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GALERKIN SPECTRAL METHOD FOR HIGHER-ORDER BOUNDARY VALUE PROBLEMS ARISING IN THERMAL CONVECTION

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In the present work we develop a Galerkin spectral technique for solving coupled higher-order boundary value problems arising in continuum mechanics. The set of the so-called beam functions are used as a basis together with the harmonic functions. As featuring examples we solve two fourth-order boundary value problems related to the convective flow of viscous liquid in a vertical slot and a coupled convective problem. We show that the rate of convergence of the series is fifth-order algebraic both for linear and nonlinear problems of fourth order. The coupled problem exhibits fourth- and fifth-order convergence for the different unknown functions. Though algebraic, the fourth order rate of convergence is fully adequate for the generic problems under consideration, which makes the new technique a useful tool in numerical approaches to convective problems.

Keywords: spectral methods, beam functions, natural convection

MSC 2000: 37L65, 74S25, 76M22, 76E06, 76R10

1. INTRODUCTION

Fourth-order boundary value problems are the standard model in continuum mechanics arising both in elasticity and in viscous liquid dynamics. The simplified 1D models are respectively the beam equations and Poiseuille flow. The method developed here can be applied to both elasticity and fluid dynamics. For the sake of definiteness we will focus our attention on thermal convection in a vertical slot, which is a generalization of the Poiseuille flow.

There is a compelling need to develop fast spectral methods that will lead to more efficient algorithms. Such algorithms would allow a rapid interrogation of parameter space in order to discover and understand mechanisms of flow and instability. The performance of a spectral method depends heavily on the type of the basis system. Naturally, a basis system of functions which does not satisfy all of the boundary conditions, such as Fourier functions, would exhibit very poor convergence near the boundaries, where the solution is supposed to satisfy four boundary conditions. An elucidating discussion on the performance of different set of functions can be found in the encyclopedic book of Boyd [2]. In the present work we embark on developing spectral techniques involving the so-called *beam functions* introduced first by Lord Rayleigh, see [10]. Along these lines we will investigate also in a future work the performance of Galerkin techniques with a basis derived from Chebyshev polynomials — something that goes, however, beyond the scope of the present work.

The application of the beam-Galerkin method to Poiseuille flow is at present well developed, see [9, 4]. We go a step further here and consider the generic boundary value problem for convective flows of viscous liquids. These are rather complex ones, hence geometrically simplified situations are considered in order to identify the physical mechanisms, e.g. straight ducts and/or slots. These mechanisms are often operative in more complicated situations. Even for the simplest geometries with plane parallel flows, the mathematical models are represented by higher-order boundary value problems in one and two dimensions and analytical solutions are not available. In the same time the parametric space of physical interest and significance is enormous (4–5 dimensionless parameters to vary). The Rayleigh number and modulation frequency can take on very high values, signaling the occurrence of boundary or internal layers of steep profiles of the field variables. This makes the development of effective numerical approaches a must.

2. THERMAL CONVECTION IN A VERTICAL SLOT

Consider the 2D flow in a vertical slot with a linear vertical temperature gradient, differentially heated walls, and subject to modulation of gravity in the vertical direction. The problem definition is well-described in the literature (refer to [1, 6] and Fig. 1 for a definition sketch), and the notation we use is standard:

$$x = \frac{x^*}{L} - 1, \quad y = \frac{y^*}{L}, \quad \omega = \omega^* \frac{L^2}{\kappa},$$
$$t = t^*\omega^*, \quad \psi = \frac{\psi^*}{\nu}, \quad \theta = \frac{T^*}{\delta T} + x - \tau_B y,$$

where ν is the kinematic viscosity, κ — the thermal diffusivity, $2L$ — the width of the slot, and δT — the horizontal temperature difference. The asterisk denotes dimensional variables, while the same notation without an asterisk stands for the respective dimensionless quantity. Note that the field $\theta(x, y, t)$ is the departure from the linear vertical and horizontal stratification. Hence one can seek solutions which are periodic in the vertical dimension.

The dimensionless boundary value problem under consideration reads

$$\frac{1}{Pr}\left(\omega\frac{\partial\Delta\psi}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right) = -Ra\left(\frac{\partial\theta}{\partial x}-1\right)[1+\varepsilon\cos(t)]+\Delta^2\psi, \tag{1}$$

$$\omega\frac{\partial\theta}{\partial t}+\left(\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right)=\frac{\partial\psi}{\partial y}+\tau_B\frac{\partial\psi}{\partial x}+\Delta\theta \tag{2}$$

