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VANISHING OF THE FIRST DOLBEAULT COHOMOLOGY GROUP OF HOLOMORPHIC LINE BUNDLES ON COMPLETE INTERSECTIONS IN INFINITE DIMENSIONAL PROJECTIVE SPACE

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We consider a complex submanifold X of finite codimension in an infinite-dimensional complex projective space $\boldsymbol{P}$ and prove that the first Dolbeault cohomology group of all line bundles $\mathcal{O}_X(n)$, $n \in \mathbb{Z}$, vanishes when X is a complete intersection and $\boldsymbol{P}$ admits smooth partitions of unity.

Keywords: Dolbeault cohomology groups, infinite-dimensional complex manifolds, projective manifolds, vanishing theorems

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1. INTRODUCTION

In this paper, we prove a vanishing theorem for the first Dolbeault cohomology group of the line bundles $\mathcal{O}_X(n)$, $n \in \mathbb{Z}$, where X is a complete intersection in an infinite-dimensional projective space $\boldsymbol{P}$ which admits smooth partitions of unity.

For a given complex Banach space V, the associated complex projective space $\boldsymbol{P}(V)$ consists of all complex lines in V. The set $\boldsymbol{P}(V)$ has a natural structure of complex manifold which is described in detail in [3]. For a submanifold X of finite codimension in $\boldsymbol{P}(V)$ the complexified tangent bundle $T_{\mathbf{C}}X$, the holomorphic tangent bundle $T^{1,0}X$, and the antiholomorphic tangent bundle $T^{0,1}X$ of X can be defined as in finite dimensions. Given a vector bundle $E \to X$, we define $(0,q)$-forms on X with values in E as bundle maps from $T^{0,q}X = \bigwedge^q T^{0,1}X$ to E. For any open set $U \subset X$, we denote by $C^r_{0,q}(U,E)$ the vector space of all r-times continuously differentiable $(0,q)$-forms with values in E, $0 \le r \le \infty$. We will also write $C^r(U,E)$ instead of $C^r_{0,0}(U,E)$, $C_{0,q}(U,E)$ instead of $C^0_{0,q}(U,E)$, and $C(U,E)$ instead of $C^0_{0,0}(U,E)$. When the vector bundle E is holomorphic, the $\overline{\partial}$-operator, $\overline{\partial} : C^r_{0,q}(U,E) \to C^{r-1}_{0,q+1}(U,E)$, $r \ge 1$, is defined by means of Cartan's formula for the exterior derivative. The Dolbeault cohomology groups $H^{0,q}(X,E)$, $q \ge 0$, of a holomorphic vector bundle $E \to X$ are defined as in finite dimensions:

$$H^{0,q}(X,E) = \frac{\{\text{closed smooth } (0,q)\text{-forms with values in } E\}}{\{\text{exact smooth } (0,q)\text{-forms with values in } E\}}.$$

We refer to [5] for a detailed treatment of (p,q)-forms with values in vector bundles and the $\overline{\partial}$-operator on infinite dimensional complex manifolds.

L. Lempert has proved in [5, Theorem 7.3] that if $E \to \boldsymbol{P}(V)$ is a holomorphic vector bundle of finite rank over localising infinite-dimensional complex projective space $\boldsymbol{P}(V)$, then $H^{0,q}(\boldsymbol{P}(V),E) = 0$, $q \ge 1$. The extra condition on the projective space $\boldsymbol{P}(V)$ has to do with the existence of bump functions. A differentiable manifold M localises if, for every nonempty open set $W \subset M$, there exists a smooth not identically zero function $\phi : M \to \mathbb{R}$ that is supported in W. Every Hilbert space localises whereas the space l^1 does not [4]. A projective space $\boldsymbol{P}(V)$ associated with a locally convex topological vector space V localises if and only if V localises [5, p. 509].

In this paper, we partially extend some of the results in [5] to complete intersections in infinite-dimensional complex projective space. The methods we use require that even stronger conditions should be imposed on the projective space $\boldsymbol{P}(V)$. Namely we have to assume that $\boldsymbol{P}(V)$ admits smooth partitions of unity. A differentiable manifold X admits smooth partitions of unity if, for any open cover $\{U_i\}_{i\in I}$ of X, there are $\theta_i \in C^\infty(X)$, supported in U_i such that $\sum_{i\in I}\theta_i = 1$, the sum being locally finite. Hilbert spaces are examples of such manifolds. Separable and reflexive Banach spaces that localise are other examples. Paracompact manifolds modeled on spaces that admit smooth partitions of unity also admit smooth partitions of unity. In particular, if V is a Banach space that admits smooth partitions of unity, then the associated projective space $\boldsymbol{P} = \boldsymbol{P}(V)$ admits smooth partitions of unity. We refer to [1] for more details about smooth partitions of unity.

Here is a brief outline of the contents of the paper.

In Section 2, we consider a closed form $f \in C^r_{0,1}(\boldsymbol{P}(V), \mathcal{O}_{\boldsymbol{P}(V)}(n))$, $1 \le r \le \infty$, $n \in \mathbb{Z}$. In Proposition 2.2.1 and Proposition 2.2.2, we prove that if V localises and $f|_W \in C^\infty_{0,1}(W, \mathcal{O}_{\boldsymbol{P}(V)}(n))$ for some none-empty open set $W \subset \boldsymbol{P}(V)$, then f is exact. Both propositions are generalisations of [5, Theorem 7.3] for $(0,1)$-forms. The difference is that the differential form f is assumed to be smooth in [5], whereas for our purposes we have to give a proof for differential forms that are smooth on a proper open subset of $\boldsymbol{P}(V)$. Let us emphasise that these results are global.The local solvability of the $\overline{\partial}$-equation can not be taken for granted in infinite dimensions - see [6] for an example of a complex Banach space V and a closed form $f \in C^{\infty}_{0,1}(V)$ which is not exact on any nonempty open subset U of V. In the proofs we use Lempert's idea to solve the $\overline{\partial}$-equation on the blow up $Bl_x\boldsymbol{P}(V)$ of $\boldsymbol{P}(V)$ at a point $x \in \boldsymbol{P}(V)$.

In Section 3, we prove the main result of this paper. The proof consists of two parts. The first part is to find local solutions to the $\overline{\partial}$-equation: we consider an arbitrary submanifold X of finite codimension in $\boldsymbol{P}(V)$ and a closed form $f \in C^{\infty}_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, and construct an open covering $\{X_i\}_{i\in I}$ of X and a collection of sections $\{u_i \in C^{\infty}(X_i, \mathcal{O}_{X_i})\}_{i\in I}$ such that $\overline{\partial} u_i = f|_{X_i}$ for $i \in I$. For this part of the proof we need to assume only that the projective space $\boldsymbol{P}(V)$ localises. The second part of the proof is to solve the Cousin problem for the cocycle $\{u_j - u_i \in H^0(U_i \cup U_j, \mathcal{O}_X)\}_{i,j\in I}$. That this is possible is proved in [3]. For the second part we have to assume that X is a complete intersection and $\boldsymbol{P}(V)$ admits smooth partitions of identity.

This paper is based on the author's Ph.D. thesis (Purdue University, 2001).

2. THE $\overline{\partial}$-EQUATION ON INFINITE-DIMENSIONAL PROJECTIVE SPACES

Let X be a complex manifold and let Y be a submanifold of X of codimension 1. We recall that there exists a holomorphic line bundle L_Y over X, and a section $u_Y \in H^0(X, L_Y)$ such that $Y u_Y^{-1}(0)$ and $\mathrm{d}u_Y(y) \not\equiv 0$ for any $y \in Y$ (where $\mathrm{d}u_Y(y)$ is calculated in some local trivialisation of L_Y at y). Let $L \to X$ be a complex line bundle. Given a section $u \in C(U, L)$ on an open set $U \subset X$, we say that u is locally bounded at a subset $X' \subset X$ if for any $x' \in X'$ there exist an open set $W \ni x'$ and a local trivialisation $\phi : L|_W \to W \times \mathbb{C}$ such that the function $p_2\phi u|_{W\cap U} : W \cap U \to \mathbb{C}$ is bounded on $W \cap U$. We say that u vanishes at X' if for any $x' \in X'$ and any real number $\epsilon > 0$ there exist a neighbourhood W of x' and a local trivialisation $\phi : L|_W \to W \times \mathbb{C}$ such that $|p_2\phi u(x)| < \epsilon$ for all $x \in W \cap U$. Given a submanifold Y of codimension 1 in X, and an integer $n \in \mathbb{Z}$, we write $u = O(|u_Y|^n)$ at Y (resp. $u = o(|u_Y|^n)$ at Y) if the restriction of $u \otimes u_Y^{-n}$ to $U \backslash Y$ is locally bounded at Y (resp. vanishes at Y). Given a differential form $f \in C_1(U, L)$, we write $f = O(|u_Y|^n)$ at Y (resp. $f = o(|u_Y|^n)$ at Y) if $f(\Omega) = O(|u_Y|^n)$ at Y (resp. $f(\Omega) = o(|u_Y|^n)$ at Y) for any vector field $\Omega \in C^{\infty}(X, T_{\mathbb{C}}X)$.

In 2.1.1 we will need the concept of a *weak solution* of the $\overline{\partial}$-equation. Let A be an open subset of a complex Banach space V. Suppose that $u \in C(A)$ and $f \in C_{0,1}(A)$. We say that $\overline{\partial} u = f$ in the *weak sense* if for any finite dimensional affine subspace $F \subset V$, $\overline{\partial}(u|_{F\cap A}) = f|_{F\cap A}$ holds in the sense of distribution theory. For example, if for any $x \in A$ and any $\xi \in V$ the directional derivative

$$\overline{\partial} u(x; \overline{\xi}) = \frac{1}{2}\, \mathrm{d}/\mathrm{d}t|_{t=0}\{u(x + t\xi) + iu(x + it\xi)\}$$

exists and $\overline{\partial}u(x;\overline{\xi}) = f(x;\overline{\xi})$, then $\overline{\partial}u = f$ in the weak sense. We will use the following fact from "elliptic regularity theory":

Proposition 2.1 If $u \in C(A)$, $f \in C^r_{0,1}(U)$, $1 \leq r \leq \infty$, and $\overline{\partial}u = f$ in the weak sense, then $u \in C^r(A)$ and $\overline{\partial}u = f$ holds according to the original definition of the $\overline{\partial}$-operator.

