\emptyset$-singular mappings. The same holds also for K and (3.4), since we can replace φ with $[\varphi, \psi]T$. The right-hand side of (3.5) is also the value of a singular mapping for the arguments φ, ψ, χ, whence by Theorem 1 we have

$$\varphi \cdot \psi \cdot \chi = \mathbf{I}([I, 2^+\varphi, 3^+\psi, 4^+\chi]C_0)T = \mathbf{I}([I, 2^+\varphi, [3^+, 4^+][T\psi, F\chi]]C_0)T$$

for a certain OS-expressible in $\{O\}$ element $C_0 \in \mathcal{F}$, and we can define C_* by the help of Corollary 3 as an element of $\mathcal{F}$ satisfying

$$C_* \cdot \varphi \cdot \vartheta = \mathbf{I}([I, 2^+\varphi, [3^+, 4^+]\vartheta]C_0)T$$

for all φ, $\vartheta \in \mathcal{F}$. □

The partially ordered algebras having one binary operation called application, five constants A,C,K,C_* and D_*, satisfying (3.2)-(3.5), and a least element O for which $D_* \cdot O \cdot O = O$ were studied before by the present author under the name of CLCA (cartesian linear combinatory algebras; [6], [7]). However, a more appropriate (for the traditions of algebraic recursion theory) terminology would be, for instance, 'applicative spaces' instead of CLCA, and we shall follow this terminology below. The applicative spaces were shown to provide a simple abstract algebraic treatment of graph models of lambda calculus, which can comprise all the basic recursive algebra of sets of natural numbers under appropriate conditions (consisting in a suitable strengthening of the supposition of existence of least solutions of all inequalities of the form $\varphi \cdot \xi \le \xi$ with respect to ξ) and which have a good variety of models inspired besides the graph models also by continuous functionals, Scott domains and others. Now the last corollary shows that applicative spaces admit a still greater variety of models, every operative space with iteration in which the equality $[O, O] = O$ holds providing such a model.

Properly speaking, we obtained a functor $\Phi : \mathbf{OSI} \to \mathbf{AS}$ from the category **OSI** of operative spaces with iteration satisfying the last equality to the category **AS** of applicative spaces (morphisms of **OSI** are the mappings preserving the basic OS-operations and the iteration, and the morphisms of **AS** are the mappings preserving the application and the basic constants A,C,K,C_*,D_* and O), which sends every object $\mathcal{F}$ of **OSI** to the applicative space $\Phi(\mathcal{F})$ described by (the proof of) Corollary 4 (in particular, $\Phi(\mathcal{F})$ has the same set of elements and the same partial order as $\mathcal{F}$) and every **OSI**-morphism $f : \mathcal{F} \to \mathcal{F}'$ to the same mapping f. This functor Φ is obviously faithful, but it is a problematical question whether Φ is full, which amounts to the question whether the original basic operations of an operative space $\mathcal{F} \in \mathbf{OSI}$ are explicitly expressible via the basic constants and operations of the applicative space $\Phi(\mathcal{F})$; and the question of equivalence of the categories **OSI** and **AS** is still more problematical.

A nice feature of the functor Φ is that it preserves the storage operations in the following sense. As it was shown in [7], the unary operation ∇ in an applicative space $\mathcal{A}$, which assigns to each element φ of $\mathcal{A}$ the least solution $\nabla(\varphi)$ of the inequality $D_* \cdot \varphi \cdot \xi \le \xi$ with respect to ξ, has the properties of the storage operations arising in semantics of linear logic, namely there are elements $I_*, M_*, Q_*, K_*, W_* \in \mathcal{A}$, such that the equalities

$$
\begin{aligned}
I_* \cdot \nabla(\varphi) &= \varphi, && (3.7)\\
M_* \cdot \nabla(\varphi) \cdot \nabla(\psi) &= \nabla(\varphi \cdot \psi), && (3.8)\\
Q_* \cdot \nabla(\varphi) &= \nabla^2(\varphi) = \nabla(\nabla(\varphi)), && (3.9)\\
K_* \cdot \psi \cdot \nabla(\varphi) &= \psi, && (3.10)\\
W_* \cdot \psi \cdot \nabla(\varphi) &= \psi \cdot \varphi \cdot \varphi && (3.11)
\end{aligned}
$$

