ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

BIVARIATE INTERPOLATION BY (m,n)–SPLINES

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We study the problem of interpolation and approximation of bivariate functions by (m,n)-splines, that is, by functions $\phi(x,y)$ for which $\phi^{(m,n)}(x,y)$ is a piece-wise constant.

Keywords: reconstruction, multivariate polynomials, interpolation, splines

2000 MSC: 41A63, 41A15

1. INTRODUCTION

The interpolation methods are a basic tool for approximation of functions. While, in the univariate case, most of the interpolation problems admit a nice treatment, often yielding a closed form expression for the interpolating function, the study of the corresponding multivariate problems encounters serious difficulties. For example, the interpolation by multivariate algebraic polynomials is not always regular. One of the central directions of investigation in this field is the construction of appropriate configurations of nodes for which the problem is regular. Similar difficulties occur in interpolation by other multivariate classes and, in particular, by splines. In this paper, we consider a standard problem of interpolation of bivariate functions on a rectangular grid by a special class of splines, which we call (m,n)-*splines*. Let us give the precise definition.

Suppose $G := [a,b] \times [c,d]$ is a given rectangular domain on the plane. Let us introduce a grid on G defined by the lines $x_i = a + i\dfrac{b-a}{M}, y_j = c + j\dfrac{d-c}{N}, i = 1,\dots,M,\ j = 1,\dots,N$. In this way we get a partition of G into a sum of small rectangles $\square_{ij} = [x_{i-1}, x_i) \times [y_{j-1}, y_j)$. We say that a function $s(x, y)$ is an (m, n)-*spline* on G if

$$\frac{\partial^{m+n} s}{\partial x^m \partial y^n}(x, y) = c_{ij} \text{ for } (x, y) \in \square_{ij},$$

where c_{ij} are constants.

In what follows, for the sake of definiteness, we assume that G is the unit square, i.e., $G = [0, 1]^2$. Let us denote by $\square_{ij}^c$ the closure of $\square_{ij}$.

We consider the following interpolation problem:

For a given sufficiently smooth function f and a set of MN distinct points $\{t_{ij}\}$ in G, find an (m, n)-spline $s(x, y)$ such that

$$s(t_{ij}) = f(t_{ij}), \quad i = 1, \ldots, M, \quad j = 1, \ldots, N, \tag{1.1}$$

and satisfying the boundary conditions

$$\frac{\partial^i s}{\partial y^i}(0, y) = \frac{\partial^i f}{\partial y^i}(0, y), \quad i = 0, \ldots, m - 1, \quad y \in [0, 1],$$

$$\frac{\partial^j s}{\partial x^j}(x, 0) = \frac{\partial^j f}{\partial x^j}(x, 0), \quad j = 0, \ldots, n - 1, \quad x \in [0, 1].$$

We show that the interpolation problem (1.1) has a unique solution for any choice of the nodes $t_{ij} = (\xi_{ij}, \eta_{ij})$ such that

$$\left\{ \begin{array}{ll} x_{i-1} < \xi_{ij} \le x_i, & i = 1, \ldots, M \\ y_{j-1} < \eta_{ij} \le y_j, & j = 1, \ldots, N \end{array} \right\}.$$

The solution is given explicitly for some small (m, n). We study also the question of approximation of the functions f by the corresponding interpolating spline s and give an error estimate for $(m, n) = (1, 1)$, $(1, 2)$ and $(2, 2)$.

2. PRELIMINARIES

The notion of a blending function is frequently used in this paper. Let us recall the definition (cf. [1]).

Functions from the space

$$C_{[0,1]^2}^{m,n} = \left\{ f : \frac{\partial^{k+l} f}{\partial x^k \partial y^l} \in C_{[0,1]^2},\ k = 1, \ldots, m,\ l = 1, \ldots, n \right\},$$

satisfying the conditions

$$\frac{\partial^{k+l} f}{\partial x^k \partial y^l} = 0,$$

are said to be *blending functions* of order (m, n).

We shall denote the space of all blending functions of order (m, n) by $B_{[0,1]^2}^{m,n}$.

The next representation (given in [2]) of any sufficiently smooth function $f(x,y)$ in terms of blending functions of order (m,n) will be used in the sequel. It is based on the Taylor-type operators T_x^m and T_y^n, defined as the Taylor expansion of $f(x,y)$ at $(0,y)$, $(x,0)$, respectively, of order m, respectively n. In other words,

$$T_x^m f := f(0,y) + \frac{1}{1!}\frac{\partial}{\partial x} f(0,y)x + \cdots + \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial x^{m-1}} f(0,y)x^{m-1}.$$

Lemma 2.1. *For any function $f \in C^{m,n}_{[0,1]^2}$, the following representation holds:*

$$\begin{aligned} f(x,y) &= T_x^m f(x,y) + T_y^n f(x,y) - T_x^m T_y^n f(x,y) \\ &+ \frac{1}{(m-1)!(n-1)!}\int_0^1\int_0^1 (x-t)_+^{m-1}(y-\tau)_+^{n-1}\frac{\partial^{m+n} f}{\partial t^m \partial \tau^n} f(t,\tau)\,dt\,d\tau. \end{aligned}$$

Proof. According to the Taylor's formula with integral representation of the reminder, we have

$$f(x,y) = T_x^m f(x,y) + \int_0^1 \frac{(x-t)_+^{m-1}}{(m-1)!}\frac{\partial^m}{\partial t^m} f(t,y)\,dt.$$

Applying again Taylor's formula to $f^{(m,0)}(t,y)$, this time with respect to y at $y=0$, we get

$$\frac{\partial^m}{\partial t^m} f(t,y) = T_y^n \frac{\partial^m}{\partial t^m} f(t,y) + \int_0^1 \frac{(y-v)_+^{n-1}}{(n-1)!}\frac{\partial^{m+n}}{\partial t^m \partial v^n} f(t,v)\,dv.$$

Inserting the last expression of $\frac{\partial^m}{\partial t^m} f(t,y)$ in the first equality and taking into account that, by the commutativity of the differentiation operator and T_n^y,

$$\int_0^1 \frac{(x-t)_+^{m-1}}{(m-1)!} T_y^n \frac{\partial^m}{\partial t^m} f(t,y)\,dt = T_y^n\Big[f(x,y) - T_x^m f(x,y)\Big],$$

we obtain the wanted equality.

Let us mention that $B_f(x,y) := T_x^m f(x,y) + T_y^n f(x,y) - T_x^m T_y^n f(x,y)$ is a blending function of order (m,n). Moreover, the restriction of B_f and its partial derivatives $B_f^{(i,j)}$ on the lines $x=0$ and $y=0$ coincide with the corresponding values of f and its derivatives there for $i=0,\ldots,m-1$, $j=0,\ldots,n-1$.