Proof. See [5, Proposition 9.3]. □

2.1. THE $\overline{\partial}$-EQUATION FOR (0, 1)-FORMS ON $\boldsymbol{P}^1$-BUNDLES

In this section, we consider first a trivial $\boldsymbol{P}^1$-bundle $\pi : X = B \times \boldsymbol{P}^1 \to B$ over a complex manifold B. Let q be the projection $X = B \times \boldsymbol{P}^1 \to \boldsymbol{P}^1$. For an integer $n \in \mathbb{Z}$, we denote by $\mathcal{O}_X(n)$ the holomorphic line bundle $q^*(\mathcal{O}_{\boldsymbol{P}^1}(n))$ over X.

Let $z : \boldsymbol{P}^1 \setminus \{\infty\} \to \mathbb{C}$ and $w = z^{-1} : \boldsymbol{P}^1 \setminus \{0\} \to \mathbb{C}$ be the local coordinates on $\boldsymbol{P}^1 \setminus \{\infty\}$ and $\boldsymbol{P}^1 \setminus \{0\}$, respectively. A section $u \in C^r(W, \mathcal{O}_X(n))$ on an open set $W \subset X$ is represented by a pair of functions $u_1 \in C^r(W \setminus B \times \{\infty\})$ and $u_2 \in C^r(U \setminus B \times \{0\})$ such that

$$u_2(b, w) = w^n u_1(b, w^{-1}), \qquad (b, w) \in W,\ w \neq 0. \tag{2.1.1}$$

Proposition 2.1.1. If $f \in C^r_{0,1}(\boldsymbol{P}^1 \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$, $y \in \boldsymbol{P}^1$, $n \in \mathbb{Z}$, $0 \leq r \leq \infty$, is such that $f = O(|u_{\{y\}}|^n)$, then there is a unique section $u \in C^r(\boldsymbol{P}^1 \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ such that $\overline{\partial}u = f$ and $u = o(|u_{\{y\}}|^n)$ at $\{y\}$.

Proof. The section u is unique because if $v \in H^0(\boldsymbol{P}^1 \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ is such that $v = o(|u_{\{y\}}|^n)$ at $\{y\}$, then $v = 0$.

To prove the existence of u, we can assume that $y = \infty$ and write $f = F(z)\,\mathrm{d}\overline{z}$ with $F \in C^r(\mathbb{C})$. Relation (2.1.1) yields $f = -w^n \overline{w}^{-2} F(w^{-1})\,\mathrm{d}\overline{w}$, $w \neq 0$. Since $f = O(|s_{\{\infty\}}|^n)$, there is a constant $C \geq 0$ such that

$$|F(z)| \leq C(1 + |z|)^{-2}, \qquad z \in \mathbb{C}. \tag{2.1.2}$$

We set

$$u_1(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{F(\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}, \qquad z \in \mathbb{C}. \tag{2.1.3}$$

Integral (2.1.3) converges by estimate (2.1.2). Moreover $u_1 \in C^r(\mathbb{C})$ and $\partial u_1/\partial\overline{z} = F$ [2, Theorem 1.2.2]. Let $u_2 \in C^r(\boldsymbol{P}^1 \setminus \{0, \infty\})$ be given by $u_2(w) = w^n u_1(w^{-1})$, $w \neq 0$. Let $u \in C^r(\boldsymbol{P}^1 \setminus \{\infty\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ be represented by the pair $u_1(z), u_2(w)$. Then $u \in C^r(\boldsymbol{P}^1 \setminus \{\infty\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ and $\overline{\partial}u = f$. To complete the proof, we have to show that

$$\lim_{w \to 0} u_1(w^{-1}) = 0\,.$$

Let $G \in C^r(\mathbb{C} \setminus \{0\})$ be given by $G(w) = -\overline{w}^{-2}F(w^{-1})$, $w \neq 0$. Estimate (2.1.2) yields

$$|G(w)| \leq C(1+|w|)^{-2}, \qquad w \neq 0\,. \tag{2.1.4}$$

Making the substitutions $z = w^{-1}$ and $\lambda = \mu^{-1}$ in (2.1.3), we obtain

$$u_1(w^{-1}) \;=\; \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{G(\mu)}{\mu - w}\,\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu} \;-\; \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{G(\mu)}{\mu}\,\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\,.$$

Let $U : \mathbb{C} \to \mathbb{C}$ be given by

$$U(w) \;=\; \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{G(\mu)}{\mu - w}\,\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu} \;=\; \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{G(\nu + w)}{\nu}\,\mathrm{d}\nu \wedge \mathrm{d}\overline{\nu}\,.$$

We claim that U is continuous on $\mathbb{C}$. It is easy to see this when G can be extended to a continuous function $\widetilde{G}$ on $\mathbb{C}$ because estimate (2.1.4) yields

$$|\nu^{-1}\widetilde{G}(\nu + w)| \leq C|\nu|^{-1}(1+|\nu+w|)^{-2} \leq C(1+|w|)^{2}|\nu|^{-1}(1+|\nu|)^{-2}\,,$$

and the function $|\nu|^{-1}(1+|\nu|)^{-2}$ is integrable on $\mathbb{C}$. To deal with the general case, we use continuous bump functions at 0 to construct a sequence of functions $G_m \in C(\mathbb{C})$, $m = 1, 2, \ldots$, such that $|G_m(\mu)| \leq |G(\mu)|$ for $\mu \neq 0$, $G_m(\nu) = G(\nu)$ for $|\mu| \geq m^{-1}$, and $G_m(\mu) = 0$ for $|\mu| \leq (2m)^{-1}$. Let

$$U_m(w) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{G_m(\mu)}{\mu - w}\,\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}, \qquad m = 1, 2, \ldots\,.$$

Now each U_m is continuous on $\mathbb{C}$ and it is not difficult to check that the sequence U_m, $m = 1, 2 \ldots$, converges uniformly to U as $m \to \infty$. Hence U is also continuous on $\mathbb{C}$. Since $u_1(w^{-1}) = U(w) - U(0)$ for $w \neq 0$, we see that $\lim_{w \to 0} u_1(w^{-1}) = 0$. □

Corollary 2.1.2. Suppose that $u \in C^r(\boldsymbol{P}^1 \setminus \{\infty\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$, $n \in \mathbb{Z}$, $1 \leq r \leq \infty$, is such that $u = o(|s_{\{\infty\}}|^n)$ at ∞ and $\overline{\partial} u = O(|s_{\{\infty\}}|^n)$. Then

$$u_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial u_1/\partial\overline{\lambda}}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}\,.$$

Proof. This follows immediately from Proposition 2.1.1 □

In the next proposition, we denote by σ a holomorphic section $\sigma : B \to X$ of a trivial $\boldsymbol{P}^1$-bundle $X = B \times \boldsymbol{P}^1 \to B$. The submanifold $\sigma(B) \subset X$ will be denoted by Y.

Proposition 2.1.3. Suppose that $f \in C^r_{0,1}(X \setminus Y, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, $1 \leq r \leq \infty$, is a closed form that satisfies the following conditions:

(i) $f = O(|u_Y|^n)$ at Y;
(ii) if $\Omega \in C(X, T_{\mathbb{C}}X)$ is a vector field that is tangent to Y, then $f(\Omega) = o(|u_Y|^n)$ at Y and $\overline{\partial}(f(\Omega)) = O(|u_Y|^n)$ at Y.
Then there exists unique $u \in C^r(X \setminus Y, \mathcal{O}_X(n))$ such that $\overline{\partial}u = f$ and $u = o(|u_Y|^n)$ at Y.

Proof. The uniqueness of u is established as in the proof of Proposition 2.1.2. To prove the existence of u, we can assume that $\sigma : B \to X$ is the section given by $\sigma(b) = (b, \infty)$, $b \in B$ and thus $Y = B \times \{\infty\}$. Then we write $f|_{\{b\}\times\mathbb{C}} = F(b,z)\,\mathrm{d}\overline{z}$ and set

$$u_1(b,z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{F(b,\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}, \tag{2.1.5}$$

where condition (i) makes sure that the integral converges. One verifies, as in the proof of Proposition 2.1.1, that u_1 is a continuous function on $X \setminus Y$ and that u_1 vanishes on Y. Let u_2 be given by

$$u_2(b,w) = w^n u_1(b, w^{-1}), \qquad w \neq 0,$$

and let u be the section of $\mathcal{O}_X(n)$ on $X \setminus Y$ that is represented by the pair $u_1(b,z), u_2(b,w)$. It is clear that $u = o(|u_Y|^n)$ at Y.

According to Proposition 2.1.1, we have $\overline{\partial}(u|_{\{b\}\times\mathbb{C}}) = f|_{\{b\}\times\mathbb{C}}$, $b \in B$. To prove that $u \in C^r(X \setminus Y, \mathcal{O}_X(n))$ and $\overline{\partial}u = f$, it is enough to show that $\overline{\partial}(u_1|_{B\times\{z\}}) = f|_{B\times\{z\}}$ weakly for any $z \in \mathbb{C}$. This implies $\overline{\partial}u = f$ weakly and then the claim follows from Proposition 2.1. Let $z \in \mathbb{C}$ and let $\omega \in C^\infty(B \times \{z\}, T^{0,1}(B \times \{z\}))$. Define a vector field $\Omega \in C^\infty(X, T^{0,1}X)$ by $\Omega(b,p) = \omega(b,z)$, $p \in \boldsymbol{P}^1$. It is clear that Ω commutes with the vector field $\partial/\partial\overline{z} \in C^\infty(X \setminus Y, T^{0,1}X)$, i.e $[\Omega, \partial/\partial\overline{z}] = 0$ on $X \setminus Y$. Since f is closed, Cartan's formula yields

$$0 = \overline{\partial}f(\Omega, \partial/\partial\overline{z}) = \Omega(f(\partial/\partial\overline{z})) - \partial/\partial\overline{z}(f(\Omega)) - f([\Omega, \partial/\partial\overline{z}]).$$

Hence $\Omega F = \Omega(f(\partial/\partial\overline{z})) = \partial/\partial\overline{z}(f(\Omega))$. Formal differentiation in (2.1.5) yields

$$\Omega u_1(b,z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\Omega F(b,\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial/\partial\overline{\lambda}(f(\Omega))(b,\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}. \tag{2.1.6}$$

Since the vector field Ω is tangent to $Y = B \times \{\infty\}$, condition (ii) holds for $f(\Omega)$ and it follows, from Corollary 2.1.2, that $\Omega u = f(\Omega)$. Hence $\overline{\partial}(u|_{B\times\{z\}}) = f|_{B\times\{z\}}$ weakly.