hold for all $\varphi, \psi \in \mathcal{A}$, provided the space $\mathcal{A}$ satisfies the conditions mentioned above and specified in [6] and [7]. The algebra of proofs for the Hilbert-type axiomatization of the fragment of linear logic restricted to the linear implication and the exponential connective 'of course' can be regarded, by the well known formulae-as-types correspondence, as a typed version of the algebras with two operations – (linear) application and storage ∇ – and seven constants A, C, I_*, M_*, Q_*, K_*, W_* satisfying (3.2), (3.3) and (3.7)-(3.11). Hence we use the term *type-free models of linear logic* for the last algebras. Now the natural storage operation ∇ defined above in the applicative space $\Phi(\mathcal{F})$ assigned to an operative space $\mathcal{F} \in$ **OSI** coinsides with the natural storage operation in the last space, which is called *translation* (in [2]) and is defined as the least solution of the inequality $[T\varphi, F\xi] \le \xi$ with respect to ξ. This follows immediately from the equality (3.6) of Corollary 4 and is the reason to say that Φ preserves the storage operations.

The type-free models of linear logic have various instances closely connected with recursion theory; that is why they were introduced and used even before the discovery of linear logic for the purposes of axiomatization of recursion (e.g. in [5]). From purely formal point of view, they provide a substitute for the (models of the) combinatory logic, which is easier to deal with, being based on the binary operation of linear application. The last operation is more natural and easier to model, and has the advantage of being free of the gross algebraic complexity of the basic laws for the traditional application operation of combinatory logic, replacing them with some kind of generalized associative and commutative laws. The usual combinatory logic can be easily modeled in the type-free linear logic by defining the application operation Ω as follows: $\Omega(\varphi, \psi) = \varphi \cdot \nabla(\psi)$.

Now the results of [7] imply that for every object $\mathcal{F}$ of **OSI** the applicative space $\Phi(\mathcal{F})$ forms a model of the type-free linear logic with respect to the storage operation defined as the least solution of the inequality $D_* \cdot \varphi \cdot \xi \le \xi$, provided the space $\Phi(\mathcal{F})$ satisfies the conditions specified in [7], which is always the case for continuous operative spaces $\mathcal{F}$. However, one can naturally construct models of the type-free linear logic in a larger class of operative spaces (not necessarily continuous), namely the iterative ones, as well as in all combinatory spaces in the sense of [4] satisfying some weak suppositions of iterativity, by a direct use of translation and other suitable storage operations. This we shall discuss in the next section.

4. STRONG STORAGE OPERATIONS IN OPERATIVE SPACES

Let $\mathcal{F}$ be an operative space with iteration and let \$ be a unary operation in $\mathcal{F}$. We shall say that \$ is a *strong storage* operation in $\mathcal{F}$ iff there are three elements $D, P, Q \in \mathcal{F}$ such that the equalities

$$\begin{aligned} \$(\varphi)n^+ &= n^+\varphi, && (4.1)\\ \$(\varphi\psi) &= \$(\varphi)\$(\psi), && (4.2)\\ \$([\varphi,\psi]) &= [\$(\varphi),\$(\psi)]D, && (4.3)\\ \$(\$(\varphi)) &= Q\$(\varphi)P, && (4.4)\\ \$(\mathbf{I}(\varphi)) &= \mathbf{I}(D\$(\varphi)) && (4.5) \end{aligned}$$

hold for all φ, $\psi \in \mathcal{F}$ and all natural numbers n. A natural example of a strong storage operation provides the operation of translation in iterative operative spaces, which are defined ([2]) as operative spaces with iteration satisfying the following additional axiom

($\mathcal{L}$) There is a unary operation $\varphi \mapsto \langle\varphi\rangle$ in $\mathcal{F}$ called translation, such that the inequality $[T\varphi, F\langle\varphi\rangle] \leq \langle\varphi\rangle$, and the implication

$$(\alpha F \leq \psi\alpha \,\&\, [\alpha T\varphi, \psi\tau] \leq \tau) \Rightarrow \alpha\langle\varphi\rangle \leq \tau$$

hold for all φ, α, ψ, $\tau \in \mathcal{F}$.