Therefore, in view of Lemma 2.1, any (m,n)-spline f can be represented as a sum of an appropriate blending function B_f of order (m,n) and a convolution of the kernel

$$K(x,y,t,\tau) := \frac{(x-t)_+^{m-1}}{(m-1)!}\frac{(y-\tau)_+^{n-1}}{(n-1)!}$$

with a piecewise constant function $c(t,\tau)$,

$$c(t,\tau) := c_{ij} \quad \text{for} \quad (t,\tau) \in \square_{ij}.$$

Next we introduce a class of (m,n)-splines with a final support, the so-called B-splines, which will be used as a basis in the space of (m,n)-splines. In order to do this, we consider an infinite rectangular net in the plane:

$$\{x_i = i/M,\ y_j = j/N,\ -\infty < i,j < \infty,\ i,j - integers\}.$$

As in the introduction, we denote

$$\Box_{ij} = \{(x,y) : x_{i-1} \le x < x_i, y_{j-1} \le y < y_j\}.$$

With any pair (i,j) of indices we associate the B-splines $B_{ij}^{(m,n)}$ of two variables of order (m,n) defined by

$$B_{ij}^{(m,n)}(x,y) := (\cdot - x)_+^n[x_i,\dots,x_{i+m+1}](\cdot - y)_+^n[y_j,\dots,y_{j+n+1}] = B_i^m(x)B_j^n(y).$$

For simplicity of the notations we shall often omit the upper indices m and n, when it is possible. Let us denote by D_{ij} the support of $B_{ij}(x,y)$. It is the Cartesian product of the supports of the univariate B-splines $B_i^{(m)}(x)$ and $B_j^{(n)}(y)$, namely, $D_{ij} = (x_i, x_{i+m+1}) \times (y_j, y_{j+n+1})$. Notice that in our notations the lower and left most rectangle, included in D_{ij}, is $\Box_{i+1,j+1}$.

Lemma 2.2. *For any finite set $I := (I_1, I_2) \subset Z \times Z$ of indices, the B-splines $B_{ij}^{(m,n)}(x,y)$, $(i,j) \in I$, are linearly independent in $\Re^2$.*

Proof. Assume the contrary. Then there exists a linear combination

$$g(x,y) := \sum_{(i,j)\in I} \alpha_{ij} B_{ij}(x,y)$$

with at least one non-zero coefficient α_{ij}, which vanishes identically on the plane $\Re^2$. We introduce the lexicografic order in I. Let (i_0, j_0) be the first member of I. If $t_{i_0j_0}$ belongs to the interior of $\Box_{i_0j_0} \subset D_{i_0j_0}$, we have $B_{i_0j_0}(t_{i_0j_0}) \neq 0$ and hence $\alpha_{i_0j_0} = 0$. Let $(\tilde{\imath}, \tilde{\jmath})$ be the next member of I. Quite analogously, we get $\alpha_{\tilde{\imath},\tilde{\jmath}} = 0$. Hence $\alpha_{ij} = 0$ for all $(i,j) \in I$. □

Lemma 2.3. *The functions $\{B_{ij}\}_{i=0\ j=0}^{M-1N-1}$ are linearly independent in $[0,1]^2$.*

The proof is similar to that of Lemma 2.2 and we omit it here.

Let us consider the subspace of (m,n)-splines

$$S_{m,n}^0 = \left\{ s \in S_{m,n} :\ \frac{\partial^i s}{\partial y^i}(0,y) = 0,\quad i = 0,\dots,m-1,\right.$$

$$\left.\frac{\partial^j s}{\partial x^j}(x,0) = 0,\quad j = 0,\dots,n-1 \right\}.$$

Corollary 2.1. *The B-splines $\{B_{ij}\}_{i=0\ j=0}^{M-1N-1}$ form a basis of $S_{m,n}^0$.*

Proof. We have proved that $\{B_{ij}\}_{i=0}^{M-1}{}_{j=0}^{N-1}$ are linearly independent. It is obvious that $B_{ij} \in S^0_{m,n}$ (since $B_{ij}(x,y) = B_i^{(m)}(x)B_j^{(n)}(y)$). Besides, $dim S^0_{m,n} = MN$=number of $B_{ij}(x,y)$.

3. THE INTERPOLATION THEOREM

The regularity of the interpolation problem by univariate splines is completely characterized by the *interlacing condition* of Schoenberg and Whitney [3], [4]. There is not yet such a characterization result in the multivariate case. In the next theorem we prove the regularity of the bivariate interpolation by (m,n)-splines for a quite general class of node configurations.

Theorem 3.1. *If*

$$\left\{ \begin{array}{l} x_{i-1} < \xi_{ij} \le x_i, \ i = 1, \ldots, M \\ y_{j-1} < \eta_{ij} \le y_j, \ j = 1, \ldots, N \end{array} \right\},$$

then the interpolation problem (1.1) *has a unique solution.*

Proof. There exists a unique blending function $b(x,y) \in B^{m,n}_{[0,1]^2}$ such that

$$\frac{\partial^i b}{\partial y^i}(0,y) = \frac{\partial f^i}{\partial y^i}(0,y), \quad i = 0, \ldots, m-1, \quad y \in [0,1],$$

$$\frac{\partial^j b}{\partial x^j}(x,0) = \frac{\partial^j f}{\partial x^j}(x,0), \quad j = 0, \ldots, n-1, \quad x \in [0,1].$$

Let us consider the values $\tilde{f}_{ij} = f(t_{ij}) - b(t_{ij})$. We claim that there exists a unique spline $s^0_{m,n} \in S^0_{m,n}$, which satisfies the interpolation conditions

$$s^0_{m,n}(t_{ij}) = \tilde{f}_{ij}.$$

Indeed, let us consider the corresponding homogeneous problem $s^0_{m,n}(t_{ij}) = 0, \ i = 1, \ldots, M, \ j = 1, \ldots, N$.

Lemma 2.3 gives a representation of $s^0_{m,n}$ in the form

$$s^0_{m,n} = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \alpha_{ij} B_{ij}.$$

Then $0 = s^0_{m,n}(t_{11}) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \alpha_{ij} B_{ij}(t_{11}) = \alpha_{00} B_{00}(t_{11})$. Since $t_{11} \in \Box_{11} \subset D_{00}$, we have $B_{00}(t_{11}) \ne 0$ and $\alpha_{00} = 0$.