Formal differentiation in (2.1.5) is justified as follows. It follows from the growth estimate $\overline{\partial}(f(\Omega)) = O(|s_Y|^n)$ that for any $b_0 \in B$ there is a neighbourhood $U \ni b_0$ and a constant $C \geq 0$ such that

$$|\partial/\partial\overline{\lambda}(f(\Omega))(b,\lambda)| \leq C(1 + |\lambda|)^{-2}, \qquad b \in U,\ \lambda \in \mathbb{C}.$$

Since the function $|\lambda - z|(1+|\lambda|)^{-2}$ is integrable on $\mathbb{C}$, integral (2.1.6) converges uniformly in $b \in U$. Thus formal differentiation in (2.1.5) is justified. $\square$

In the next two propositions, we consider a (not necessarily trivial) $\boldsymbol{P}^1$-bundle $\pi : X \to B$ over a complex manifold B which has a holomorphic section $\sigma : B \to X$. The codimension 1 submanifold $\sigma(B) \subset X$ will be denoted by Y.

Proposition 2.1.4. Let $L \to X$ be a holomorphic line bundle such that for any $b \in B$ there is a neighbourhood $U \ni b$ for which $L|_{\pi^{-1}(U)} \cong \mathcal{O}_{\pi^{-1}(U)}(n)$ for some fixed integer $n < 0$. Then for any closed form $f \in C^1_{0,1}(X, L) \cap C^r_{0,1}(X \setminus Y, L)$, $1 \leq r \leq \infty$, there exists a unique section $u \in C^r(X \setminus Y, L)$ such that $\overline{\partial} u = f$ and $u = o(|u_Y|^n)$ at Y.

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of B such that $L|_{\pi^{-1}(U_i)} \cong \mathcal{O}_{\pi^{-1}(U_i)}(n)$, $i \in I$. Denote $Y \cap \pi^{-1}(U_i)$ by Y_i. Conditions (i) and (ii) of Proposition 2.1.3 hold trivially for $f|_{\pi^{-1}(U_i)}$ and u_{Y_i} because $n < 0$. For each $i \in I$ Proposition 2.1.3 yields a unique section $u_i \in C^r(\pi^{-1}(U_i) \setminus Y, L)$ such that $\overline{\partial} u_i = f$ on $\pi^{-1}(U_i) \setminus Y_i$ and $u_i = o(|u_{Y_i}|^n)$. For $i, j \in I$ the restrictions $u_i|_{\pi^{-1}(U_i) \cap \pi^{-1}(U_j)}$ and $u_j|_{\pi^{-1}(U_i) \cap \pi^{-1}(U_j)}$ are the same because $u_i|_{Y_i \cap Y_j} = o(|u_{Y_i \cap Y_j}|^n)$ and $u_j|_{Y_i \cap Y_j} = o(|u_{Y_i \cap Y_j}|^n)$. Hence the sections u_i, $i \in I$, paste together to a section $u \in C^r(X \setminus Y, L)$ such that $\overline{\partial} u = f$ on $X \setminus Y$ and $u = o(|u_Y|^n)$ at Y. The section u is unique because if a holomorphic section $s \in H^0(X \setminus Y, L)$ is such that $s = o(|u_Y|^n)$ at Y, then $s = 0$. $\square$

We recall that a smooth vector-valued function u on a real differentiable manifold X vanishes at $x \in X$ of order $k+1$ if all differentials $\mathrm{d}^0 u, \mathrm{d}^1 u, \ldots, \mathrm{d}^k u$ vanish at x. Given a vector bundle $E \to X$ and a section $u \in C^\infty(X, E)$, we say that u vanishes at $x \in X$ of order $k+1$ if for some (or any) local trivialisation $\phi : E|_U \to U \times \mathbb{R}^n$ of E about x the vector-valued function $p_2 \phi u|_U : U \to \mathbb{R}^n$ vanishes at x of order $k+1$. Given a differential form $f \in C^\infty_1(X, E)$, we say that f vanishes of order $k+1$ at $x \in X$, if for any neighbourhood U of x and any vector field $\Omega \in C^\infty(U, TX)$ the section $f(\Omega) \in C^\infty(U, E)$ vanishes of order $k+1$ at x. Let X' be a subset of X. We will say that $f \in C^\infty_1(X, E)$ vanishes of order $k+1$ at X' if f vanishes of order $k+1$ at x for any $x \in X'$.

Proposition 2.1.5. Let $L \to X$ be a holomorphic line bundle such that for any $b \in B$ there is a neighbourhood $U \ni b$ for which $L|_{\pi^{-1}(U)} \cong \mathcal{O}_{\pi^{-1}(U)}(n)$ for some fixed integer $n \geq 0$. Suppose that $f \in C^r_{0,1}(X, L)$, $1 \leq r \leq \infty$, is a closed form such that

(i) $f \in C^\infty_{0,1}(W, L)$ for some open set $W \supset Y$;

(ii) f vanishes of order n at Y;

(iii) $f(\Omega)$ vanishes of order $n+1$ at Y for any vector field $\Omega \in C^\infty(X, T_{\mathbb{C}}(X))$ that is tangent to Y.

Then there is a unique $u \in C^r(X \setminus Y, L) \cap C^\infty(W \setminus Y, L)$ such that $\overline{\partial} u = f$ on $X \setminus Y$ and $u = o(|u_Y|^n)$ at Y.

Proof. Condition (ii) yields $f = O(|u_Y|^n)$. Condition (iii) yields $f(\Omega) = o(|u_Y|^n)$ and $\overline{\partial}(f(\Omega)) = O(|u_Y|^n)$ for any vector field $\Omega \in C^\infty(X, T_{\mathbb{C}}(X))$ that is tangent to Y. Let $\{U_i\}_{i\in I}$ and Y_i, $i \in I$, be as in the proof of Proposition 2.1.4. Then conditions (i) and (ii) of Proposition 2.1.3 hold for $f|_{\pi^{-1}(U_i)}$ and u_{Y_i}, $i \in I$, and the rest of the proof is analogous to the proof of Proposition 2.1.4. □

2.2. THE $\overline{\partial}$-EQUATION FOR $(0,1)$-FORMS ON PROJECTIVE SPACE

In this subsection, we consider a projective space $\boldsymbol{P}(V)$, corresponding to a complex Banach space V, and apply the results from the previous subsection to construct a solution of the equation $\overline{\partial} u = f$ for $(0,1)$-forms on $\boldsymbol{P}(V)$ with values in the line bundle $\mathcal{O}_{\boldsymbol{P}(V)}(n)$, $n \in \mathbb{Z}$. For a description of the complex structure of $\boldsymbol{P}(V)$, we refer to [3, Sec. 3] . In the proofs we will use the blow up manifold $Bl_x(\boldsymbol{P}(V))$ of $\boldsymbol{P}(V)$ at a point $x \in \boldsymbol{P}(V)$, which is described as follows. For a given $x = [v_0] \in \boldsymbol{P}(V)$, we denote by V' the factor-space $V/[v_0]$, and by q the factoring linear map from V to V'. To simplify the notation, we will write $\boldsymbol{P}$ (resp. $\boldsymbol{P}'$) instead of $\boldsymbol{P}(V)$ (resp. $\boldsymbol{P}(V')$). The blow up $Bl_x(\boldsymbol{P})$ of $\boldsymbol{P}$ at x is the set

$$Bl_x(\boldsymbol{P}) = \{([v],[v']) \in \boldsymbol{P} \times \boldsymbol{P}' : q(v) \in [v']\}\,.$$

Let π (resp. ρ) be the restriction of the projection $\boldsymbol{P}\times\boldsymbol{P}' \to \boldsymbol{P}'$ (resp. $\boldsymbol{P}\times\boldsymbol{P}' \to \boldsymbol{P}$) to $Bl_x(\boldsymbol{P})$. To make $Bl_x(\boldsymbol{P})$ into a complex manifold, we first choose a bounded linear functional l on V such that $x \in \boldsymbol{P}_l$ and then, for any $l' \in V'^*$, $l' \neq 0$, define a coordinate map $\phi_{l'} : \pi^{-1}(\boldsymbol{P}'_{l'}) \to \boldsymbol{P}'_{l'} \times \boldsymbol{P}^1$ by the formula $\phi_{l'}([v],[v']) = ([v'],[l'(p(v)) : l(v)])$. The family of coordinate maps $\phi_{l'}$, $l' \in V'^*$, $l' \neq 0$, endows $Bl_x(\boldsymbol{P})$ with a structure of a complex manifold such that the maps π and ρ are holomorphic. Furthermore, the map π is a locally trivial projective line bundle over $\boldsymbol{P}'$, and the map ρ is biholomorphic outside the *exceptional divisor* $E = \rho^{-1}(x)$ of $Bl_x(\boldsymbol{P})$. We note that the map $\sigma : \boldsymbol{P}' \to Bl_x(\boldsymbol{P})$, given by the formula $\sigma([v']) = ([v_0],[v'])$ for $[v'] \in \boldsymbol{P}'$, is a holomorphic section of π such that $\sigma(\boldsymbol{P}') = E$.

Proposition 2.2.1. Let $f \in C^r_{0,1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$, $r \geq 1$, $n < 0$, be a closed form. If $\dim \boldsymbol{P} > 1$, then there exists a unique section $u \in C^r(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u = f$.

Proof. Since $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n)) = 0$ for $n < 0$, the equation $\overline{\partial} u = f$ cannot have two distinct solutions. To prove the existence of a solution u, it is enough to show that, for any $x \in \boldsymbol{P}$, there exists a section $u_x \in C^r(\boldsymbol{P} \setminus \{x\}, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$. Indeed, let $y \in \boldsymbol{P}$, $x \neq x$, and $u_y \in C^r(\boldsymbol{P} \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}}(n))$ be such that $\overline{\partial} v_y = f$ on $\boldsymbol{P} \setminus \{y\}$. Then $s = u_x - u_y \in H^0(\boldsymbol{P} \setminus \{x,y\}, \mathcal{O}_{\boldsymbol{P}}(n))$ extends to a global holomorphic section $\tilde{s}$ of $\mathcal{O}_{\boldsymbol{P}}(n)$ by Hartogs' theorem. Let

$u \in C^r(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$ be given by u_x on $\boldsymbol{P} \setminus \{x\}$ and by $u_y + \tilde{s}$ on $\boldsymbol{P} \setminus \{y\}$. Then $\overline{\partial} u = f$.