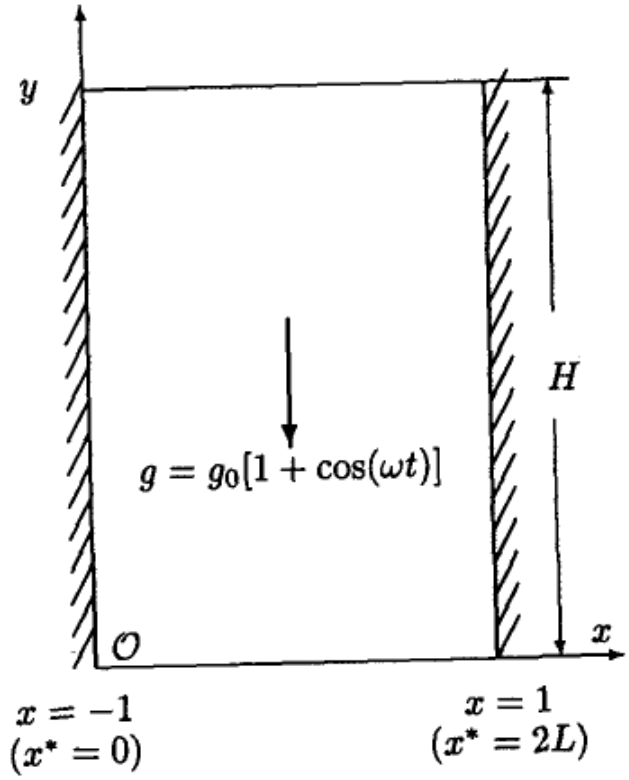


Fig. 1. Flow geometry

with boundary conditions

$$\psi=\frac{\partial\psi}{\partial x}=\theta=0 \quad \text{for} \quad x=\pm1\,, \tag{3}$$

and periodic conditions in vertical direction

$$\begin{aligned}
\psi(x,0,t) &= \psi(x,H,t),\\
\psi_y(x,0,t) &= \psi_y(x,H,t),\\
\psi_{yy}(x,0,t) &= \psi_{yy}(x,H,t),\\
\psi_{yyy}(x,0,t) &= \psi_{yyy}(x,H,t),\\
\theta(x,0,t) &= \theta(x,H,t),\\
\theta_y(x,0,t) &= \theta_y(x,H,t),
\end{aligned} \tag{4}$$

where $H = H^*/L = 2\pi/\alpha$ is the dimensionless height of the vertical box: equivalently, α is the dimensionless vertical wave number of the periodic solutions.

The Rayleigh number Ra, the Prandtl number Pr, and stratifications parameter, γ, are defined as:

$$Ra = \frac{\beta g_0 \delta T L^3}{\nu\kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad 4\gamma^4 = \tau_B Ra,$$

where β is the coefficient of thermal expansion of the liquid, g_0 — the mean gravity, ε — the dimensionless amplitude of gravity modulations, ω — the dimensionless frequency, and τ_B is the dimensionless vertical temperature gradient. Using a difference approximation and an operator splitting, the 2D flow is investigated numerically in [5]. We focus our attention on the 1D case for the purposes of developing the new numerical technique.

Under the selected boundary conditions the problem also admits a plane--parallel solution of the form $\Psi(x,t)$, $\Theta(x,t)$ for which the governing system reduces to the following:

$$\frac{\omega}{Pr}\frac{\partial^3\Psi}{\partial t\partial x^2} = -Ra\left[1+\frac{\partial\Theta}{\partial x}\right][1+\varepsilon\cos(t)] + \frac{\partial^4\Psi}{\partial x^4}, \tag{5}$$

$$\omega\frac{\partial\Theta}{\partial t} = \tau_B\frac{\partial\Psi}{\partial x} + \frac{\partial^2\Theta}{\partial x^2}, \tag{6}$$

with the same boundary conditions (3).

The 1D flow was first treated in [6], where different régimes of flow were studied. The parametric bifurcation of the 1D solutions was studied in detail in [5] by means of a fully implicit difference scheme and a related 1D problem in [12].

A way out of these difficulties is to use spectral decomposition with respect to complete orthonormal (CON) systems in x-direction. The performance of a spectral method depends heavily on the type of the basis system of functions. The scope of this paper is to implement these ideas for the one-dimensional in space and time-dependent problem (5), (6), (3).

In order to assess the approximation, convergence rate and truncation error, it is enough to consider a model ODE which contains all of the different terms of the time dependent system. A simplified first step is to consider just one ODE of fourth order and to compile the rest of the technique.