Proposition 1. *In every iterative operative space the operation of translation is a strong storage operation such that the corresponding constants D,P and Q are explicitly expressible by means of the basic operations, iteration and translation.*

Proof. This is proved in [2]. Namely, the equality (4.1) for the translation operation is Proposition 5.6 in the quoted book; the equalities (4.2), (4.3) and (4.4) are Propositions 6.21, 6.36 and 6.40, respectively; and the equality (4.5) is Proposition 6.37 in view of the expressions $\mathbf{I}(\varphi) = [\varphi]F = [\varphi]\varphi$ and $[\varphi] = (I, \mathbf{I}(\varphi))$ of the operations of iteration in the sense of the present paper and that in the sense of [2] with each other, which is easy to check directly and which also occurs, for instance, in [8], pp. 1739-1740. □

Generally speaking, there are many other fixed point definable strong storage operations in every iterative operative space, but the translation is one of the simplest of them.

Another important example of strong storage operations is provided by the notion of combinatory space of Skordev [4]. Consider a combinatory space $\mathcal{S} = \langle \mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F \rangle$; we shall use the notations and the terminology concerning combinatory spaces from [4], and we shall suppose that $T, F \in \mathcal{C}$, which does not make an essential difference with the original definition in [4]. We shall call the space $\mathcal{S}$ *weakly iterative* (compare with the notion of *iterative* combinatory space from [4]) iff for all φ, $\chi \in \mathcal{F}$ the least solution $[\varphi, \chi] \in \mathcal{F}$ of the inequality $(\chi \to \xi\varphi, I) \leq \xi$ with respect to ξ exists, and for all $\alpha \in \mathcal{F}$ the element $\alpha[\varphi, \chi]$ is the least solution of $(\chi \to \xi\varphi, \alpha) \leq \xi$ with respect to ξ in $\mathcal{F}$.

Proposition 2. *Suppose in the combinatory space $\mathcal{S}$ the equality $(L, R) = I$ holds. Then there are elements G, T_+ , F_+, D, P, $Q \in \mathcal{F}$ elementary in $\emptyset$ such that: the poset $\mathcal{F}$ forms an operative space $\mathcal{S}_+$ with respect to the same unit I and a multiplication operation as in $\mathcal{S}$, the operation $[-,-]$ defined by*

$$[\varphi, \psi] = (L \to \varphi R, \psi R)G,$$

and the elements T_+ and F_+ as the basic OS-constants T and F, respectively; and the operation \$ defined by $\$(\varphi) = (L, \varphi R)$ satisfies (4.1)- (4.4) *in $\mathcal{S}_+$. Moreover, if the space $\mathcal{S}$ is weakly iterative, then the operative space $\mathcal{S}_+$ has an iteration and* (4.5) *also holds, i.e. \$ is a strong storage operation in $\mathcal{S}_+$.*

Proof. The proof makes use of the technique for combinatory spaces developed in [4] and [2]. More specifically, it is proved in [2] that the partially ordered monoid $\mathcal{F}$ is an operative space $\mathcal{S}_*$ (called the companion operative space of $\mathcal{S}$) with respect to the pairing operation $\varphi, \psi \mapsto (L \to \varphi R, \psi R)$ and the elements $T' = (T, I)$ and $F' = (F, I)$ as the basic constants T and F, respectively; and the elements $C = (LR \to T'\$(R), F'\$(R))$, $P = ((L, LR), RR)$ and $Q = (LL, (RL, R))$ satisfy the equalities (4.2)-(4.4) in $\mathcal{S}_*$, replacing D by C (see Propositions 27.13, 10.12, 10.13 and 10.16 in [2]). Then define $T_+ = PT'T'$, $F_+ = PF'$ and $G = (LL \to T'R, F'(RL, R))$, and note that Proposition 27.8 in [2] combined with the supposition $(L, R) = I$ implies the following fact:

(*) *For all φ, $\psi \in \mathcal{F}$ such that $\varphi(c, I) \leq \psi(c, I)$ for all $c \in \mathcal{C}$ we have $\varphi \leq \psi$.*