So let $x \in \boldsymbol{P}$ and let $\tilde{f} = \rho^* f \in C^r_{0,1}(Bl_x(\boldsymbol{P}), \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$. By Proposition (2.1.4), there is a section $\tilde{u} \in C^r(Bl_x \boldsymbol{P} \setminus E, \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial}\tilde{u} = \tilde{f}$. Now the map ρ is biholomorphic on $Bl_x(\boldsymbol{P}) \setminus E$, and $u_x = (\rho^{-1})^* \tilde{f} \in C^r(\boldsymbol{P} \setminus \{p\}, \mathcal{O}_{\boldsymbol{P}}(n))$ is such that $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$. □

To deal with (0,1)-forms with values in the line bundles $\mathcal{O}_{\boldsymbol{P}}(n)$, $n \geq 0$, we consider a special class of Banach spaces. They have the property that for any nonempty open subset $W \subset V$, there exists a not identically zero function $\omega \in C^\infty(V)$ that is supported in U. A differentiable manifold M that has this property is called localising (cf. [5, Sec. 7]), or we will say that M localises. The projective space $\boldsymbol{P}(V)$ localises if and only if the Banach space V localises [5, Sec. 7]. All Hilbert spaces localise because the square of the norm is a smooth function. The Banach space l^1 is an example of a space that is not localising [4].

Proposition 2.2.2. Let V be a Banach space that localises. Suppose that $f \in C^r_{0,1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$, $n \geq 0$, $r \geq 1$, is a closed form such that $f \in C^\infty_{0,1}(W, \mathcal{O}_{\boldsymbol{P}}(n))$ for some nonempty open set $W \subset \boldsymbol{P}$. Then there exists a $u \in C^r(P, \mathcal{O}_{\boldsymbol{P}}(n)) \cap C^\infty(W, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u = f$.

Proof. We will assume that V is an infinite dimensional Banach space (for $\dim V < \infty$ the proposition is well known under much weaker conditions on the regularity of f). Then it is enough to show that for any $x \in W$ there is a $u_x \in C^r(\boldsymbol{P} \setminus \{x\}, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$ (cf. the proof of Proposition (2.2.1)

So let $x \in W$ and $\boldsymbol{P}'$ be a hyperplane in $\boldsymbol{P}$ which does not contain x. Let W' be a neighbourhood of x such that $W' \subset W$ and $W' \cap \boldsymbol{P}' = \emptyset$. Since the line bundle $\mathcal{O}_{\boldsymbol{P}}(n)$ trivialises on W', there exists $u' \in C^\infty(W', \mathcal{O}_{\boldsymbol{P}}(n))$ such that $f|_{W'} - \overline{\partial} u'$ vanishes of order n at x (see [5, Theorem 3.6]). Let $\omega \in C^\infty(\boldsymbol{P})$ be a cut-off function that is supported in W', and equal to 1 in a neighbourhood of x and let $g = f - \overline{\partial}(\omega v)$. Then $g \in C^r_{0,1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n)) \cap C^\infty_{0,1}(W, \mathcal{O}_{\boldsymbol{P}}(n))$ is a closed form that vanishes of order n at x. Consequently, $\tilde{g} = \rho^* g \in C^r_{0,1}(Bl_p(\boldsymbol{P}), \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$ is a closed form that is smooth on $\rho^{-1}(W)$ and vanishes of order n at E. Moreover, if Ω is a smooth vector field on $Bl_x(\boldsymbol{P})$ that is tangent to E, then $\tilde{g}(\Omega)$ vanishes of order $n+1$ at E because $\rho_*(T_{\mathbf{C}} E) = 0$. By Proposition 2.1.5, there is a $\tilde{u} \in C^r(Bl_x(\boldsymbol{P}) \setminus E, \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial}\tilde{u} = \tilde{g}$ on $Bl_x(\boldsymbol{P}) \setminus E$. Set $u_x = (\rho^{-1})^* \tilde{u} + \omega v \in C^r(\boldsymbol{P} \setminus \{x\}, \mathcal{O}_{\boldsymbol{P}}(n))$. Then $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$, which completes the proof. □

3. ANALYSIS OF REGULARITY

Let X be a submanifold of finite codimension n and degree d in $\boldsymbol{P} = \boldsymbol{P}(V)$ and (V', W, z) be an admissible triple for X (see [3, Section 3]). Let $p : X \rightarrow$

$\boldsymbol{P}(V')$ be the map induced by the projection $V = W \times V' \to V'$. According to [3, Proposition 4.2], for any given $f \in C^r_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, there exist unique forms $f_j \in C^r_{0,1}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(n-j))$, $j = 0, \dots, d-1$ such that $f|_{X_D} = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j$.

The differential forms f_j, $j = 0, \dots, d-1$, should not be expected to behave in a regular manner along the divisor $\mathcal{D} = Z(D)$ (see Example 1 below). However, their behaviour can be improved to a degree by twisting with powers of D. Thus for any natural number N, we define new forms

$$\tilde{f}_j^N \in C_{0,1}(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(n - j + N \deg D)), \quad j = 0, \dots, d-1,$$

in the following way:

$$\tilde{f}_j^N(b, \xi) = \begin{cases} D(b)^N f_j(b, \xi) & \text{for } b \in \boldsymbol{P}(V')_D \text{ and } \xi \in T_b^{0,1}\boldsymbol{P}(V'), \\ 0 & \text{for } b \notin \boldsymbol{P}(V')_D \text{ and } \xi \in T_b^{0,1}\boldsymbol{P}(V'). \end{cases}$$

The following proposition is the main result in this section.

Proposition 3.1. Let $f \in C^r_{0,1}(X, \mathcal{O}_X(n))$, $1 \leq r < \infty$, $n \in \mathbb{Z}$. If $N > 4r + 3$, then

$$\tilde{f}_j^N \in C^r_{0,q}(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(n - j + N \deg D)), \quad j = 0, \dots, d-1.$$

The proof of Proposition 3.1 will be given later in this section.

Since the vector bundle $\pi : \boldsymbol{P} \backslash \boldsymbol{P}(W) \to \boldsymbol{P}(V')$ trivialises over every affine open set $\boldsymbol{P}(V')_h$, $0 \neq h \in V'^*$, we will prove first an affine version of Proposition 3.1.

From now on we will assume that X is an algebraic manifold of finite codimension n in a Banach space V. Let (W, V') be an admissible factorisation for X, and let $Z_1, \dots, Z_n$ be a basis of W^* such that $z = Z_1 + I(X) \in C[X]$ generates the field of fractions of $C[X]$ over the field of fractions of the $C[V']$. We denote by D the discriminant of the minimal polynomial of z over the field of fractions of $C[V']$. The restriction of the projection $p : X \to V'$ to X_D will be denoted by p_D. By [3, Proposition 2.3] the holomorphic map $p_D : X_D \to V_D$ is a covering of degree $d = \deg F$. For a given $f \in C^r_{0,1}(X)$, let $g = f|_{X_D}$. By [3, Proposition 4.3] there exist unique forms $g_j \in C^r_{0,1}(V'_D)$ $j = 0, \dots, d-1$ such that $g = \sum_{j=0}^{d-1} z^j \pi^* g_j$. For any natural number N we define new forms $\tilde{g}_j^N \in C_{0,1}(V')$, $j = 0, \dots, d-1$, in the following way:

$$\tilde{g}_j^N(b, \xi) = \begin{cases} D(b)^N g_j(b, \xi) & \text{for} \quad b \in V'_D \quad \text{and} \quad \xi \in T_b^{0,1}V' \\ 0 & \text{for} \quad b \notin V'_D \quad \text{and} \quad \xi \in T_b^{0,1}V'. \end{cases}$$

Proposition 3.2. Let $f \in C^r_{0,1}(X)$, $1 \leq r < \infty$, and let $g = f|_{X_D}$. If $N > 4r + 3$, then $\tilde{g}_j^N \in C^r_{0,1}(V')$, $j = 0, \dots, d-1$,

The proof of this proposition will be given later in the section.

The following example shows a typical behaviour of the forms g_j, along the divisor $\mathcal{D} = Z(D)$. Let $V = \mathbb{C}^2$ and let $X = \{(Y,Z) \in \mathbb{C}^2 : Z^2 = Y/4\}$. Set $z = Z|_X \in C[X]$. Let $W = \{(0,Z) \in \mathbb{C}^2 : Z \in \mathbb{C}\}$ and $V' = \{(Y,0) \in \mathbb{C}^2 : Y \in \mathbb{C}\}$. Then the projection $p : X \to \mathbb{C}$ given by $p(Y,Z) = Y$ is finite and surjective, and z generates $C[X]$ over $C[V'] = \mathbb{C}[Y]$. The discriminant of the minimal polynomial $F = Z^2 - Y/4$ is $D = Y$. Thus $\mathcal{D} = \{0\}$, $V'_D = \mathbb{C} \setminus \{0\}$, and $X_D = X \setminus \{(0,0)\}$.

Example 1. Let $X \subset \mathbb{C}^2$ be the quadric described above and $f = \mathrm{d}\overline{z}|_X$. Then $8\overline{z}\,\mathrm{d}\overline{z} = p^*(\mathrm{d}\overline{Y})$, and solving for $\mathrm{d}\overline{z}$, we obtain

$$g \;=\; \frac{1}{8\overline{z}}\, p_D^*(\mathrm{d}\overline{Y}) \;=\; \frac{z}{8z\overline{z}}\, p_D^*(\mathrm{d}\overline{Y}) \;=\; z\, p_D^*(\frac{1}{2|Y|}\mathrm{d}\overline{Y}).$$

Hence $g_0 = 0$ and $g_1 = 2^{-1}|Y|^{-1}\mathrm{d}\overline{Y}$. It is easy to see that, for every natural number r, there exists a natural number N_r such that $\tilde{g}_1^N = 2^{-1}|Y|^{-1}Y^N\,\mathrm{d}\overline{Y} \in C^r(\mathbb{C})$ for $N > N_r$. However, there is no natural number N such that $\tilde{g}_1^N \in C^\infty(\mathbb{C})$.