Further, $0 = s^0_{m,n}(t_{12}) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \alpha_{ij} B_{ij}(t_{12}) = \alpha_{00} B_{00}(t_{12}) + \alpha_{01} B_{01}(t_{12}) = 0 + \alpha_{01} B_{01}(t_{12})$, hence $\alpha_{01} = 0$. We continue the process and finally get a diagonal matrix with $B_{i-1,j-1}(t_{ij})$ sitting in the diagonal. These numbers are different from zero since $t_{ij} \in \Box_{ij} \subset D_{i-1,j-1}$. Hence the homogeneous problem has only the trivial zero solution. This means that the non-homogenous problem has a unique solution. We assert that $s(x,y) = s^0_{m,n}(x,y) + b(x,y)$ is a solution of the original interpolation problem. Indeed,

$$\begin{aligned} s^{(m,n)}(x,y) &= \left(s^0_{(m,n)}\right)^{(m,n)}(x,y) + b^{(m,n)}(x,y) \\ &= \left(s^0_{(m,n)}\right)^{(m,n)}(x,y) + 0 = c_{i,j} \end{aligned}$$

when $(x,y) \in \square_{ij},\ i = 1,\dots,M,\ j = 1,\dots,N$. Hence $s(x,y) \in S_{m,n}$. Besides,

$$s(t_{ij}) = s^0_{m,n}(t_{ij}) + b(t_{ij}) = f(t_{ij}) - b(t_{ij}) + b(t_{ij}) = f(t_{ij})$$

for $i = 1,\dots,M,\ j = 1,\dots,N$. Obviously, $s(x,y)$ satisfies also the matching conditions along the segments $[0,y]$, $0 \le y \le 1$, and $[x,0]$, $0 \le x \le 1$, since $s^0_{(m,n)}(x,y)$ was chosen to satisfy the zero conditions.

Let us suppose that there are two solutions of the interpolation problem: $s_1(x,y)$ and $s_2(x,y)$. If $s(x,y) = s_1(x,y) - s_1(x,y)$, then s satisfies the zero boundary conditions (on the segments). Hence $s \in S^0_{m,n}$ and from the condition $s(t_{ij}) = 0,\ i = 1,\dots,M,\ j = 1,\dots,N$, we get that $s \equiv 0$. The uniqueness is proved.

4. PARTICULAR CASES

4.1. AN ESTIMATE OF THE ERROR IN THE CASE $(m,n) = (1,1)$ AND $t_{ij} = (i/M, j/N)$

We analyse further the interpolating spline in case of low orders m, n. Consider the rectangular net of points

$$x_i = i/M,\ y_j = j/N,\ i = 1,\dots,M,\ j = 1,\dots,N.$$

Let us denote

$$\begin{aligned} \Delta_{xy} &= f(x,y) + f(0,0) - f(x,0) - f(0,y), \\ \Delta_{ij} &= f(x_i,y_j) + f(x_{i-1},y_{j-1}) - f(x_i,y_{j-1}) - f(x_{i-1},y_j), \\ \Delta_1 &= f(x_{i-1},y) + f(0,y_{j-1}) - f(x_{i-1},y_{j-1}) - f(0,y), \\ \Delta_2 &= f(x,y_{j-1}) + f(x_{i-1},0) - f(x,0) - f(x_{i-1},y_{j-1}), \\ \Delta_3 &= f(x,y) + f(x_{i-1},y_{j-1}) - f(x,y_{j-1}) - f(x_{i-1},y). \end{aligned}$$

We shall approximate the function $f(x,y)$ by interpolating $(1,1)$-splines, that is, by functions of the form

$$s(f,x,y) = f(x,0) + f(0,y) - f(0,0) + \int_0^x \int_0^y c(u,v)\,du\,dv,$$

where

$$c(u,v) = \{c_{ij} \text{ } for \text{ } (u,v) \in \Box_{ij}, \ i = 1,\dots,M, \ j = 1,\dots,N\}$$

and the constants $\{c_{ij}\}_{i=1}^{M} {}_{j=1}^{N}$ are chosen to satisfy the interpolating conditions

$$s(f, x_i, y_j) = f(x_i, y_j), \quad i = 1,\dots,M, \quad j = 1,\dots,N.$$

We shall derive an expression for the error of approximation in terms of modulus of continuity $\omega(f, \delta_1, \delta_2)$. Recall that

$$\omega(f, \delta_1, \delta_2) = \sup_{|h_1| \le \delta_1} \sup_{|h_2| \le \delta_2} |f(x + h_1, y + h_2) - f(x, y)|.$$

In order to estimate the error, we need the values of $\{c_{ij}\}$, which we calculate below using the above-mentioned interpolatory conditions. By the first interpolatory condition we have

$$s(f, x_1, y_1) = f(x_1, 0) + f(0, y_1) - f(0,0) + \int_0^{x_1} \int_0^{y_1} c_{11} \, du \, dv \; = f(x_1, y_1),$$

which is easily reduced to $\Delta_{11} = hc_{11}$, where $h = \dfrac{1}{MN}$. Hence $c_{11} = \dfrac{\Delta_{11}}{h}$. From the interpolatory conditions at the point (x_1, y_2) we get

$$f(x_1, y_2) = f(x_1, 0) + f(0, y_2) - f(0,0) + \int_0^{x_1} \int_0^{y_1} c_{11} \, du \, dv + \int_0^{x_1} \int_{y_1}^{y_2} c_{12} \, du \, dv.$$

To find c_{12}, we use the above formula and the value of c_{11}, just found. We obtain $c_{12} = \dfrac{\Delta_{12}}{h}$. Similarly, we get that $c_{1j} = \dfrac{\Delta_{1j}}{h}$ for $j = 1,\dots,N$. We continue with the calculations of c_{21} up to c_{2N} and so on, till c_{MN}. In this way , we get that $c_{ij} = \dfrac{\Delta_{ij}}{h}$, $i = 1,\dots,M$, $j = 1,\dots,N$.

Now we are prepared to estimate the error. Let us suppose that the point (x, y) is in $\Box_{ij}$. Consider the identity

$$\begin{aligned} f(x,y) - s(f,x,y) &= f(x,y) + f(0,0) - f(x,0) - f(0,y) - \int_0^{x} \int_0^{y} c(u,v) \, du \, dv \\ &= \Delta_{xy} - \int_0^{x} \int_0^{y} c(u,v) \, du \, dv. \end{aligned}$$

Let us denote, respectively:

- by h_1 – the area of the rectangle with vertices (x_1, y), $(0, y_{j-1})$, (x_1, y_{j-1}) and $(0, y)$;

- by h_2 – the area of the rectangle with vertices (x, y_1), $(x_{i-1}, 0)$, $(x, 0)$ and (x_{i-1}, y_1);

- by h_3 – the area of the rectangle with vertices (x, y), $(x_{i-1},\ y_{j-1})$, (x, y_{j-1}) and (x_{i-1}, y).