Using this fact and the equalities

$$GF_+(c, I) = GP(F, (c, I)) = G((F, c), I) = F'(RL, R)((F, c), I) = F'(c, I)$$

we obtain $GF_+ = F'$, and similarly,

$$GT_+ = GPT'T' = GP(T, (T, I)) = G((T, T), I) = T'R((T, T), I) = T'.$$

Hence

$$[\varphi, \psi]F_+ = (L \to \varphi R, \psi R)GF_+ = (L \to \varphi R, \psi R)F' = \psi,$$

and similarly, $[\varphi,\psi]T_+ = \varphi$, which shows that $\mathcal{S}_+$ is an operative space, the equality $\chi[\varphi,\psi] = [\chi\varphi,\chi\psi]$ being obvious. Using again (*) and the equalities

$$\$(\varphi)(c,I) = (L,\varphi R)(c,I) = (c,\varphi) = (c,I)\varphi, \tag{4.6}$$

which hold for all $\varphi \in \mathcal{F}$ and $c \in \mathcal{C}$, we can prove the equality $\$(\varphi)F_+ = F_+\$(\varphi)$ as follows:

$$\begin{aligned}\$(\varphi)F_+(c,I) &= \$(\varphi)PF'(c,I) = \$(\varphi)((L,LR),RR)(F,(c,I))\\ &= (L,\varphi R)((F,c),I) = ((F,c),\varphi) = ((F,c),I)\varphi\\ &= F_+(c,I)\varphi = F_+\$(\varphi)(c,I).\end{aligned}$$

Similarly, $\$(\varphi)PT' = PT'\$(\varphi)$, whence

$$\begin{aligned}\$(\varphi)T_+ &= \$(\varphi)PT'T' = PT'\$(\varphi)T' = PT'(L,\varphi R)(T,I)\\ &= PT'(T,\varphi) = PT'(T,I)\varphi = T_+\varphi.\end{aligned}$$

The equalities $\$(\varphi)F_+ = F_+\$(\varphi)$ and $\$(\varphi)T_+ = T_+\varphi$ imply (4.1) for the operative space $\mathcal{S}_+$; and (4.2) and (4.4) are the same as in the companion space $\mathcal{S}_*$. The equality (4.3) follows from the same one in $\mathcal{S}_*$:

$$\begin{aligned}\$([\varphi,\psi]) &= \$((L\to\varphi R,\psi R))\$(G) = (L\to\$(\varphi)R,\$(\psi)R)C\$(G)\\ &= [\$(\varphi),\$(\psi)](L\to T_+R,F_+R)C\$(G) = [\$(\varphi),\$(\psi)]D,\end{aligned}$$

where

$$D = (L\to T_+R,F_+R)C\$(G) = (LR\to T_+\$(R),F_+\$(R))\$(G).$$

Now suppose the combinatory space $\mathcal{S}$ is weakly iterative. To prove that the operative space $\mathcal{S}_+$ has iteration, it is enough to show that the companion space $\mathcal{S}_*$ has an iteration, as it follows easily from the definition of the pairing operation in $\mathcal{S}_+$. We shall see that the iteration $\mathbf{I}_*$ in $\mathcal{S}_*$ can be defined by $\mathbf{I}_*(\varphi) = R[(L\to F'R,T'R)\varphi R,\ L]T'$, i.e. that for all $\varphi,\ \alpha \in \mathcal{F}$ the element $\alpha\mathbf{I}_*(\varphi)$ is the least solution of

$$(L\to\alpha R,\xi R)\varphi \le \xi \tag{4.7}$$

with respect to ξ in $\mathcal{F}$. Indeed, for all $\vartheta\in\mathcal{F}$ we have $(L\to[\vartheta R,L]\vartheta R,I) = [\vartheta R,L]$, whence

$$[\vartheta R,L]T' = [\vartheta R,L]\vartheta.$$

Using the last equality and writing shortly E for $(L\to F'R,T'R)$, we can check that $\alpha\mathbf{I}_*(\varphi)$ satisfies (4.7) as follows:

$$\begin{aligned}(L\to\alpha R,\alpha\mathbf{I}_*(\varphi)R)\varphi &= (L\to\alpha R,\alpha R[E\varphi R,\ L]T'R)\varphi\\ &= (L\to\alpha R[E\varphi R,\ L]T'R,\alpha R)(L\to F'R,T'R)\varphi\\ &= \alpha R(L\to[E\varphi R,\ L]T'R,\ I)E\varphi\\ &= \alpha R(L\to[E\varphi R,\ L](L\to F'R,T'R)\varphi R,\ I)E\varphi\\ &= \alpha R[E\varphi R,\ L]E\varphi = \alpha R[E\varphi R,\ L]T' = \alpha\mathbf{I}_*(\varphi).\end{aligned}$$