Suppose that $r : U \to X$ is a right inverse to p_D on some open set $U \subset V'_D$, i.e. $p_D \circ r = \mathrm{id}_U$. Let $e_1, \ldots, e_n$ be the basis of W which is dual to the basis $Z_1, \ldots, Z_n$. Let $R_j = z_j \circ r \in H^0(U, \mathcal{O}_U)$, $j = 1, \ldots, n$. Then

$$r(b) = (\, b\,,\, \sum_{j=1}^{n} R_j(b) e_j\,) \;\in\; X \subset V' \times W$$

for all $b \in U$. Since $F(z) = 0$, where F is the minimal polynomial of z, we obtain $F(R_1) = 0$. As in [3, Lemma 2.3], there exist polynomials $F_j \in C[V'][Z]$, $j = 2, \ldots, n$, such that

$$R_j = D^{-1} F_j(R_1)\,, \qquad j = 2, \ldots, n\,. \tag{3.1}$$

The holomorphic map $r : U \to X \subset V$ induces a complex linear map r_* from the complexified tangent space of $b \in U$ to the complexified tangent space of $r(b) \in X$, $r_* : T_b^{\mathbb{C}} U \to T_{r(b)}^{\mathbb{C}} X \subset T_{r(b)}^{\mathbb{C}} V$, for all $b \in U$. For $\xi \in T_b^{\mathbb{C}} U$, we denote by $r_*(b,\xi)$ the image of ξ in $T_{r(b)}^{\mathbb{C}} X$. Since V' and V are vector spaces, we can naturally identify $T_b^{\mathbb{C}} U$ and $T_{r(b)}^{\mathbb{C}} V$ with $V' \oplus \overline{V'}$ and $V \oplus \overline{V} = (V' \oplus \overline{V'}) \times (W \oplus \overline{W})$, respectively. Since the map r is holomorphic, $r_*(T_b^{1,0} U) \subset T_{r(b)}^{1,0} X$ and $r_*(T_b^{0,1} U) \subset T_{r(b)}^{0,1} X$. The restriction of r_* to $T_b^{1,0} U$ will be denoted by dr, and the restriction of r_* to $T_b^{0,1} U$ will be denoted by $d\overline{r}$. For any vector $\xi \in T_b^{1,0} U$, its conjugate vector $\overline{\xi}$ is in $T_b^{0,1} U$, and $d\overline{r}(b, \overline{\xi}) = \overline{dr(b,\xi)}$. It is clear that for $\xi \in T_b^{1,0} U$ we have

$$r_*(b,\xi) = (\,\xi\,,\, \sum_{j=1}^{k} dR_j(b,\xi)\, \frac{\partial}{\partial Z_j}\,) \in V' \times W\,, \tag{3.2}$$

where dR_j is the differential of the holomorphic function R_j, $j = 1, \ldots, n$.

The next lemma is the main step in the proof of Proposition 3.2.

Lemma 3.3. There exists a smooth function $H : V \times V' \to V$ such that:
(i) for any open set U in V'_D and any right inverse $r : U \to X$ to p_D on U we have $dr(b,\xi) = D(b)^{-2}H(r(b),\xi)$ for $b \in U$ and $\xi \in V'$;
(ii) $H(x,\xi) \in T^{1,0}_x X$ for $x \in X$ and $\xi \in V'$;
(iii) H is linear in $\xi \in V'$.

Proof. We want to find the coefficients $dR_j(b,\xi)$, $j = 1,\dots,n$, in (3.2). Since the function R_1 satisfies the equation $F(R_1) = 0$, we use implicit differentiation to find $dR_1(b,\xi)$. Then we differentiate (3.1) to find $dR_j(b,\xi)$, $j = 2,\dots,n$.

Let $F = Z^d + a_1 Z^{d-1} + \dots + a_d$, where $a_m \in C[V']$, $m = 1,\dots,d$, and let $F' \in C[V'][Z]$ be the derivative of F with respect to Z.

Since $F(R_1) = 0$, we obtain

$$F'(R_1(b))\, dR_1(b,\xi) + \sum_{m=1}^{d} da_m(b,\xi)\, R_1(b)^{d-m} = 0\,. \tag{3.3}$$

It is well known (see for example [7]) that there exist polynomials $A, B \in C[V'][Z]$ such that

$$AF + BF' = D\,.$$

Hence

$$F'(R_1(b))^{-1} = D(b)^{-1}B(R_1(b)) \tag{3.4}$$

for $b \in U$ and $\xi \in V'$. Let $H_1 \in C^\infty(V \times V')$ be the function given by

$$H_1(v,\xi) = -B(Z_1(v)) \sum_{m=1}^{d} da_m(\pi(v),\xi) Z_1(v)^{d-m}\,.$$

It follows from (3.3) and (3.4) that

$$dR_1(b,\xi) = D(b)^{-1}H_1(r(b),\xi) \tag{3.5}$$

for $b \in U$ and $\xi \in V'$.

Let $F_j = \sum_{m=0}^{d-1} a_{mj} Z^{d-m-1} \in C[V'][Z]$, $j = 2,\dots,n$, where $a_{mj} \in C[V']$ for $j = 2,\dots,n$, and $m = 0,\dots,d-1$. Let $F'_j \in C[V'][Z]$, $j = 2,\dots,n$, be the derivative of F_j with respect to Z. Since $R_j = D^{-1}F_j(R_1)$, $j = 2,\dots,n$, (by equation (3.1)), we obtain

$$\begin{aligned} dR_j(b,\xi) &= -D(b)^{-2}dD(b,\xi)F_j(R_1(b)) + D(b)^{-1}dR_1(b,\xi)F'_j(R_1(b)) + \\ &\quad + D(b)^{-1}\sum_{m=0}^{d-1} da_{mj}(b,\xi)R_1(b)^{d-m-1} \qquad \text{(by (3.5))} \\ &= -D(b)^{-2}dD(b,\xi)F_j(R_1(b)) + D(b)^{-2}H_1(r(b),\xi)F'_j(R_1(b)) + \\ &\quad + D(b)^{-1}\sum_{m=0}^{d-1} da_{mj}(b,\xi)R_1(b)^{d-m-1}\,. \end{aligned} \tag{3.6}$$

Let $H_j \in C^\infty(V \times V')$, $j = 2, \dots, k$, be the function given by

$$H_j(v,\xi) = -dD(\pi(v),\xi)\, F_j(Z_1(v)) + H_1(w,\xi)\, F_j'(Z_1(v)) + \\ + D(p(v)) \sum_{m=0}^{d-1} da_{mj}(p(v),\xi) Z_1(v)^{d-m-1}.$$

It follows from (3.6) that $dR_j(b,\xi) = D(b)^{-2} H_j(r(b),\xi)$ for $j = 2,\dots,n$.

Finally, let $H : V \times V' \to V$ be the smooth function given by

$$H(v,\xi) = (\, D(\pi(v))^2 \xi \,,\, D(\pi(v)) H_1(v,\xi) \frac{\partial}{\partial Z_1} + \sum_{j=2}^{k} H_j(v,\xi) \frac{\partial}{\partial Z_j}\,).$$

It follows from (3.1), (3.5), and (3.6), that (i) holds for H.

To prove that (ii) holds for H, we notice first that if $x \in X_D$ then there exists an open neighbourhood $U \subset V_D'$ of $p(x)$ and a right inverse $r : U \to X$ to p_D on U such that $r(p(x)) = x$. By part (i) we have $H(x,\xi) = D(p(x))^2\, r_*(p(x),\xi) \in T_x^{1,0} X$ for all $\xi \in V'$. Thus (ii) holds for H if $x \in X_D$. Let $\widetilde{H} : X \times V' \to T^{1,0}V = V \times V$ be given by the formula $\widetilde{H}(x,\xi) = (x, H(x,\xi))$ for $(y,\xi) \in X \times V'$. It is clear that $\tilde{H}$ is a continuous map and $\widetilde{H}(X_D \times V') \subset T^{1,0}X$. Since $T^{1,0}X$ is a closed subset of $T^{0,1}V$, and $X_D \times V'$ is a dense subset of $X \times V'$, we see that $\widetilde{H}(X \times V') \subset T^{1,0}X$. Hence condition (ii) holds for H. Finally, condition (iii) also holds for H because all functions H_j, $1 = 2, \dots, n$, are linear in ξ. □

We will need a similar result for the restriction of r_* to the bundle $T^{0,1}U$.

Lemma 3.4. There exists a smooth function $\overline{H} : V \times \overline{V'} \to \overline{V}$ such that:
(i) for any open set U in V_D' and any right inverse $r : U \to X$ to p_D on U we have $d\overline{r}(b,\overline{\xi}) = \overline{D(b)}^{-2}\, \overline{H}(r(b),\overline{\xi})$ for $b \in U$ and $\overline{\xi} \in \overline{V'}$;
(ii) $\overline{H}(x,\overline{\xi}) \in T_x^{0,1} X$ for $x \in X$ and $\overline{\xi} \in \overline{V'}$;
(iii) $\overline{H}$ is linear in $\overline{\xi} \in \overline{V'}$.

Proof. Let $\overline{H} : V \times \overline{V'} \to \overline{V}$ be given by $\overline{H}(w,\overline{\xi}) = \overline{H(w,\xi)}$ for $w \in W$, $\xi \in W'$. It follows from Lemma 3.3 that (i), (ii) and (iii) hold for the map $\overline{H}$ because $d\overline{r}(b,\overline{\xi}) = \overline{dr(b,\xi)}$. □

We denote by $X_{\mathbb{R}}$ the real manifold associated with the complex manifold X. For any $x \in X$ there is a natural inclusion

$$T_x X_{\mathbb{R}} \to T_x^{\mathbb{C}} X = \mathbb{C} \otimes_{\mathbb{R}} T_x X_{\mathbb{R}} = T_x^{1,0} X \oplus T_x^{0,1} X$$

given by

$$T_x X_{\mathbb{R}} \ni \eta \mapsto \frac{1}{2}(1 \otimes \eta + \frac{1}{i} \otimes i\eta) \oplus \frac{1}{2}(1 \otimes \eta - \frac{1}{i} \otimes i\eta) \in T_x^{1,0} X \oplus T_x^{0,1} X.$$

For a given vector $\eta \in T_x X_{\mathbb{R}}$, we denote by $\eta^{1,0}$ and $\eta^{0,1}$ the vectors $\frac{1}{2}(1\otimes\eta+\frac{1}{i}\otimes i\eta)$ and $\frac{1}{2}(1\otimes\eta-\frac{1}{i}\otimes i\eta)$, respectively.