Then

$$\int_0^x \int_0^y c(u,v)\,du\,dv = h\sum_{k=1}^{i-1}\sum_{l=1}^{j-1} c_{kl} + h_1 \sum_{k=1}^{i-1} c_{kj} + h_2 \sum_{l=1}^{j-1} c_{il} + h_3 c_{ij}$$
$$= \sum_{k=1}^{i-1}\sum_{l=1}^{j-1} \Delta_{kl} + \frac{h_1}{h}\sum_{k=1}^{i-1}\Delta_{kj} + \frac{h_2}{h}\sum_{l=1}^{j-1}\Delta_{il} + \frac{h_3}{h}\Delta_{ij}.$$

Hence

$$f(x,y) - s(f,x,y) = \Delta_1 + \Delta_2 + \Delta_3 - \frac{h_1}{h}\sum_{k=1}^{i-1}\Delta_{kj} - \frac{h_2}{h}\sum_{l=1}^{j-1}\Delta_{il} - \frac{h_3}{h}\Delta_{ij}$$
$$= -\frac{h_1}{h}\left[f(x_{i-1},y_j) + f(0,y) - f(x_{i-1},y) - f(0,y_j)\right] + \left(1 - \frac{h_1}{h}\right)\Delta_1$$
$$-\frac{h_2}{h}\left[f(x_i,y_{j-1}) + f(x,0) - f(x_i,0) - f(x,y_{j-1})\right] + \left(1 - \frac{h_2}{h}\right)\Delta_2 + \Delta_3 - \frac{h_3}{h}\Delta_{ij}.$$

Using the properties of the modulus of continuity, we obtain the estimate

$$|f(x,y) - s(f,x,y)| \le 2\left[\frac{h_1}{h}\omega_{1,1}\left(f,1,\frac{1}{N}\right) + \left(1-\frac{h_1}{h}\right)\omega_{1,1}\left(f,1,\frac{1}{N}\right)\right.$$
$$\left. + \frac{h_2}{h}\omega_{1,1}\left(f,\frac{1}{M},1\right) + \left(1-\frac{h_2}{h}\right)\omega_{1,1}\left(f,\frac{1}{M},1\right) + 2\omega_{1,1}\left(f,\frac{1}{M},\frac{1}{N}\right)\right].$$

In this simple case we can give explicitely the Lagrangian basis for the interpolation problem

$$S^0_{1,1} = \{ \quad s: \quad \frac{\partial^2 s}{\partial x^1 \partial y^1}(x,y) = c_{ij},\ (x,y) \in \Box_{ij},$$
$$i = 1,\ldots,M,\ j = 1,\ldots,N,\ s(0,y) = 0,\ s(x,0) = 0\}.$$

More precisely , we construct functions $\delta_{pq}(x,y) \in S^0_{1,1}$ such that

$$\delta_{pq}\left(\frac{k}{M},\frac{l}{N}\right) = \delta_{pk}\delta_{ql}\ ,\ k = 1,\ldots,M,\ l = 1,\ldots,N.$$

(Obviously, $\{\delta_{pq}\}_{p=1}^{M}{}_{q=1}^{N}$ are linearly independent and since their number is $M.N$, they indeed form a basis of $S^0_{1,1}$.) We seek $\delta_{pq}(x,y)$ in the form

$$\delta_{pq}(x,y) = \int_0^1 \int_0^1 (x-t)^0_+ (y-v)^0_+ c(t,v)\,dt\,dv,$$

where $c(t,v) = c_{ij}$ for $(t,v) \in \Box_{ij}, \quad i = 1,\ldots,M,\ j = 1,\ldots,N.$

We derive that

$$\delta_{pq}(x,y)=\begin{cases} MN\left(x-\frac{p-1}{M}\right)\left(y-\frac{q-1}{N}\right)\ ,\ (x,y)\in\Box_{pq}, \\ MN\left(\frac{p+1}{M}-x\right)\left(y-\frac{q-1}{N}\right)\ ,\ (x,y)\in\Box_{p+1,q}, \\ MN\left(x-\frac{p-1}{M}\right)\left(\frac{q+1}{N}-y\right),\ (x,y)\in\Box_{p,q+1}, \\ MN\left(\frac{p+1}{M}-x\right)\left(\frac{q+1}{N}-y\right),\ (x,y)\in\Box_{p+1,q+1}, \\ 0,\ \text{elsewhere.} \end{cases}$$

The calculation of δ_{pq} is straightforward and we omit it here. Instead, we show how to compute the basic functions in the more complicated case $(m,n)=(1,2)$.

Having computed δ_{pq}, one can give explicitely the solution of the interpolation problem as

$$s(f,x,y)=f(0,y)+f(x,0)-f(0,0)+\sum_{p=1}^{M}\sum_{q=1}^{N}f(t_{pq})\delta_{pq}(x,y).$$

4.2. THE CASE $(m,n)=(1,2)$ AND $t_{ij}=(i/M,j/N)$

Here we construct the Lagrangian basic functions $\delta_{pq}(x,y)\in S^0_{1,2}$. By definition, they satisfy the conditions

$$\delta_{pq}\left(\frac{k}{M},\frac{l}{N}\right)=\delta_{pk}\delta_{ql}\ ,\ k=1,\dots,M,\ l=1,\dots,N. \tag{4.1}$$

We seek $\delta_{pq}(x,y)$ in the form

$$\delta_{pq}(x,y)=\int_0^1\int_0^1 (x-t)^0_+(y-v)^1_+c(t,v)\,dt\,dv,$$

where $c(t,v)=c_{ij}$ for $(t,v)\in\Box_{ij},\quad i=1,\dots,M,\ j=1,\dots,N.$

Our next purpose is to calculate the constants c_{ij}. We will determine them using the interpolatory conditions (4.1).

Assume first that $p>1$, $q>1$. Using the condition $0=\delta_{pq}\left(\frac{1}{M},\frac{1}{N}\right)$, we get

$$\begin{aligned} 0 &= \int_0^1\int_0^1\left(\frac{1}{M}-t\right)^0_+\left(\frac{1}{N}-v\right)^1_+c(t,v)\,dt\,dv \\ &= c_{11}\int_0^{1/M}\left(\frac{1}{M}-t\right)^0dt\int_0^{1/N}\left(\frac{1}{N}-v\right)dv=c_{11}\frac{1}{2MN^2}, \end{aligned}$$

i.e., $c_{11} = 0$. Moreover, the condition $\delta_{pq}\left(\frac{1}{M}, \frac{l}{N}\right) = 0$ gives

$$\begin{aligned} 0 &= \int_0^{1/M} \left(\frac{1}{M} - t\right)^0 dt \int_0^{1/N} \left(\frac{l}{N} - v\right) dv c_{11} \\ &+ \int_0^{1/M} \left(\frac{1}{M} - t\right)^0 dt \int_{1/N}^{2/N} \left(\frac{l}{N} - v\right) dv c_{12} + \cdots \\ &+ \int_0^{1/M} \left(\frac{1}{M} - t\right)^0 dt \int_{(l-1)/N}^{l/N} \left(\frac{l}{N} - v\right) dv c_{1l}. \end{aligned}$$

For $l = 2$ we get $0 = \alpha_1.0 + \alpha_2.c_{12}$ and hence $c_{12} = 0$. Analogously, we find that $c_{13} = 0, \dots, c_{1l} = 0$ for $l = 1, \dots, N$. If $p > 2$, in the same manner we show that $c_{2l} = 0$ for $l = 1, \dots, N$. Moreover , for all $k < p$, we have $c_{kl} = 0$ for $l = 1, \dots, N$. Quite analogously, we get the same for all $l < q$ and $k = 1, \dots, M$. In the case $p = 1$ (or $q = 1$) these null columns (rows) are missing.