Assuming that ξ is an arbitrary solution of (4.7) in $\mathcal{F}$, we have as well

$$\begin{aligned}&(L \to (L \to \xi R, \alpha R)E\varphi R, \alpha R)\\&\quad = (L \to (L \to \alpha R, \xi R)\varphi R, \alpha R) \le (L \to \xi R, \alpha R),\end{aligned}$$

whence by the weak iterativity of $\mathcal{S}$ we obtain

$$\alpha R[E\varphi R, L] \le (L \to \xi R, \alpha R)$$

and

$$\alpha \mathbf{I}_*(\varphi) = R[E\varphi R, L]T' \le (L \to \xi R, \alpha R)T' = \xi,$$

completing the proof that $\mathbf{I}_*$ is an iteration in $\mathcal{S}_*$. Hence the operation $\mathbf{I}_+$ defined by $\mathbf{I}_+(\varphi) = \mathbf{I}_*(G\varphi)$ is an iteration in $\mathcal{S}_+$. To establish the equality (4.5) for $\mathcal{S}_+$, we shall do this first for $\mathcal{S}_*$, namely we shall show that

$$\$(\mathbf{I}_*(\varphi)) = \mathbf{I}_*(C\$(\varphi))$$

for all $\varphi \in \mathcal{F}$. Indeed, using the equalities (4.3) (for $\mathcal{S}_*$) and $\$(I) = (L, R) = I$, we have

$$\begin{aligned}(L \to R, \$(\mathbf{I}_*(\varphi))R)C\$(\varphi) &= (L \to \$(I)R, \$(\mathbf{I}_*(\varphi))R)C\$(\varphi)\\ &= \$((L \to R, \mathbf{I}_*(\varphi)R)\varphi) = \$(\mathbf{I}_*(\varphi)),\end{aligned}$$

which shows that $\mathbf{I}_*(C\$(\varphi)) \le \$(\mathbf{I}_*(\varphi))$. To prove the reverse inequality, take an arbitrary $c \in C$ and note, by the help of the definition of the constant C and the equalities (4.6), that

$$C(c, I) = (LR \to T'\$(R), F'\$(R))(c, I) = (L \to T'(c, I)R, F'(c, I)R).$$

Then

$$\begin{aligned}&(L \to (c, I)R, \mathbf{I}_*(C\$(\varphi))(c, I)R)\varphi\\ &\quad= (L \to R, \mathbf{I}_*(C\$(\varphi))R)(L \to T'(c, I)R, F'(c, I)R)\varphi\\ &\quad= (L \to R, \mathbf{I}_*(C\$(\varphi))R)C(c, I)\varphi\\ &\quad= (L \to R, \mathbf{I}_*(C\$(\varphi))R)C\$(\varphi)(c, I) = \mathbf{I}_*(C\$(\varphi))(c, I),\end{aligned}$$

whence, $\mathbf{I}_*$ being iteration in $\mathcal{F}_*$, we obtain

$$(c, I)\mathbf{I}_*(\varphi) \le \mathbf{I}_*(C\$(\varphi))(c, I),$$

which by (4.6) and (*) implies $\$(\mathbf{I}_*(\varphi)) \le \mathbf{I}_*(C\$(\varphi))$ and completes the proof of the equality $\$(\mathbf{I}_*(\varphi)) = \mathbf{I}_*(C\$(\varphi))$. The last one implies

$$\$(\mathbf{I}_+(\varphi)) = \$(\mathbf{I}_*(G\varphi)) = \mathbf{I}_*(C\$(G\varphi)) = \mathbf{I}_*(C\$(G)\$(\varphi)).$$