If E is a complex vector space, then the real tangent bundle $TE_{\mathbb{R}} \to E_{\mathbb{R}}$ is naturally isomorphic to the trivial bundle $E_{\mathbb{R}} \times E_{\mathbb{R}} \to E_{\mathbb{R}}$. Thus, for any $b \in E_{\mathbb{R}}$, we can canonically identify $T_b E_{\mathbb{R}}$ with $E_{\mathbb{R}}$.

Lemma 3.3 and Lemma 3.4 are combined in the next lemma, to prove a similar result for the map $r_* : TU_{\mathbb{R}} \to TV_{\mathbb{R}}$.

Lemma 3.5 *There exists a smooth function $R : V_{\mathbb{R}} \times V'_{\mathbb{R}} \to V_{\mathbb{R}}$ such that:*
(i) for any open set U in V'_D and any right inverse $r : U \to X$ to p_D on U we have $r_*(b,\eta) = D(b)^{-2}\,\overline{D(b)}^{-2} R(r(b),\eta)$ for $b \in U_{\mathbb{R}}$ and $\eta \in V'_{\mathbb{R}}$;
(ii) $R(x,\eta) \in T_x X_{\mathbb{R}}$ for $x \in X_{\mathbb{R}}$ and $\eta \in V'_{\mathbb{R}}$;
(iii) R is $\mathbb{R}$-linear in $\eta \in V'_{\mathbb{R}}$.

Proof. For any $b \in U_{\mathbb{R}}$ and any $\eta \in V'_{\mathbb{R}}$

$$\begin{aligned} r_*(b,\eta) &= r_*(b,\eta^{1,0}) + r_*(b,\eta^{0,1}) \\ &= dr(b,\eta^{1,0}) + d\overline{r}(b,\eta^{0,1}) \\ &= D(b)^{-2} H(r(b),\eta^{1,0}) + \overline{D(b)}^{-2}\overline{H}(r(b),\eta^{0,1}) \\ &= D(b)^{-2}\overline{D(b)}^{-2}\left\{\overline{D(b)}^{2} H(r(b),\eta^{1,0}) + D(b)^2\overline{H}(r(b),\eta^{0,1})\right\}. \end{aligned} \tag{3.7}$$

Let $R : V_{\mathbb{R}} \times V'_{\mathbb{R}} \to V_{\mathbb{R}}$ be the map given by

$$R(v,\eta) = \overline{D(\pi(v))}^{2} H(v,\eta^{1,0}) + D(\pi(v))^2\,\overline{H}(v,\eta^{0,1}), \quad v \in V_{\mathbb{R}}, \eta \in V'_{\mathbb{R}}. \tag{3.8}$$

Eq. (3.7) shows that condition (i) holds for R. If $x \in X_{\mathbb{R}}$, then the vectors $\overline{D(p(x))}^{2} H(x,\eta^{1,0})$ and $D(p(x))^2\overline{H}(x,\eta^{0,1})$ are conjugate to each other. Hence $R(x,\eta) \in T_x X_{\mathbb{R}}$ for $x \in X_{\mathbb{R}}$. Thus condition (ii) holds for R. It is clear from (3.8) that R is $\mathbb{R}$-linear in η. □

Lemma 3.6. Suppose that $g \in C^r_{0,1}(X_D)$ is such that $g = f|_{X_D}$ for some $f \in C^r_{0,1}(X)$. Then there exists a function $G \in C^r(X \times \overline{V'})$ such that:
(i) for any open set U in V'_D and any right inverse $r : U \to X$ to p_D on U we have $r^*g\,(b,\overline{\xi}) = \overline{D(b)}^{-2} G(r(b),\overline{\xi})$ for $b \in U$ and $\overline{\xi} \in \overline{V'}$;
(ii) G is linear in $\overline{\xi} \in \overline{V'}$.

Proof. For $x \in X$ and $\overline{\xi} \in \overline{V'}$, let $G(x,\overline{\xi}) = f(x,\overline{H}(x,\overline{\xi}))$, where $\overline{H}$ is the map defined in Lemma 3.4. We note that the right-hand side makes sense because $\overline{H}(x,\overline{\xi}) \in T^{0,1}_x X$ by part (ii) of Lemma 3.4. Let us verify that (i) holds for G:

$$r^*g\,(b,\overline{\xi}) = f(r(b), d\overline{r}(b,\overline{\xi})) = f(r(b), \overline{D(b)}^{-2}\,\overline{H}(r(b),\overline{\xi})) = \overline{D(b)}^{-2} G(r(b),\overline{\xi}).$$

The definition of G shows that it is linear in $\overline{\xi} \in \overline{V'}$. $\square$

Proposition 3.7. Let $f \in C^r_{0,1}(X)$ and $g = f|_{X_D}$. Suppose that $g_j \in C^r_{0,1}(V'_D)$, $j = 0, \ldots, d-1$, are such that $g = \sum_{j=0}^{d-1} z^j\, \pi^* g_j$. Then there exist functions $G_j \in C^r(X^d \times \overline{V'})$, $j = 0, \ldots d-1$, such that:
(i) each function G_j is symmetric in $(x_1, \ldots, x_d) \in X^d$ and linear in $\overline{\xi} \in \overline{V'}$;
(ii) $g_j(b, \overline{\xi}) = D(b)^{-1}\overline{D(b)}^{-2} G_j(r_1(b), \ldots, r_d(b), \overline{\xi})$ for $j = 0, \ldots, d-1$, $b \in V'_D$, $\overline{\xi} \in \overline{V'}$, where $\{r_1(b), \ldots, r_d(b)\}$ is the fiber of $p_D : X_D \to V'_D$ over $b \in V'_D$.

Proof. Let $b \in V'_D$ and let $U \subset V'_D$ be a neighbourhood of b such that the covering map $p_D : X_D \to V'_D$ has d distinct right inverses $r_i : U \to X_D$ on U, $p \circ r_i = \mathrm{id}_U$, $i = 1, \ldots, d$. Then, according to [3, Proposition 4.3],

$$g_j(b, \overline{\xi}) = D(b)^{-1}\Delta(b) \det A_j(b, \overline{\xi})$$

where

$$\Delta(b) = \prod_{1 \le i_1 < i_2 \le d} \left(z(r_{i_2}(b)) - z(r_{i_1}(b))\right),$$

and $A_j(b, \overline{\xi})$ is the $d \times d$ matrix

$$\begin{pmatrix} 1 & z(r_1(b)) & \cdots & z(r_1(b))^{j-1} & r_1^* g\,(b, \overline{\xi}) & z(r_1(b))^{j+1} & \cdots & z(r_1(b))^{d-1} \\ 1 & z(r_2(b)) & \cdots & z(r_2(b))^{j-1} & r_2^* g\,(b, \overline{\xi}) & z(r_2(b))^{j+1} & \cdots & z(r_2(b))^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(r_d(b)) & \cdots & z(r_d(b))^{j-1} & r_d^* g\,(b, \overline{\xi}) & z(r_d(b))^{j+1} & \cdots & z(r_d(b))^{d-1} \end{pmatrix}.$$

According to Lemma 3.6 $r_i^* g\,(b, \overline{\xi}) = \overline{D(b)}^{-2} G(r_i(b), \overline{\xi})$ for $i = 1, \ldots, d$. Hence $\det A_j(b, \overline{\xi}) = \overline{D(b)}^{-2} \det B_j(b, \overline{\xi})$, where $B_j(b, \overline{\xi})$, $j = 0, \ldots, d-1$, is the matrix

$$\begin{pmatrix} 1 & z(r_1(b)) & \cdots & z(r_1(b))^{j-1} & G(r_1(b), \overline{\xi}) & z(r_1(b))^{j+1} & \cdots & z(r_1(b))^{d-1} \\ 1 & z(r_2(b)) & \cdots & z(r_2(b))^{j-1} & G(r_2(b), \overline{\xi}) & z(r_2(b))^{j+1} & \cdots & z(r_2(b))^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(r_d(b)) & \cdots & z(r_d(b))^{j-1} & G(r_d(b), \overline{\xi}) & z(r_d(b))^{j+1} & \cdots & z(r_d(b))^{d-1} \end{pmatrix}.$$

Let $\delta : X^d \to \mathbb{C}$ be the smooth function given by

$$\delta(x_1, \ldots, x_d) = \prod_{1 \le i_1 < i_2 \le d} \left(z(x_{i_2}) - z(x_{i_1})\right).$$

Let $C_j(x_1, \ldots, x_d, \overline{\xi})$, $j = 0, \ldots, d-1$, be the matrix

$$\begin{pmatrix} 1 & z(x_1) & \cdots & z(x_1)^{j-1} & G(x_1, \overline{\xi}) & z(x_1)^{j+1} & \cdots & z(x_1)^{d-1} \\ 1 & z(x_2) & \cdots & x(y_2)^{j-1} & G(x_2, \overline{\xi}) & x(y_2)^{j+1} & \cdots & z(x_2)^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(x_d) & \cdots & z(x_d)^{j-1} & G(x_d, \overline{\xi}) & z(x_d)^{j+1} & \cdots & z(x_d)^{d-1} \end{pmatrix}.$$

Finally, let $G_j : X^d \times \overline{V'} \to \mathbb{C}$, $j = 0, \dots, d-1$, be given by

$$G_j(x_1, \dots, x_d, \overline{\xi}) = \delta(x_1, \dots, x_d) \det C_j(x_1, \dots, x_d, \overline{\xi})$$

for $j = 0, \dots, d-1$. The definition of G_j shows that it satisfies (i). Since $\Delta(b) = \delta(r_1(b), \dots, r_d(b))$ and $B_j(b, \overline{\xi}) = C_j(r_1(b), \dots, r_d(b), \overline{\xi})$, we see that (ii) also holds for G_j, $j = 0, \dots, d-1$. □

Let $\omega \in C^r_{0,1}(V'_D)$ and let $\eta_1, \dots, \eta_m$ be vectors in $V'_{\mathbb{R}}$, $(m \le r)$. The derivative of order m of ω at the point $(b, \overline{\xi}) \in (V'_D)_{\mathbb{R}} \times \overline{V'}$ in the directions $\eta_1, \dots, \eta_m$ will be denoted by $d^m_b \omega(b, \overline{\xi}; \eta_1, \dots, \eta_m)$. For any $\omega \in C^r_{0,1}(V'_D)$ and any natural number N, we will denote by ω^N the form $D^N \omega \in C^r_{0,1}(V'_D)$. The next lemma is an extension of Proposition 3.7 to the derivatives of the forms g_j, $j = 0, \dots, d-1$.