To this end we consider the following three boundary value problems (b.v.p.):

1. B.v.p. containing both fourth and second-order derivatives:

$$\frac{d^4u}{dx^4} + 2\frac{d^2u}{dx^2} + u = 1, \quad u(-1) = u(1) = 0, \quad u'(-1) = u'(1) = 0, \tag{7}$$

which possesses an analytical solution:

$$u(x) = 1 - \frac{2\cos x[\cos 1 + \sin 1] - 2x\sin 1\sin x}{2+\sin 2}. \tag{8}$$

2. A nonlinear version of the above b.v.p.:

$$\frac{d^4u}{dx^4}+2\frac{d^2u}{dx^2}+u=1-100u^2(x),$$
$$u(-1)=u(1)=0, \quad u'(-1)=u'(1)=0, \tag{9}$$

where the large coefficient 100, multiplying the nonlinear term, is selected for the sake of making the nonlinearity more appreciable.

3. The higher-order coupled b.v.p. for an ODE system, which retains all of the important terms in the full-fledged unsteady problem for the thermal convection in a vertical slot:

$$\frac{d^4\Psi}{dx^4}=Ra\left[-1+\frac{d\Theta}{dx}\right]+\frac{1}{Pr}\frac{\partial^2\Psi}{\partial x^2},$$
$$\Theta-\frac{d\Psi}{dx}=\frac{d^2\Theta}{dx^2}, \quad \Psi=\Psi_x=\Theta=0, \quad \text{for} \quad x=\pm 1. \tag{10}$$

We find the above system generically representative of the problem under consideration, because it retains the second spatial derivatives. In a sense, it can be considered as a simplification of an Euler time-stepping scheme with time increment equal to one.

3. THE SPECTRAL TECHNIQUE

The expansion in x direction is nontrivial because of the higher-order boundary value problem for the stream function. The right CON system for a fourth-order problem was introduced by Lord Rayleigh for the problem of vibration of elastic beams. For the specific boundary conditions arising in viscous liquid dynamics the system and its completeness were discussed in [3]. The product formulas as well as the expansion formulas for the derivatives of different orders were derived in a preceding authors work [4]. The product formula is essential for the application to a nonlinear problem.

3.1. BEAM FUNCTIONS

Consider the Sturm-Liouville problem

$$\frac{d^4u}{dy^4}=\lambda^4 u, \quad u=\frac{du}{dy}=0, \quad \text{for} \quad x=\pm 1. \tag{11}$$

The nontrivial solutions (eigen-functions) of this problem are given by

$$s_m=\frac{1}{\sqrt{2}}\left[\frac{\sinh\lambda_m x}{\sinh\lambda_m}-\frac{\sin\lambda_m x}{\sin\lambda_m}\right], \qquad \operatorname{cotanh}\lambda_m-\operatorname{cotan}\lambda_m=0, \tag{12}$$

$$c_m=\frac{1}{\sqrt{2}}\left[\frac{\cosh\kappa_m x}{\cosh\kappa_m}-\frac{\cos\kappa_m x}{\cos\kappa_m}\right], \qquad \tanh\kappa_m+\tan\kappa_m=0. \tag{13}$$

These functions have been introduced by Lord Rayleigh to solve problems arising in beam theory and they are sometimes called beam functions. A major step in the advancement of the application of the beam functions to fluid-dynamics problems was made by Poots [9]. The magnitudes of the different eigenvalues can be found in most of the above cited works from the literature.

Chandrasekhar [3] derived their counterparts for problems with cylindrical symmetry. For applications to stability problems, see also [7, 11].

The expressions for developing the nonlinear terms into series with respect to the system appeared simultaneously in [8] and [4] though in different form. We stick here to the notations of [4] as more explicit and easier to verify.

3.2. EXPANSIONS FOR THE DERIVATIVES

The different derivatives can be expressed in series with respect to the system as follows:

$$c_n' = \sum_{m=1}^{\infty} a_{nm} s_m \,, \qquad a_{nm} = \frac{4\kappa_n^2 \lambda_m^2}{\kappa_n^4 - \lambda_m^4} \,, \tag{14}$$

$$s_n' = \sum_{m=1}^{\infty} \bar{a}_{nm} c_m \,, \qquad \bar{a}_{nm} = \frac{4\kappa_m^2 \lambda_n^2}{-\kappa_m^4 + \lambda_n^4} \,, \tag{15}$$

$$c_n'' = \sum_{m=1}^{\infty} \beta_{nm} c_m, \quad s_n'' = \sum_{m=1}^{\infty} \bar{\beta}_{nm} s_m, \tag{16}$$

$$\beta_{nm} = \begin{cases} \dfrac{4\kappa_n^2 \kappa_m^2}{\kappa_m^4 - \kappa_n^4} \left(\kappa_m \tanh \kappa_m - \kappa_n \tanh \kappa_n\right), & m \neq n, \\ \kappa_n \tanh \kappa_n - \left(\kappa_n \tanh \kappa_n\right)^2 , & m = n, \end{cases} \tag{17}$$