Assume that $p = 1, \dots, M, \quad q = 1, \dots, N$. From the equality $1 = \delta_{pq}\left(\frac{p}{M}, \frac{q}{N}\right)$ we get

$$1 = c_{pq} \int_{(p-1)/M}^{p/M} \left(\frac{p}{M} - t\right)^0 dt \int_{(q-1)/N}^{q/N} \left(\frac{q}{N} - v\right)^1 dv = c_{pq} \frac{1}{M} \frac{1}{2N^2},$$

i.e., $c_{pq} = 2MN^2$. Let now $k \geq p$ and $l \geq q$, $(k-p)^2 + (l-q)^2 \neq 0$. Then

$$\begin{aligned} 0 &= \delta_{pq}\left(\frac{k}{M}, \frac{l}{N}\right) \\ &= \sum_{j=q}^{l} \sum_{i=p}^{k} c_{ij} \int_{(i-1)/M}^{i/M} \left(\frac{k}{M} - t\right)^0 dt \int_{(j-1)/N}^{j/N} \left(\frac{l}{N} - v\right)^1 dv \\ &= \sum_{j=q}^{l} \sum_{i=p}^{k} c_{ij} \frac{1}{M} \frac{(2l - 2j + 1)}{2N^2}. \end{aligned}$$

We get

$$c_{kl} = -\left[\sum_{j=q}^{l-1} \sum_{i=p}^{k} c_{ij}(2l - 2j + 1) + \sum_{i=p}^{k-1} c_{il}\right]. \tag{4.2}$$

If the upper index is less than the lower one in any of the sums, we interprete this sum as equal to zero. For $l = q, \ k = p + 1, \dots, M$, we have

$$c_{kq} = -\sum_{i=p}^{k-1} c_{iq}. \tag{4.3}$$

For $k = p, \ l = q + 1, \dots, N$,

$$c_{pl} = -\sum_{j=q}^{l-1} c_{pj}(2l - 2j + 1). \tag{4.4}$$

We put in (4.3) $k = p+1$ and get $c_{p+1,q} = -c_{pq} = -2MN^2$. For $k = p+2$: $c_{p+2,q} = -(c_{pq} + c_{p+1,q}) = 0$, for $k = p+3$: $c_{p+3,q} = -(c_{pq} + c_{p+1,q} + c_{p+2,q}) = 0$, and so on, $c_{kp} = 0$ for $k = p+2, \ldots, M$.

From (4.4) for $l = q+1$ we have that $c_{p,q+1} = -3c_{pq} = -6MN^2$. It can be shown by induction that $c_{pl} = (-1)^{l-q}4c_{pq} = (-1)^{l-q}8MN^2$ for $l = q+2, \ldots, N$. Indeed, from (4.4) for $l = q+2$,

$$c_{p,q+2} = -[5c_{pq} + 3(-3c_{pq})] = -(1)^{q+2-q}4c_{pq}.$$

Let us suppose that the assertion holds for every natural number from $(q+2)$ till $(l-1)$. Then we shall prove that it is true also for $l = q+2n$. The case $l = q+2n+1$ holds analogously. To this purpose, we use (4.4):

$$\begin{aligned} c_{pl} = c_{p,q+2n} &= -[(4n+1)c_{pq} + (4n-1)c_{p,q+1} + (4n-3)c_{p,q+2} + \cdots + 3c_{p,l-1}] \\ &= -c_{pq}[-3.4 + 5.4 - 7.4 + \cdots + (4n-3).4 + (4n-1).3 + (4n+1)] \\ &= 4c_{pq} = -(1)^{l-q}4c_{pq} = -(1)^{l-q}8MN^2. \end{aligned}$$

By induction (on l) we shall show that $c_{pl} = -c_{p+1,l}$, $c_{kl} = 0$ for $k = p+2, \ldots, M$ and $l = q, \ldots, N$. The assertion holds for $l = q$. Let it hold for every natural number from q till $(l-1)$. Then, by (4.2),

$$c_{kl} = \left[\sum_{j=q}^{l-1} (c_{pj} + c_{p+1,j})(2l-2j+1) + \sum_{j=q}^{l-1}\sum_{i=p+2}^{k-1} c_{ij}(2l-2j+1) + \sum_{i=p}^{k-1} c_{il}\right] = -\sum_{i=p}^{k-1} c_{il},$$

i.e., we get a relation similar to (4.3). Putting in it $k = p+1$, we get $c_{p+1,l} = -c_{pl}$. The substitution $k = p+2$ gives $c_{p+2,l} = 0$ and so on , we get $c_{Ml} = 0$. The assertion is proved.

So we have calculated $\{c_{ij}\}_{i=1}^{M}{}_{j=1}^{N}$ for $\delta_{pq}(x,y)$.

Now we are ready to give the explicite form of $\delta_{pq}(x,y)$. Let $(x,y) \in \square_{kl}$. Having in mind that $c(t,v) = c_{ij}$ for $(x,y) \in \square_{ij}$ and $c_{ij} = 0$ for some (i,j), we get

$$\begin{aligned} \delta_{pq}(x,y) &= \sum_{j=q}^{l-1}\sum_{i=p}^{k-1} c_{ij} \int_{(i-1)/M}^{i/M} (x-t)^0\, dt \int_{(j-1)/N}^{j/N} (y-v)^1\, dv \\ &+ \sum_{j=q}^{l-1} c_{kj} \int_{(k-1)/M}^{x} (x-t)^0\, dt \int_{(j-1)/N}^{j/N} (y-v)^1\, dv \\ &+ \sum_{i=p}^{k-1} c_{il} \int_{(i-1)/M}^{i/M} (x-t)^0\, dt \int_{(l-1)/N}^{y} (y-v)^1\, dv \end{aligned}$$

$$
\begin{aligned}
&+ \quad c_{kl} \int_{(k-1)/M}^{x} (x-t)^0\, dt \int_{(l-1)/N}^{y} (y-v)^1\, dv \\
&= \quad \sum_{j=q}^{l-1} \sum_{i=p}^{k-1} c_{ij} + \frac{1}{MN}\left(y - \frac{2j-1}{2N}\right) \\
&+ \quad \sum_{j=q}^{l-1} c_{kj} \frac{1}{N}\left(x - \frac{k-1}{M}\right)\left(y - \frac{2j-1}{2N}\right) \\
&+ \quad \sum_{i=p}^{k-1} c_{il} \frac{1}{2M}\left(y - \frac{l-1}{N}\right)^2 + c_{kl}\left(x - \frac{k-1}{M}\right)\frac{1}{2}\left(y - \frac{l-1}{N}\right)^2.
\end{aligned}
$$