Lemma 3.8. Let $f \in C^r_{0,1}(X)$, $1 \le r < \infty$, and $g = f|_{X_D}$. Suppose that $g_j \in C^r_{0,1}(V'_D)$, $j = 0, \dots, d-1$, are such that $g = \sum_{j=0}^{d-1} z^j \pi^* g_j$. Then for any $0 \le m \le r$, $0 \le j \le d-1$ and any natural number N there exists a function $G^N_{jm} \in C^{r-m}(X^d \times \overline{V'} \times {V'}^m_{\mathbb{R}})$ such that:
(i) G^N_{jm} is symmetric in $x_1, \dots, x_d \in X^d$ and linear in $\overline{\xi} \in \overline{V'}$;
(ii) for any $b \in V'_D$, $\overline{\xi} \in \overline{V'}$, and $\eta_1, \dots, \eta_m \in T_b V'_{\mathbb{R}} = V'_{\mathbb{R}}$ we have

$$d^m_b g^N_j(b, \overline{\xi}; \eta_1, \dots, \eta_m) =$$
$$D(b)^{N-2m-1} \overline{D(b)}^{-2m-2} G^N_{jm}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_m).$$

Proof. The proof is by induction on m. For $m = 0$ the lemma is true by Proposition 3.7. We are going to show that if $1 \le m \le r$ and there exists a function $G^N_{j\,m-1} \in C^{r-m+1}(X^d \times \overline{V'} \times {V'}^{m-1}_{\mathbb{R}})$ such that (i) and (ii) hold for $G^N_{j\,m-1}$, then there exists a function $G^N_{jm} \in C^{r-m}(Y^d \times \overline{V'} \times {V'}^m_{\mathbb{R}})$ such that (i) and (ii) hold for G^N_{jm}.

Let $b \in V'_D$ and let $U \subset V'_D$ be a neighbourhood of b such that the covering map $p_D : X_D \to V'_D$ has d distinct right inverses $r_i : U \to X_D$, $i = 1, \dots, d$, on U, $\pi \circ r_i = \mathrm{id}_U$, $i = 1, \dots, d$.

To find $d^m_b g^N_j(b, \overline{\xi}; \eta_1, \dots, \eta_m)$, we differentiate the function

$$D(b)^{N-2m+1} \overline{D(b)}^{-2m} G^N_{j\,m-1}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1})$$

in the direction $\eta_m \in T_b V'_{\mathbb{R}} = V'_{\mathbb{R}}$. After applying the product rule and the chain rule, we obtain the following terms:
I. $A_{Nm}\, D(b)^{N-2m}\, dD(b, \eta_m)\, \overline{D(b)}^{-2m}\, G^N_{j\,m-1}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1})$, where $A_{Nm} = N - 2m - 1$ and $dD(b, \eta_m)$ is the derivative of D in the direction $\eta_m \in T_b V'_{\mathbb{R}} = V'_{\mathbb{R}}$.

Let $\mathcal{F}_{jm}^N \in C^{r-m+1}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function given by

$$\mathcal{F}_{jm}^N(x_1,\ldots,x_d,\overline{\xi},\eta_1,\ldots,\eta_m) = \\ A_{Nm}\, D(b)\, \overline{D(b)}^2\, dD(b,\eta_m)\, G_{j\,m-1}^N(x_1,\ldots,x_d,\overline{\xi},\eta_1,\ldots,\eta_{m-1})\,. \tag{3.9}$$

II. $B_m\, D(b)^{N-2m+1}\, \overline{D(b)}^{\,-2m-1}\, d\overline{D}(b,\eta_m)\, G_{j\,m-1}^N(r_1(b),\ldots,r_d(b),\overline{\xi},\eta_1,\ldots,\eta_{m-1})$, where $B_m = -2m-1$ and $d\overline{D}(b,\eta_m)$ is the derivative of $\overline{D}$ in the direction $\eta_m \in T_b V_{\mathbb{R}}' = V_{\mathbb{R}}'$.
Let $\mathcal{G}_{jm}^N \in C^{r-m+1}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function given by

$$\mathcal{G}_{jm}^N(x_1,\ldots,x_d,\overline{\xi},\eta_1,\ldots,\eta_m) = \\ B_m\, D(b)^2\, \overline{D(b)}\, d\overline{D}(b,\eta_m)\, G_{j\,m-1}(x_1,\ldots,x_d,\overline{\xi},\eta_1,\ldots,\eta_{m-1})\,. \tag{3.10}$$

III. $D(b)^{N-2m+1}\, \overline{D(b)}^{\,-2m}\, \dfrac{\partial G_{j\,m-1}^N}{\partial y_i}(r_1(b),\ldots,r_d(b),\overline{\xi},\eta_1,\ldots,\eta_{m-1})(r_{i*}(b,\eta_m))$, $i = 1,\ldots,d$, where

$$\frac{\partial G_{j\,m-1}^N}{\partial y_i}(r_1(b),\ldots,r_d(b),\overline{\xi},\eta_1,\ldots,\eta_{m-1}) : T_{r_i(b)}X_{\mathbb{R}} \to \mathbb{C}$$

is the "the partial derivative" of $G_{j\,m-1}^N$ with respect to x_i, $i = 1,\ldots,d$, and

$$r_{i*}(b,\cdot) : T_b V_{\mathbb{R}}' \to T_{r_i(b)}X_{\mathbb{R}}\,, \qquad i = 1,\ldots,d,$$

is the $\mathbb{R}$-linear map from $T_b V_{\mathbb{R}}'$ to $T_{r_i(b)}X_{\mathbb{R}}$ that is induced by $r_i : U \to X_D$ for $i = 1,\ldots,d$. By Lemma 3.5,

$$r_{i*}(b,\eta_m) = D(b)^{-2}\, \overline{D(b)}^{\,-2}\, R(r_i(b),\eta_m)\,, \qquad i = 1,\ldots,d\,.$$

Let $\mathcal{H}_{jmi}^N \in C^{r-m}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function given by

$$\mathcal{H}_{jmi}^N(x_1,\ldots,x_d,\overline{\xi},\eta_1,\ldots,\eta_m) = \\ = \frac{\partial G_{j\,m-1}^N}{\partial y_i}(x_1,\ldots,x_d,\overline{\xi},\eta_1,\ldots,\eta_{m-1})(R(y_i,\eta_m)) \tag{3.11}$$

for $i = 1,\ldots,d$.

Finally, let $G_{jm}^N \in C^{r-m}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function

$$G_{jm}^N = \mathcal{F}_{jm}^N + \mathcal{G}_{jm}^N + \sum_{i=1}^d \mathcal{H}_{jmi}^N\,. \tag{3.12}$$

It follows from (3.9), (3.10), and (3.11) that (ii) holds for G_{jm}. We note that the functions $\mathcal{F}_{jm}^N$ and $\mathcal{G}_{jm}^N$ are symmetric in $x_1,\ldots,x_d$. Since the function $\sum_{i=1}^d \mathcal{H}_{jmi}^N$ is also symmetric in $x_1,\ldots,x_d$, we see that G_{jm}^N is symmetric in $x_1,\ldots,x_d$. The function G_{jm}^N is linear in $\overline{\xi} \in \overline{V'}$ because all terms on the right-hand side of (3.12) are linear, too. Thus (i) holds for G_{jm}^N. □

In the proof of Proposition 3.2 we will need the following simple lemma.

Lemma 3.9. Let X, Y, and W be metric spaces and let $p : Y \to Z$ be a proper map. Let d be a natural number and suppose that G is a continuous function on $Y^d \times W$. Then, for any $z_0 \in Z$ and any $w_0 \in W$, there are neighbourhoods $\mathcal{U}$ and $\mathcal{W}$ of z_0 and w_0, respectively such that the function G is bounded on the set $p^{-1}(\mathcal{U})^d \times \mathcal{W}$.

Proof. Since p is a proper map, the fiber $F = p^{-1}(z_0)$ is compact. Since $F^d \times \{w_0\}$ is a compact subset of $Y^d \times W$, there exists an open set $\mathcal{A} \subset Y^d \times W$ which contains $F^d \times \{w_0\}$, and is such that G is bounded on $\mathcal{A}$. By the tube lemma from topology there exist an open set $\mathcal{V} \subset Y$ that contains F and a neighbourhood $\mathcal{W}$ of w_0 such that $\mathcal{V}^d \times \mathcal{W} \subset \mathcal{A}$. Since p is a proper map, there exists a neighbourhood $\mathcal{U}$ of z_0 such that $p^{-1}(\mathcal{U}) \subset \mathcal{V}$. The function G is bounded on $p^{-1}(\mathcal{U})^d \times \mathcal{W}$ because $p^{-1}(\mathcal{U})^d \times \mathcal{W} \subset \mathcal{A}$. □

Proof of Proposition 3.2. Let $(b_0, \overline{\xi}_0, \eta_0) \in \mathcal{D} \times \overline{V'} \times V'^m$ for $0 \le m \le r$. We will prove that if $N > 4r + 3$, then:

(i) For any $0 \le m \le r - 1$ and any sequence $\{b_n\}_{n=1}^\infty \subset V'_D$ such that $\lim_{n\to\infty} b_n = b_0$

$$\lim_{n\to\infty} \frac{d_b^m g_j^N(b_n, \overline{\xi}_0, \eta_0)}{\|b_n - b_0\|} = 0\,, \qquad\qquad j = 0, \ldots, d-1\,.$$

This shows that $\tilde{g}_j^N$ has a derivative of order $m + 1$ at $(b_0, \overline{\xi}_0, \eta_0)$, and that this derivative vanishes at $(b_0, \overline{\xi}_0, \eta_0)$.

(ii) For any $0 \le m \le r$ and any sequence $\{(b_n, \overline{\xi}_n, \eta_n)\}_{n=1}^\infty \subset V'_D \times \overline{V'} \times V'^m$ such that $\lim_{n\to\infty}(b_n, \overline{\xi}_n, \eta_n) = (b_0, \overline{\xi}_0, \eta_0)$

$$\lim_{n\to\infty} d_b^m g_j^N(b_n, \overline{\xi}_n, \eta_n) = 0, \qquad\qquad j = 0, \ldots, d-1\,.$$

This shows that all derivatives $d_b^m g_j^N$, $m = 0, \ldots, r$, are continuous.