$$\bar{\beta}_{nm} = \begin{cases} \dfrac{4\lambda_n^2 \lambda_m^2}{\lambda_n^4 - \lambda_m^4} \left(\lambda_n \operatorname{cotanh} \lambda_n - \lambda_m \operatorname{cotanh} \lambda_m\right), & m \neq n, \\ \lambda_n \operatorname{cotanh} \lambda_n - \left(\lambda_n \operatorname{cotanh} \lambda_n\right)^2 , & m = n, \end{cases} \tag{18}$$

$$c_n''' = \sum_{m=1}^{\infty} d_{nm} s_m \,, \qquad d_{nm} = \frac{4\kappa_n^3 \lambda_m^3}{-\kappa_n^4 + \lambda_m^4} \tanh \kappa_n \operatorname{cotanh} \lambda_m, \tag{19}$$

$$s_n''' = \sum_{m=1}^{\infty} \bar{d}_{nm} c_m \,, \qquad \bar{d}_{nm} = \frac{4\kappa_m^3 \lambda_n^3}{-\kappa_m^4 + \lambda_n^4} \tanh \kappa_m \operatorname{cotanh} \lambda_n. \tag{20}$$

3.3. PRODUCTS OF BEAM FUNCTIONS

The most important for the present work are the product formulae

$$
\begin{aligned}
c_n(x)c_m(x) = \sum_{k=1}^{\infty} h_k^{nm} c_k(x), \quad \sqrt{2}h_k^{nm} &= \sqrt{2}\int_{-1}^{1} c_n(x)c_m(x)c_k(x)\mathrm{d}x \qquad (21)\\
&= \frac{-(\kappa_m+\kappa_k)(\tanh\kappa_m+\tanh\kappa_k)-\kappa_n\tanh\kappa_n}{(\kappa_m+\kappa_k)^2-\kappa_n^2}\\
&+\frac{-(\kappa_m-\kappa_k)(\tanh\kappa_m-\tanh\kappa_k)+\kappa_n\tanh\kappa_n}{-(\kappa_m-\kappa_k)^2+\kappa_n^2}\\
&+\frac{-(\kappa_m+\kappa_k)(\tanh\kappa_m+\tanh\kappa_k)+\kappa_n\tanh\kappa_n}{(\kappa_m+\kappa_k)^2+\kappa_n^2}\\
&+\frac{-(\kappa_m-\kappa_k)(\tanh\kappa_m-\tanh\kappa_k)+\kappa_n\tanh\kappa_n}{(\kappa_m-\kappa_k)^2+\kappa_n^2}\\
&+\frac{-(\kappa_n+\kappa_k)(\tanh\kappa_n+\tanh\kappa_k)+\kappa_m\tanh\kappa_m}{(\kappa_n+\kappa_k)^2+\kappa_m^2}\\
&+\frac{-(\kappa_n-\kappa_k)(\tanh\kappa_n-\tanh\kappa_k)+\kappa_m\tanh\kappa_m}{(\kappa_n-\kappa_k)^2+\kappa_m^2}\\
&+\frac{-(\kappa_n+\kappa_m)(\tanh\kappa_n+\tanh\kappa_m)+\kappa_k\tanh\kappa_k}{(\kappa_n+\kappa_m)^2+\kappa_k^2}\\
&+\frac{-(\kappa_n-\kappa_m)(\tanh\kappa_n-\tanh\kappa_m)+\kappa_k\tanh\kappa_k}{(\kappa_n-\kappa_m)^2+\kappa_m^2},
\end{aligned}
$$

$$
\begin{aligned}
s_n c_m = \sum_{k=1}^{\infty} f_k^{nm} s_k,\ s_n s_m = \sum_{k=1}^{\infty} f_m^{nk} c_k, \quad \sqrt{2}f_k^{nm} &= \sqrt{2}\int_{-1}^{1} s_n c_m s_k \mathrm{d}x \qquad (22)\\
&= \frac{(\lambda_n+\lambda_k)(\coth\lambda_n+\coth\lambda_k)-\kappa_m\tanh\kappa_m}{(\lambda_k+\lambda_n)^2-\kappa_m^2}\\
&+\frac{-(\lambda_k-\lambda_n)(\coth\lambda_k-\coth\lambda_n)+\kappa_m\tanh\kappa_m}{(\lambda_k-\lambda_n)^2-\kappa_m^2}\\
&+\frac{-(\lambda_k+\kappa_m)(\coth\lambda_k+\tanh\kappa_m)+\lambda_n\coth\lambda_n}{(\lambda_k+\kappa_m)^2+\lambda_n^2}\\
&+\frac{-(\lambda_k-\kappa_m)(\coth\lambda_k-\tanh\kappa_m)+\lambda_n\coth\lambda_n}{(\lambda_k-\kappa_m)^2+\lambda_n^2}\\
&+\frac{-(\lambda_n+\kappa_m)(\coth\lambda_n+\tanh\kappa_m)+\lambda_k\coth\lambda_k}{(\lambda_n+\kappa_m)^2+\lambda_k^2}\\
&+\frac{-(\lambda_n-\kappa_m)(\coth\lambda_n-\tanh\kappa_m)+\lambda_k\coth\lambda_k}{(\lambda_n-\kappa_m)^2+\lambda_k^2}\\
&+\frac{-(\lambda_n+\lambda_k)(\coth\lambda_n+\coth\lambda_k)+\kappa_m\tanh\kappa_m}{(\lambda_k+\lambda_n)^2+\kappa_m^2}\\
&+\frac{(\lambda_k-\lambda_n)(\coth\lambda_n-\coth\lambda_k)+\kappa_m\tanh\kappa_m}{(\lambda_k-\lambda_n)^2+\kappa_m^2}.
\end{aligned}
$$