We will consider various cases for k and l :

1. $(x,y) \in \square_{pq}$, i.e., $k = p,\ l = q$:

$$
\delta_{pq}(x,y) = c_{pq}\left(x - \frac{p-1}{M}\right)\frac{1}{2}\left(y - \frac{q-1}{N}\right)^2 = MN^2\left(x - \frac{p-1}{M}\right)\left(y - \frac{q-1}{N}\right)^2.
$$

2. $(x,y) \in \square_{p,q+1}$, i.e., $k = p,\ l = q+1$:

$$
\begin{aligned}
\delta_{pq}(x,y) &= c_{pq}\left(x - \frac{p-1}{M}\right)\frac{1}{N}\left(y - \frac{2q-1}{2N}\right) + c_{p,q+1}\left(x - \frac{p-1}{M}\right)\frac{1}{2}\left(y - \frac{q}{N}\right)^2 \\
&= 3MN^2\left(x - \frac{p-1}{M}\right)\left(\frac{q+1}{N} - y\right)\left(y - \frac{q}{N} + \frac{1}{3N}\right).
\end{aligned}
$$

3. $(x,y) \in \square_{p+1,q}$:

$$
\begin{aligned}
\delta_{pq}(x,y) &= c_{pq}\frac{1}{2M}\left(y - \frac{q-1}{N}\right)^2 + c_{p+1,q}\left(x - \frac{p}{M}\right)\frac{1}{2}\left(y - \frac{q-1}{N}\right)^2 \\
&= MN^2\left(y - \frac{q-1}{N}\right)^2\left(\frac{p+1}{M} - x\right).
\end{aligned}
$$

4. $(x,y) \in \square_{p+1,q+1}$:

$$
\begin{aligned}
\delta_{pq}(x,y) &= c_{pq}\frac{1}{MN}\left(y - \frac{2q-1}{2N}\right) + c_{p+1,q}\left(x - \frac{p}{M}\right)\frac{1}{N}\left(y - \frac{2q-1}{2N}\right) \\
&+ c_{p,q+1}\frac{1}{2M}\left(y - \frac{q}{N}\right)^2 + c_{p+1,q+1}\left(x - \frac{p}{M}\right)\frac{1}{2}\left(y - \frac{q}{N}\right)^2 \\
&= 3MN^2\left(\frac{p+1}{M} - x\right)\left(\frac{q+1}{N} - y\right)\left(y - \frac{q}{N} + \frac{1}{3N}\right).
\end{aligned}
$$

5. $(x,y) \in \square_{k,l}$, for $k \le p-1$ or $l \le q-1$: Let, for instance, $k \le p-1$. Then

$$
\delta_{pq}(x,y) \quad = \quad \sum_{j=q}^{l-1} c_{kj}\left(x - \frac{k-1}{M}\right)\frac{1}{N}\left(y - \frac{2j-1}{2N}\right)
$$

$$+ \quad c_{kl}\left(x-\frac{k-1}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2=0,$$

since $c_{kj}=0$ for $j=q,\ldots,l$.

6. $(x,y)\in\square_{p,l}$, for $l\geq q+2$:

$$\delta_{pq}(x,y)=\sum_{j=q}^{l-1}c_{pj}\left(x-\frac{p-1}{M}\right)\frac{1}{N}\left(y-\frac{2j-1}{2N}\right)+c_{pl}\left(x-\frac{p-1}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2$$

$$=\left(x-\frac{p-1}{M}\right)\left[\sum_{j=q}^{l-1}c_{pj}\frac{1}{N}\left(y-\frac{2j-1}{2N}\right)c_{pl}\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2\right],$$

since $c_{pq}=2MN^2$, $c_{p,q+1}=-6MN^2$, $c_{p,q+t}=(-1)^t 8MN^2$, the second multiplier A is equal to

$$2MN\left[y-\frac{2q-1}{2N}-3\left(y-\frac{2q+1}{2N}\right)+4B\right]+4MN^2(-1)^{l-q}\left(y-\frac{l-1}{N}\right)^2,$$

where

$$B=y-\frac{2(q+2)-1}{2N}-y+\frac{2(q+3)-1}{2N}+\cdots+(-1)^{l-q-1}\left(y-\frac{2(l-1)-1}{2N}\right),$$

i.e.,

$$A=4MN\left[\left(\frac{q+1}{N}-y\right)+2B\right].$$

We will calculate B first for the case of even summands, i.e., when $l-1-(q+2)+1=l-q-2$ is even. Then $(l-q)$ is even and

$$B=\frac{2}{2N}[-(q+2)+(q+3)-(q+4)+\cdots-(l-2)+(l-1)]=\frac{l-q-2}{2N}.$$

If $(l-q)$ is odd, then

$$B=\left[\frac{l-1-q-2}{2N}+y-\frac{2(l-1)-1}{2N}\right]=\left[y-\frac{l+q}{2N}\right],$$

and thus

$$\delta_{pq}=(-1)^{l-q+1}\left(x-\frac{p-1}{M}\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)4MN^2.$$

7. $(x,y)\in\square_{p+1,l}$, for $l\geq q+2$:

$$\delta_{pq}(x,y)=\sum_{j=q}^{l-1}c_{pj}\frac{1}{MN}\left(y-\frac{2j-1}{2N}\right)+\sum_{j=q}^{l-1}c_{p+1,j}\frac{1}{N}\left(x-\frac{p}{M}\right)\left(y-\frac{2j-1}{2N}\right)$$

$$+c_{pl}\frac{1}{2M}\left(y-\frac{l-1}{N}\right)^2+c_{p+1,l}\left(x-\frac{p}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2.$$

Since $c_{p+1,j}=-c_{pj}$ for $j=1,\ldots,N$, then

$$\delta_{pq}(x,y)=\left(\frac{p+1}{M}-x\right)\left[\sum_{j=q}^{l-1}c_{pj}\frac{1}{N}\left(y-\frac{2j-1}{2N}\right)+c_{pl}\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2\right]$$

and as in item **6.**,

$$\delta_{pq}(x,y)=(-1)^{l-q+1}4MN^2\left(\frac{p+1}{M}-x\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right).$$