Let us prove (i). Let $H_{jm}^N \in C^{r-m}(X^d)$ be the function given by

$$H_{jm}^N(x_1, \ldots, x_d) = G_{jm}^N(x_1, \ldots, x_d, \overline{\xi}_0, \eta_0)$$

for $(x_1, \ldots, x_d) \in X^d$. By Lemma 3.9, there exists a neighbourhood $\mathcal{U}$ of b_0 such that H_{jm}^N is bounded on $p^{-1}(\mathcal{U})^d$. Since $\lim_{n\to\infty} b_n = b_0$, the sequence

$$\{G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0)\}_{n=1}^\infty$$

is bounded. By Lemma 3.8,

$$\frac{d_b^m g_j^N(b_n, \overline{\xi}_0, \eta_0)}{\|b_n - b_0\|} =$$

$$= \frac{D(b_n)^{N-2m-1}\overline{D(b_n)}^{-2m-2}}{\|b_n - b_0\|} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0) =$$

$$= \frac{D(b_n)^{N-4m-3}}{\|b_n - b_0\|} \times \left\{ \left[D(b_n) \,/\, \overline{D(b_n)} \right]^{2m+2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0) \right\}.$$

We note that $N - 4m - 3 = N - 4r - 3 + 4(r - m) \geq 2$ because $N > 4r + 3$ and $0 \leq m \leq r - 1$. Hence

$$\lim_{n\to\infty} \frac{D(b_n)^{N-4m-3}}{\|b_n - b_0\|} = 0\,.$$

Since the sequence

$$\left\{ \left[D(b_n) \,/\, \overline{D(b_n)} \right]^{2m+2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0) \right\}_{n=1}^{\infty}$$

is bounded, we see that

$$\lim_{n\to\infty} \frac{d_b^m g_j^N(b_n, \overline{\xi}_0, \eta_0)}{\|b_n - b_0\|} = 0\,.$$

The proof of part (ii) is similar. By Lemma 3.9 there exist neighbourhoods $\mathcal{U}$ and $\mathcal{W}$ of b_0 and $(\overline{\xi}_0, \eta_0)$, respectively such that $G_{jm}^N \in C^{r-m}(X^d \times \overline{V'} \times V'^m)$ is bounded on the set $p^{-1}(\mathcal{U})^d \times \mathcal{W}$. Since $\lim_{n\to\infty} = (b_n, \overline{\xi}_n, \eta_n)(b_0, \overline{\xi}_0, \eta_0)$ the sequence

$$\{G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n)\}_{n=1}^{\infty}$$

is bounded. By Lemma 3.8,

$$d_b^m g_j^N(b_n, \overline{\xi}_n, \eta_n) =$$

$$= D(b_n)^{N-2m-1}\overline{D(b_n)}^{-2m-2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n)$$

$$= D(b_n)^{N-4m-3} \left\{ \left[D(b_n)/\overline{D(b_n)} \right]^{2m+2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n) \right\}.$$

We note that $N - 4m - 3 \geq 1$ because we assume that $0 \leq m \leq r$ and $N > 4r + 3$. Since the sequence

$$\left\{ \left[D(b_n)/\overline{D(b_n)} \right]^{2m+2q} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n) \right\}_{n=1}^{\infty}$$

is bounded, we obtain $\lim_{n\to\infty} d_b^m g_j^N(b_n, \overline{\xi}_n, \eta_n) = 0$ which finishes the proof. □

Proposition 3.1 is derived from Proposition 3.2 as follows.

Proof of Proposition 3.1. For any $h \in V'^*$, $h \neq 0$, we set $D_h = D/h^{d(d-1)} \in C[\boldsymbol{P}(V')_h]$ and $g = f/h^n \in C^r_{0,1}(X_h)$. Let $g_j \in C^r_{0,1}(\boldsymbol{P}(V')_h \cap \boldsymbol{P}(V')_D)$, $j = 0, \ldots, d-1$, be such that

$$g|_{X_{D_h}} = \sum_{j=0}^{d-1} (z/h)^j \, (p|_{X_{D_h}})^* g_j \, .$$

It is easily seen that $\tilde{g}_j^N = h^{-n+j-N \deg D} \, \tilde{f}_j^N|_{\boldsymbol{P}(V')_h}$, $j = 0, \ldots, d-1$. According to Proposition 3.2, we have $\tilde{g}_j^N \in C^r_{0,1}(\boldsymbol{P}(V')_h)$ for $N > 4r+3$ and $j = 0, \ldots, d-1$. Hence $\tilde{f}_j^N|_{\boldsymbol{P}(V')_h} \in C^r_{0,q}(\boldsymbol{P}(V')_h, \mathcal{O}_{\boldsymbol{P}(V')}(n - j + N \deg D))$ for $N > 4r+3$ and $j = 0, \ldots, d-1$. Since the open sets $\{\boldsymbol{P}(V')_h : 0 \neq h \in V'^*\}$ cover $\boldsymbol{P}(V')$, Proposition 3.1 has been proved. □

Corollary 3.10. If (V', W, z) is an admissible triple for a submanifold X of finite codimension in $\boldsymbol{P}$, and $f \in C^\infty_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, is a closed form, then there exists $u \in C^\infty(X_D, \mathcal{O}_X(n))$ such that:
(i) $\overline{\partial} u = f|X_D$;
(ii) $D^8 u = v|_{X_D}$ for some $v \in C^1(X, \mathcal{O}_X(n + 8 \deg D))$.

Proof. Let $f_j \in C^\infty_{0,1}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(n-j))$, $j = 0, \ldots, d-1$, be such that $f|_{X_D} = \sum_{j=0}^{d-1} (p|_{X_D})^* f_j \otimes z^j$. Then $\tilde{f}_j^8 \in C^1_{0,1}(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(n + 8 \deg D))$ for $j = 0, \ldots, d-1$, by Proposition 3.1, and $\tilde{f}_j^8$, $j = 0, \ldots, d-1$, is an exact form that is smooth on the open set $\boldsymbol{P}(V')_D$. By Proposition 2.2.1 (if $n + 8 \deg D < 0$) and Proposition 2.2.2 (if $n + 8 \deg D \geq 0$), there exist sections

$$u_j \in C^1(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(n + 8 \deg D))$$

such that $\overline{\partial} u_j = \tilde{f}_j^8$ for $j = 0, \ldots, d-1$. Proposition 2.1 shows that all sections u_j, $j = 0, \ldots, d-1$, are smooth on $\boldsymbol{P}(V')_D$. Let

$$u = D^{-8} \sum_{j=0}^{d-1} (p|_{X_D})^* (u_j|_{\boldsymbol{P}(V')_D}) \otimes z^j \in C^\infty(X_D, \mathcal{O}_X(n))$$

and

$$v = \sum_{j=0}^{d-1} (p|_X)^* u_j \otimes z^j \in C^1(X, \mathcal{O}_X(n + 8 \deg D)) \, .$$

Then u and v satisfy (i) and (ii). □

Now we can prove the main result of this paper. In the proof we use the complex $\mathcal{C}(X, \mathcal{O}_X(n))$ which was defined in [3, Section 5].

Theorem 3.11. Let V be a Banach space that admits smooth partitions of unity and $\boldsymbol{P} = \boldsymbol{P}(V)$. Then $H^{0,1}(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, for any complete intersection X in $\boldsymbol{P}$.

Proof. Let $f \in \mathcal{C}^{\infty}_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, be a given closed form. According to [3, Corollary 3.10], there is a collection $\{(V_i', W_i, z_i)\}_{i \in I}$ of admissible triples for X in $\boldsymbol{P}$ such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i \in I}$ is a covering of $\boldsymbol{P}$. By Corollary 3.10 in this paper, for any $i \in I$ there exist $u_i \in C^{\infty}(X_i, \mathcal{O}_X(n))$ and $v_i \in C^1(X, \mathcal{O}_X(n + 8d_i))$ such that $\overline{\partial} u_i = f|_{X_i}$ and $u_i = D_i^{-8}(v_i|_{X_i})$. For any $i, j \in I$, let

$$\varphi_{ij} = u_j|_{X_{ij}} - u_i|_{X_{ij}} \in \mathcal{C}^{\infty}(X_{ij}, \mathcal{O}_X(n)),$$
$$\tilde{\varphi}_{ij} = D_i^8 \tilde{\varphi}_j - D_j^8 \tilde{\varphi}_i \in \mathcal{C}^1(X, \mathcal{O}_X(n + 8d_i + 8d_j)).$$

Then $\overline{\partial} \varphi_{ij} = 0$ for any $i, j \in I$, which implies that $\varphi_{ij} \in H^0(X_{ij}, \mathcal{O}_X(n))$ for any $i, j \in I$. Furthermore, the global section $\tilde{\varphi}_{ij}$ is holomorphic on X_{ij} for any $i, j \in I$ because $(D_i D_j)^8 \varphi_{ij} = \tilde{\varphi}_{ij}|_{X_{ij}}$. Since $\tilde{\varphi}_{ij}$ is continuous on X, the Riemann removable singularity theorem yields $\tilde{\varphi}_{ij} \in H^0(X, \mathcal{O}_X(n + 8d_i + 8d_j))$ for any $i, j \in I$. Therefore the cocycle $\varphi = \{\varphi_{ij}\}_{i,j \in I}$ belongs to the group $\mathcal{C}^1_8(X, \mathcal{O}_X(n))$ (see [3, Section 6]). Since φ is a closed cocycle and $H^1(\mathcal{C}(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, by [3, Theorem 6.5], there exists a collection of holomorphic sections

$$\psi = \{w_i \in H^0(X_i, \mathcal{O}_X(n))\}_{i \in I} \in \mathcal{C}^0(X, \mathcal{O}_X(n))$$

such that $\delta \psi = \varphi$. This means that $(u_i - w_i)_{X_i \cap X_j} = (u_j - w_j)_{X_i \cap X_j}$ for all $i, j \in I$. Let $u \in C^{\infty}(X, \mathcal{O}_X(n))$ be given by $u|_{X_i} = u_i - w_i$, $i \in I$. Since $(\overline{\partial} u)|_{X_i} = \overline{\partial}(u_i - w_i) = \overline{\partial} u_i = f|_{X_i}$ for any $i \in I$, we obtain $\overline{\partial} u = f$. □

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