The most obvious test to verify the correctness and consistency of the above derived formulas for the products is to take the product of some two particular functions c_n and c_m and to compare pointwise the products $c_n c_m$ and $s_n c_m$ with their Galerkin expansions into c_k and s_k, respectively. For the products of even functions this comparison is shown in Fig. 2.

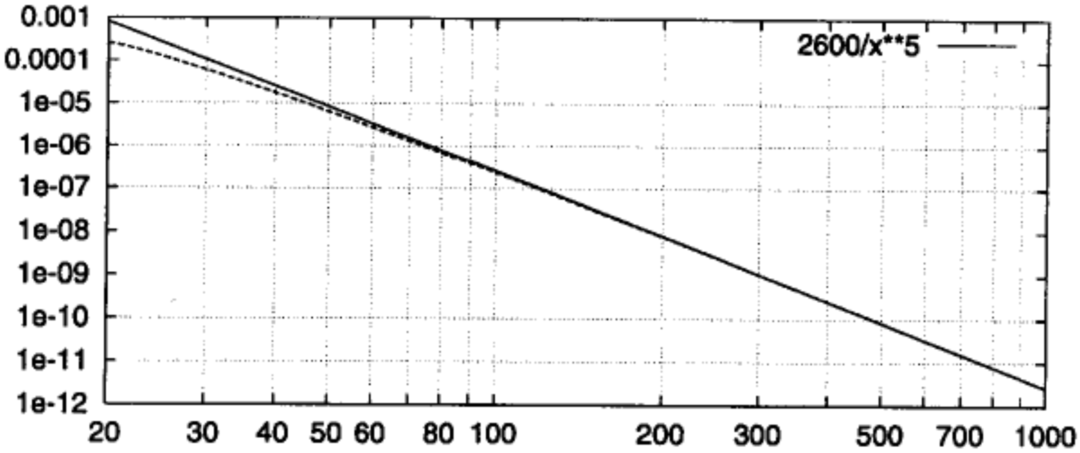


Fig. 2. The convergence of the series for the product $c_6 c_3$. Solid line: h_i^{63}; dashed line: the best fit curve $h_i^{63} = 2600\, i^{-5}$

Our numerical experiments with different products of beam functions invariably led us to the fifth-order convergence

$$f_k^{mn} \sim \hat{f}(m,n)k^{-5}, \qquad h_k^{mn} \sim \hat{h}(m,n)k^{-5}.$$

Thus a conjecture is in order that the fifth order of convergence of the series for a quadratic nonlinear term is a general property of the system of beam functions.

3.4. EXPANSION OF UNITY

We also expanded the unity into a c_m series as follows:

$$1 = \sum_{k=1}^{\infty} h_k c_k(x), \quad h_k = \int_{-1}^{1} c_k(x)\mathrm{d}x = \frac{2\sqrt{2}\tanh \kappa_k}{\kappa_k}. \tag{23}$$

The convergence of this expansion is algebraic of first order. This is due to the fact that the unity does not satisfy the boundary conditions for the beam functions and as a result a strong Gibbs effect is observed near the boundaries.

Yet the overall rate of convergence of the method is fifth-order algebraic, because in the left-hand side of the problems under consideration the fourth power of the respective eigen-value appears as a multiplier.