8. $(x,y)\in\square_{k,l}$, for $k\geq p+2$ and $l\geq q$: We represent δ_{pq} in the form

$$\sum_{j=q}^{l-1}(c_{pj}+c_{p+1,j})\frac{1}{MN}\left(y-\frac{2j-1}{2N}\right)+\sum_{j=q}^{l-1}\sum_{i=p+2}^{k-1}c_{ij}\frac{1}{MN}\left(y-\frac{2j-1}{2N}\right)$$

$$+\left[(c_{pj}+c_{p+1,j})+\sum_{i=p+2}^{k-1}c_{il}\right]\frac{1}{2M}\left(y-\frac{l-1}{N}\right)^2+c_{kl}\left(x-\frac{k-1}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2$$

and conclude that $\delta_{pq}(x,y)=0$ in this case. Therefore, we arrive at the following expressions for $\delta_{pq}(x,y)$:

$$MN^2\left(x-\frac{p-1}{M}\right)\left(y-\frac{q-1}{N}\right)^2 \text{ for } (x,y)\in\square_{pq};$$

$$MN^2\left(\frac{p+1}{M}-x\right)\left(y-\frac{q-1}{N}\right)^2 \text{ for } (x,y)\in\square_{p+1,q};$$

$$3MN^2\left(x-\frac{p-1}{M}\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right) \text{ for } (x,y)\in\square_{p,q+1};$$

$$3MN^2\left(\frac{p+1}{M}-x\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right) \text{ for } (x,y)\in\square_{p+1,q+1};$$

$$(-1)^{l-q+1}4MN^2\left(x-\frac{p-1}{M}\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

for $(x,y)\in\square_{pl},\quad N\geq l\geq q+2;$

$$(-1)^{l-q+1}4MN^2\left(\frac{p+1}{M}-x\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

for $(x,y)\in\square_{p+1,q},\quad N\geq l\geq q+2;$

0, elsewhere.

This is also true for $p = M$ and $q = N$, but in these cases we consider only such indices that are less than or equal to M or N, respectively.

Consider the operator $I_{mn}[f]$, which puts in correspondence to a function f its interpolating (m,n)-spline at a fixed set of nodes $\{t_{pq}\}$. In the $(1,2)$-case we have constructed the Lagrangian basis and thus the interpolating spline $I_{1,2}[f]$ can be represented in the form

$$I_{1,2}[f](x,y) = \sum_{p=1}^{M}\sum_{q=1}^{N} \delta_{pq} f(t_{pq}).$$

Thus, for the norm $\|I_{mn}\|$ of this operator in the space $C(G)^\circ$ of continuous functions bounded by 1 in the unit square G, we get

$$\|I_{mn}\| = \sup_{f\in C(G)^\circ} \|\sum_{p=1}^{M}\sum_{q=1}^{N} \delta_{pq} f(t_{pq})\| \le \sum_{p=1}^{M}\sum_{q=1}^{N} \|\delta_{pq}(x,y)\|.$$

Bounds of the norm $\|I_{mn}\|$ are useful for estimating the error of approximation. That is why we give below such estimates in the case of the most frequently used norms. For simplicity of notation, we will omit the indices mn of I_{mn} and also we will write $\sum$ instead of $\sum_{p=1}^{M}\sum_{q=1}^{N}$.

For every spline s we have

$$\begin{aligned} \|f - If\|_X &= \|f - s + Is - If\|_X \le \|f - s\|_X + \|I\|_{L_\infty \to X}\|f - s\|_{L_\infty} \\ &\le \|f - s\|_{L_\infty}(1 + \|I\|_{L_\infty\to X}). \end{aligned}$$

In the case $X = L_1$ we get the following bounds:

$$\|\delta_{pq}\|_{L_1} = \frac{2}{3MN}(N - q + 1) \quad \text{for } 1 \le p < M,\ 1 \le q < N,$$

$$\|\delta_{Mq}\|_{L_1} = \frac{N - q + 1}{3MN} \quad \text{for } 1 \le p < M,$$

$$\|\delta_{pN}\|_{L_1} = \frac{1}{3MN} \quad \text{for } 1 \le p < M,$$

$$\|\delta_{MN}\|_{L_1} = \frac{1}{6MN}.$$

Therefore,

$$\sum_{p=1}^{M}\sum_{q=1}^{N} \|\delta_{pq}\|_{L_1} = \frac{N}{3}\left(1 - \frac{1}{2M}\right)\left(1 + \frac{1}{N} - \frac{1}{N^2}\right) \sim \frac{N}{3},$$

when M and N tend to infinity.

Hence

$$\|f - If\|_{L_1} \le \left[\frac{N}{3}\left(1 - \frac{1}{2M}\right)\left(1 + \frac{1}{N} - \frac{1}{N^2}\right) + 1\right] E_f^\infty,$$

where E_f^∞ is the best L_∞-approximation of f by $(1,2)$-splines.

In the case $X = L_2$ we calculate $\|\delta_{pq}\|_{L_2}$. For $1 \le p < M,\ 1 \le q < N$ we get

$$\|\delta_{pq}\|_{L_2}^2 = \frac{16}{45}\left(N - q + \frac{3}{2}\right).$$

Besides, for $1 \le q < N$,

$$\|\delta_{Mq}\|_{L_2}^2 = \frac{8}{45MN}\left(N - q + \frac{3}{2}\right),$$

for $1 \le p < M$,

$$\|\delta_{pN}\|_{L_2}^2 = \frac{2}{45MN} \qquad \text{and} \qquad \|\delta_{MN}\|_{L_2}^2 = \frac{1}{15MN}.$$

Then

$$\sum \|\delta_{pq}\|_{L_2}^2 = \frac{(N+2)(2-1/M)}{45} \sim \frac{2N}{45}.$$

Using the inequality $x_1 + \cdots + x_n \le \sqrt{n(x_1^2 + \cdots + x_n^2)}$, we find the estimate

$$\begin{aligned}\sum \|\delta_{pq}\|_{L_2} &\le \sqrt{MN\sum \|\delta_{pq}\|_{L_2}^2} = \sqrt{(M-1/2)(N+2)N}\sqrt{\frac{2}{45}} \\ &\sim \sqrt{M}N\sqrt{\frac{2}{45}} \quad (\text{as } M, N \to \infty).\end{aligned}$$

Hence

$$\|I\|_{L_2} \le \sum \|\delta_{pq}\|_{L_2} \le \sqrt{(M-1/2)(N+2)N}\sqrt{\frac{2}{45}} \sim \sqrt{M}N\sqrt{\frac{2}{45}},$$

$$\|f - If\|_{L_2} \le \left(\sqrt{(M-1/2)(N+2)N}\sqrt{\frac{2}{45}} + 1\right) E_f^\infty.$$