On the other hand

$$(L \to \varphi R, \psi R) = [\varphi, \psi](L \to T_+R, F_+R)$$

whence

$$\mathbf{I}_*(\varphi) = \mathbf{I}_+((L \to T_+R, F_+R)\varphi)$$

for all $\varphi \in \mathcal{F}$, and

$$\$(\mathbf{I}_+(\varphi)) = \mathbf{I}_+((L \to T_+R, F_+R)C\$(G)\$(\varphi)) = \mathbf{I}_+(D\$(\varphi)).\square$$

Remark. The supposition $(L, R) = I$ in the last proposition was made for the sake of simplicity – we could avoid it at the expense of some complications of the exposition, especially in Section 2. On the other hand, this supposition is natural, but no special reasons are discussed in the books [3] and [4] for its abandonment; it seems it was abandoned just for the reason of its not being necessary for the exposition presented in those books. The paper [9] however, combined with Proposition 2 above, indicates a better exposition which can be simplified by adding the supposition in question to the axioms. Also, the examples of combinatory spaces occurring in the quoted books do not give reasons to consider the abandonment of $(L, R) = I$ as essential for the scope of the theory: all of them have more or less obvious variants in which the last equality is true.

Now, returning to the general case of operative spaces, we have the following

Corollary 5. *Suppose $\mathcal{F}$ is an operative space with iteration, and $\$$ is a strong storage operation in $\mathcal{F}$ with corresponding constants D, P, $Q \in \mathcal{F}$ satisfying* (4.1)-(4.5). *Then the poset $\mathcal{F}$ forms a model of the type-free linear logic with respect to the application operation defined by* (3.1), *the operation $\$$ as the storage ∇, and certain constants A, C, I_*, M_*, Q_*, K_* and W_* which are OS-expressible in $\{D, P, Q, O\}$.*

Proof. The existence of A and C satisfying (3.2) and (3.3) is established in Corollary 4. By Corollary 3 there is $I_0 \in \mathcal{F}$ OS-expressible in $\{O\}$ and such that $\varphi = I_0 \cdot \varphi$ for all $\varphi \in \mathcal{F}$, whence by (4.1)

$$\varphi = \mathbf{I}([I, 2^+\varphi]I_0)T = \mathbf{I}([I, \$(\varphi)2^+]I_0)T,$$

and taking I_* to represent the unary singular operation f_0 defined by

$$f_0(\xi) = \mathbf{I}([I, \xi 2^+]I_0)T$$

in the sense of Corollary 3 we obtain (3.7). By (4.2),(4.5) and (4.3) we have as well

$$\nabla(\varphi \cdot \psi) = \$(\mathbf{I}([I, 2^+\psi]\varphi)T) = \mathbf{I}(D[\$(I), \$(2^+)\$(\psi)]D\$(\varphi))\$(T)),$$

whence the element M_* representing the binary singular operation f_1 defined by

$$f_1(\xi, \eta) = \mathbf{I}(D[\$(I), \$(2^+)\eta]D\xi)\$(T))$$

in the sense of Corollary 3 satisfies (3.8). Similarly, the equality (4.4) shows that the element Q_* representing in the sense of Corollary 3 the unary operation f_2 defined by $f_2(\xi) = Q\xi P$ satisfies (3.9); and to satisfy (3.10) we can obviously take the constant K from Corollary 4 for K_*. Finally, by Theorem 1 we have an element $W_0 \in \mathcal{F}$ OS-expressible in $\{O\}$ such that $\mathbf{I}([I, 2^+\xi, 3^+\eta, 4^+\zeta]W_0)T = \xi \cdot \eta \cdot \zeta$ for all $\xi,\eta,\zeta \in \mathcal{F}$. Then

$$\psi \cdot \varphi \cdot \varphi = \mathbf{I}([I, 2^+\psi, 3^+\varphi, 4^+\varphi]W_0)T = \mathbf{I}([I, 2^+\psi, \$(\varphi)[3^+, 4^+]]W_0)T,$$

and taking W_* to represent in the sense of Corollary 3 the binary singular operation f_3 defined by

$$f_3(\xi, \eta) = \mathbf{I}([I, 2^+\xi, \eta[3^+, 4^+]]W_0)T$$

we obtain (3.11). □

Properly speaking, the last corollary yields a faithful functor Ψ from the category of operative spaces with iteration and strong storage operation to the category of type-free models of linear logic; but, as with the functor Φ, the questions of whether Ψ is full and of existence of equivalence of the last two categories are open.

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