For the convective problem under consideration the difficulties arise from the fact that the boundary value problem for temperature function is of second order, which means that the system of beam functions is not suitable for expanding the temperature field. It is clear that the best suited to the task system are the trigonometric *sines* and *cosines*. Hence we need to develop expressions for expanding the beam functions into trigonometric functions and vice versa:

$$\sin l\pi x = \sum_{k=1}^{\infty} \sigma_{lk} s_k(x), \qquad \sigma_{lk} = \frac{2\sqrt{2} l\pi (\lambda_k)^2 (-1)^l}{l^4\pi^4 - \lambda_k^4}, \tag{24}$$

$$\cos l\pi x = \sum_{k=1}^{\infty} \chi_{lk} c_k(x), \qquad \chi_{lk} = \frac{2\sqrt{2}\kappa_k^3 (-1)^{l+1} \tanh \kappa_k}{l^4\pi^4 - \kappa_k^4}, \tag{25}$$

$$c_n(x) = \sum_{l=1}^{\infty} \hat{\chi}_{nl} \cos l\pi x, \qquad \hat{\chi}_{nl} = \frac{2\sqrt{2}\kappa_n^3 (-1)^{l+1} \tanh \kappa_n}{l^4\pi^4 - \kappa_n^4}, \tag{26}$$

$$s_n(x) = \sum_{l=1}^{\infty} \hat{\sigma}_{nl} \sin l\pi x, \qquad \hat{\sigma}_{nl} = \frac{2\sqrt{2} l\pi (\lambda_n)^2 (-1)^l}{l^4\pi^4 - \lambda_n^4}. \tag{27}$$

Once again we point out that the convergence when expanding $\cos(l\pi x)$ into c_k series is first order k^{-1} (see (25)) due to the fact that it does not satisfy both b.c. for the beam functions. It satisfies the condition on the derivatives but fails to satisfy the conditions on the function itself. Clearly, the situation with the $\sin(l\pi x)$ is better and the rate of convergence is of second order k^{-2} (see (24)), because the *sine* functions satisfy the boundary conditions on the functions and the disagreement is more subtle since the conditions on the first derivative are not satisfied. The situation with the expansions of s_x and c_k in Fourier series is reversed. The order of convergence for c_k is l^{-4} (see (26)), and for s_x is l^{-3} (see (27)). As it will be shown in what follows, this property is of crucial importance for the overall rate of convergence.

4. THE GALERKIN METHOD

In this section we present the numerical tests and verifications of the Galerkin technique using as featuring examples the three boundary value problems outlined in Section 2.

4.1. SOLVING THE MODEL FOURTH-ORDER PROBLEM

We solve (7) numerically using the developed here beam-Galerkin expansion with respect to the complete orthonormal (CON) system of functions $c_n(x)$, $s_n(x)$. Because of the nature of the boundary conditions, we can constrain ourselves to the subset of even functions c_n (a fact verified also by the analytic solution) and expand the sought function into series with respect to $c_n(x)$:

$$u(x) = \sum_{1}^{N} b_n c_n(x). \tag{28}$$

Making use of the above compiled formulas we obtain for the coefficients b_n the following linear algebraic system of N equations with N unknowns:

$$(1+\kappa_i^4)b_i + 2\sum_{j=1}^{N} b_j\beta_{ij} = \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i}, \qquad i = 1, \ldots, N, \tag{29}$$

with β_{ij} defined in (17).

The last system is solved by means of LAPACK routine *dgesv*.

We found that the coefficients b_i decay with the number of the term i as i^{-5}, which is clearly seen in Fig. 3.

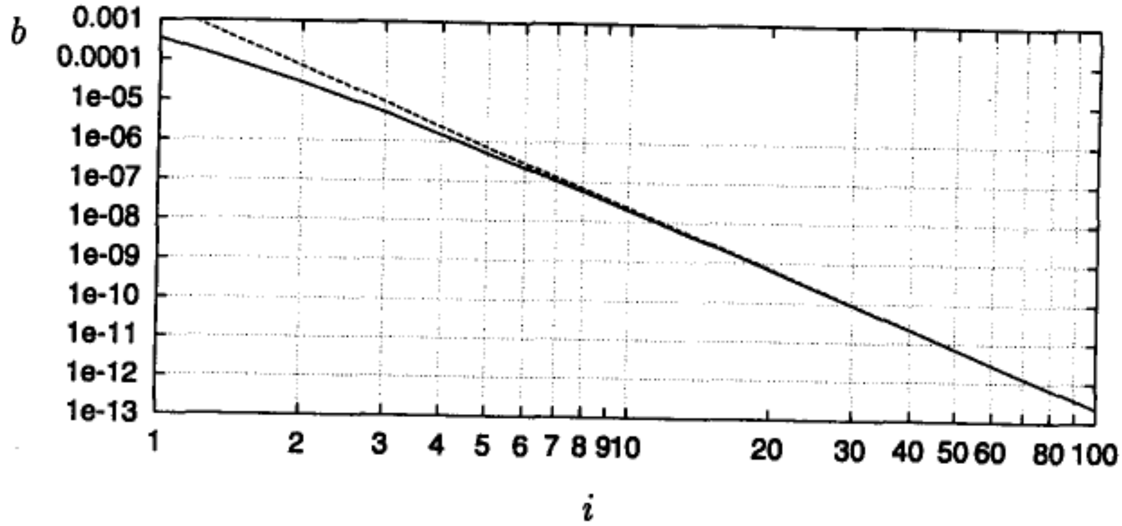


Fig. 3. Convergence of the beam-Galerkin series for the model equation (7). Solid line: b_i; dashed line: the best fit curve $b_i = 0.0023\, i^{-5}$