Let $X = L_\infty$. Using that $\|\delta_{pq}\|_{L_\infty} = 1$ we get $\|If\|_{L_\infty} \le \sum 1.1 = MN$. Hence

$$\|I\|_{L_\infty \to L_\infty} \le MN$$

and

$$\|f - If\|_{L_\infty} \le (MN+1)\, E_f^\infty.$$

We are going to use the estimate for $\sum \|\delta_{pq}\|_{L_2}^2$ to get a better estimate for $\|f - If\|_{L_1}$. Let $\chi_{pq}(x,y)$ be the characteristic function of the support of $\delta_{pq}(x,y)$. Then

$$\begin{aligned}\|If\|_{L_1} &= \|\sum \delta_{pq}\chi_{pq} f(t_{pq})\|_{L_1} \le \sum \|\delta_{pq}\chi_{pq} f(t_{pq})\|_{L_1} \\ &\le \sum \|\delta_{pq}\|_{L_2}\|\chi_{pq}\|_{L_2} \le \left(\sum \|\delta_{pq}\|_{L_2}^2\right)^{1/2}\left(\sum \|\chi_{pq}\|_{L_2}^2\right)^{1/2}.\end{aligned}$$

Since

$$\|\chi_{pq}\|_{L_2}^2 = \frac{2}{M}\left(1 - \frac{q-1}{N}\right)$$

for $1 \le p < M,\ 1 \le q \le N$ and

$$\|\chi_{Mq}\|_{L_2}^2 = \frac{1}{M}\left(1 - \frac{q-1}{N}\right),$$

we get

$$\left(\sum \|\chi_{pq}\|_{L_2}^2\right)^{1/2} = \sqrt{\left(2 - \frac{1}{M}\right)\frac{N+1}{2}}.$$

Then

$$\|I\|_{L_\infty \to L_1} = \sup_{\|f\|_{L_\infty} \le 1} \|If\|_{L_1} \le \frac{(2 - 1/M)\sqrt{(N+1)(N+2)}}{3\sqrt{10}} \sim \frac{2}{3\sqrt{10}} N$$

when M and N tend to infinity. Hence

$$\|f - If\|_{L_1} \le \left(\frac{(2 - 1/M)\sqrt{(N+1)(N+2)}}{3\sqrt{10}} + 1\right) E_f^\infty.$$

4.3. THE CASE $(m,n) = (2,2)$ AND $t_{ij} = (i/M, j/N)$

We seek $\delta_{pq}(x,y)$ of the form

$$\delta_{pq}(x,y) = \int_0^1 \int_0^1 (x-t)_+^1 (y-v)_+^1 c(t,v)\, dt\, dv,$$

where $c(t,v) = c_{ij}$ for $(t,v) \in \square_{ij},\ \ i = 1,\dots,M,\ j = 1,\dots,N$. The constants c_{ij} are determined by the interpolatory conditions

$$\delta_{pq}(t_{ij}) = \delta_{pi}\delta_{qj}, \quad i = 1,\dots,M,\ j = 1,\dots,N.$$

As in the previous section 4.2, we get

$$c_{pq} = 4M^2N^2,\ c_{p+1,q} = -3c_{pq},\ c_{kq} = (-1)^{k-p} 4c_{pq} \quad \text{for } k \ge p+2.$$

Moreover,

$$c_{k,q+1} = -3c_{kq} \text{ and } c_{kl} = (-1)^{l-q}4c_{kq} \text{ for } k \geq p+2.$$

Using the above expressions for c_{ij}, one can obtain that $\delta_{pq}(x,y)$ is equal to

$$M^2N^2\left(x-\frac{p-1}{M}\right)^2\left(y-\frac{q-1}{N}\right)^2 \text{ if } (x,y)\in\square_{pq};$$

$$3M^2N^2\left(\frac{p+1}{M}-x\right)\left(x-\frac{p}{M}+\frac{1}{3M}\right)\left(y-\frac{q-1}{N}\right)^2 \text{ if } (x,y)\in\square_{p+1,q};$$

$$3M^2N^2\left(x-\frac{p-1}{M}\right)^2\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right) \text{ if } (x,y)\in\square_{p,q+1};$$

$$9M^2N^2\left(\frac{p+1}{M}-x\right)\left(x-\frac{p}{M}+\frac{1}{3M}\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right)$$

if $(x,y)\in\square_{p+1,q+1}$;

$$(-1)^{l-q+1}4M^2N^2\left(x-\frac{p-1}{M}\right)^2\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

if $(x,y)\in\square_{pl},\quad q+2\leq l\leq N$;

$$(-1)^{l-q+1}12M^2N^2\left(\frac{p+1}{M}-x\right)\left(x-\frac{p}{M}+\frac{1}{3M}\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

if $(x,y)\in\square_{p+1,l},\ q+2\leq l\leq N$;

$$(-1)^{k-p+1}4M^2N^2\left(\frac{k}{M}-x\right)\left(x-\frac{k-1}{M}\right)\left(y-\frac{q-1}{N}\right)^2$$

if $(x,y)\in\square_{kq},\ p+2\leq k\leq M$;

$$(-1)^{k-p+1}12M^2N^2\left(x-\frac{k-1}{M}\right)\left(\frac{k}{M}-x\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right)$$

if $(x,y)\in\square_{k,q+1},\ p+2\leq k\leq M$;

$$(-1)^{l-q+k-p}16M^2N^2\left(x-\frac{k-1}{M}\right)\left(\frac{k}{M}-x\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

if $(x,y)\in\square_{k,l},\ p+2\leq k\leq M,\ q+2\leq l\leq N$.

Some technical calculations show that

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_1} = \frac{1}{9MN}\left(M^2+M-1\right)\left(N^2+N-1\right) \sim \frac{MN}{9}$$

when M and N tend to infinity.

Besides,

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_2}^2 = \frac{64}{225MN}\left(M^2+2M-\frac{21}{8}\right)\left(N^2+2N-\frac{21}{8}\right).$$

Hence

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_2} \le \frac{8}{15}\sqrt{\left(M^2+2M-\frac{21}{8}\right)\left(N^2+2N-\frac{21}{8}\right)} \sim \frac{8}{15}MN.$$

It is easy to see that

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_\infty} = MN.$$

The same way as in Section 4.2, one can get

$$\|f-If\|_{L_1} \le \left[\frac{1}{9MN}\left(M^2+M-1\right)\left(N^2+N-1\right)+1\right]E_f^\infty,$$

where E_f^∞ is the best L_∞ approximation of f with $(2,2)$-splines,

$$\|f-If\|_{L_2} \le \left[\frac{8}{15}\sqrt{\left(M^2+2M-\frac{21}{8}\right)\left(N^2+2N-\frac{21}{8}\right)}+1\right]E_f^\infty$$

Acknowledgments. The author is grateful to Peter Petrov for his help in the proof of the error estimate in Section 4.1.

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Received on November 28, 2001

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