The obtained spectral solution is compared to the analytical one and the overall truncation error is estimated. As it is to be expected for a series with fifth-order algebraic convergence, the truncation error for $N = 100$ is of order of $O(10^{-10})$.

4.2. THE NONLINEAR MODEL PROBLEM

The nonlinear problem (9) results into the following nonlinear algebraic system:

$$(1+\kappa_i^4)b_i + 2\sum_{j=1}^{N} b_j\beta_{ij} = \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i} - 100\sum_{m=1}^{N}\sum_{n=1}^{N} b_m b_n h_i^{mn}, \qquad i = 1, \ldots, N, \tag{30}$$

where h_i^{mn} is defined in formula (21). We solve the latter with semi-implicit method and iterations.

The results about the convergence of the spectral solution are shown in Fig. 4. The convergence is once again algebraic of fifth order.

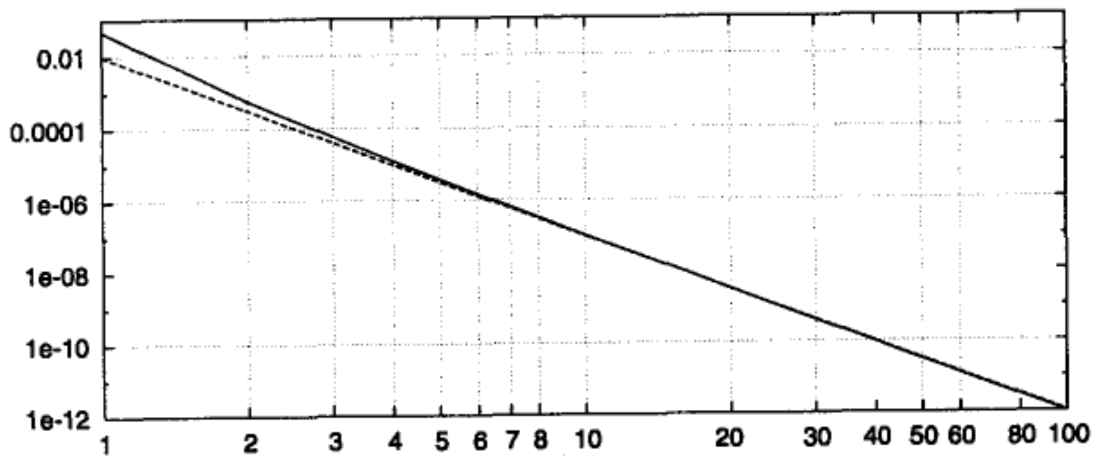


Fig. 4. Rate of convergence for the solution of the nonlinear equation. Solid line: b_i; dashed line: $b_i = 0.01\, i^{-5}$

4.3. THE COUPLED SYSTEM

In this case we consider the coupled system of one fourth-order equation for ψ and one second-order equation for θ (10). Because of the obvious symmetry of the boundary value problem under consideration, we can seek a solution in which the stream function is even and the temperature is an odd function. Acknowledging the symmetry of the problem, we develop the sought function into the series

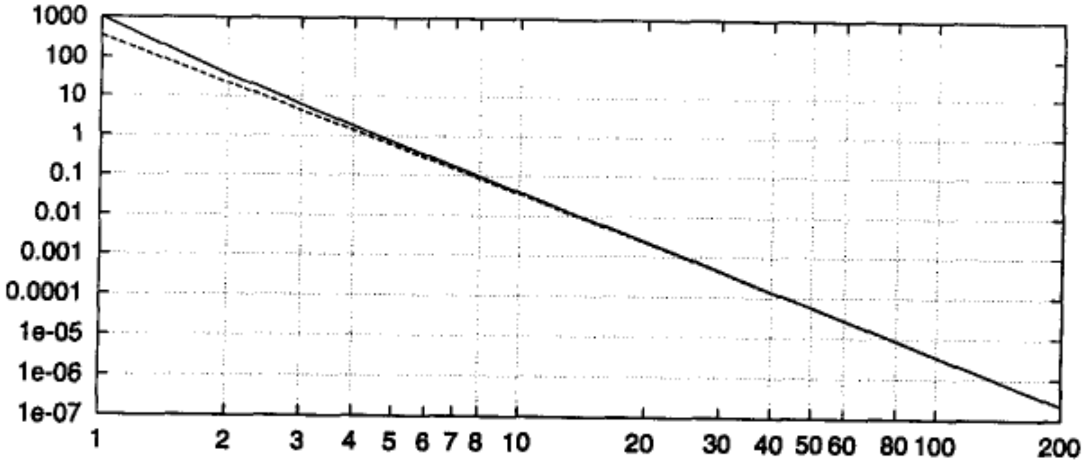
$$\Psi(x,t) = \sum_{k=1}^{K} p_k c_k(x), \quad \Theta(x,t) = \sum_{k=1}^{K} d_k \sin(k\pi x). \tag{31}$$

Upon introducing these expansions into (5), (6) and making use of the above compiled formulae, an algebraic system for the coefficients d_k and p_k is derived:

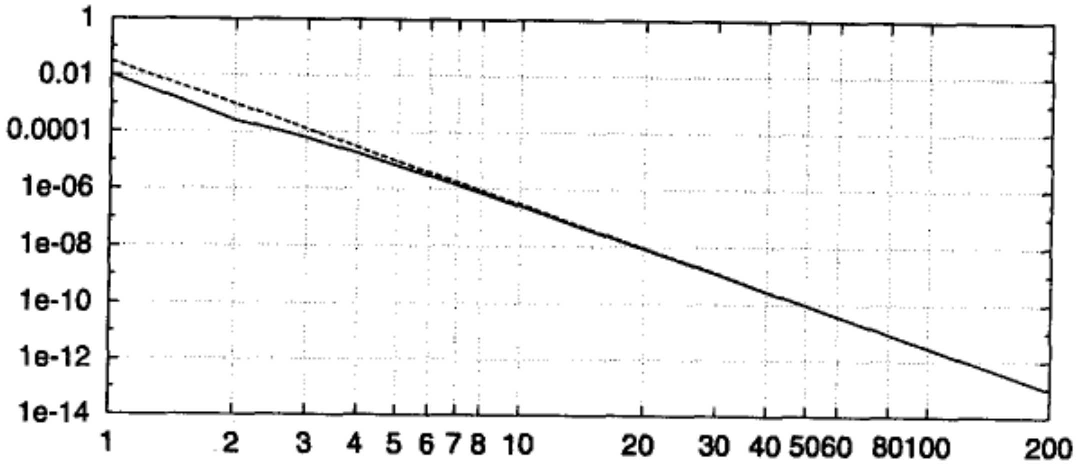
$$\begin{aligned} &-\kappa_i^4 p_i + \frac{1}{Pr}\sum_{j=1}^{N} p_j \beta_{ij} \\ &= -Ra\left[\sum_{m=1}^{N} d_m \frac{m\pi 2\sqrt{2}(-1)^{m+1}\kappa_i^3 \tanh\kappa_i}{m^4\pi^4 - \kappa_i^4} - \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i}\right], \\ &\qquad i = 1, \dots, N, \end{aligned} \tag{32}$$

$$\begin{aligned} &(1 + l^2\pi^2) d_l = \tau_B \sum_{n=1}^{N}\sum_{m=1}^{N} p_n \frac{8\sqrt{2}\kappa_n^2\kappa_m^2 l\pi(-1)^l}{(\kappa_n^4 - \kappa_m^4)(l^4\pi^4 - \kappa_m^4)}, \\ &\qquad l = 1, \dots, N. \end{aligned} \tag{33}$$

The results for the coefficients p_i and d_l are presented in Fig. 5. The peculiar finding is that the rate of convergence for Θ is algebraic of fifth order, while the rate for Ψ is one order lower (fourth-order). The analytical explanation of this phenomena will be the object of a separate study. Here it will suffice to mention that the off-diagonal elements in (32) can degrade the rate of convergence, while in the equation (33) for Θ no off-diagonal elements are present and the convergence is of fifth order as in the previous examples.



Solid line: p_i; Dashed line: $p_i = 350i^{-4}$



Solid line: d_i; Dashed line: $d_i = 0.03i^{-5}$

Fig. 5. The rate of convergence for the coupled system for $Ra = 6000$, $Pr = 1$ and $\tau_B = 0.001$. The upper panel shows the spectral coefficient for the function Psi; the lower panel shows Θ

The fourth order for the rate of convergence means that a number of terms $N = 100$ is fully adequate to obtain results with a very high precision 10^{-8}.

5. CONCLUSIONS

In the present work a new Galerkin technique is developed for coupled thermo-convective flows in a vertical slot. The well-known beam functions are used as basis set together with the trigonometric functions. The formulas for the cross expansion of the two systems are not available from the literature and are derived here. The construction of the numerical algorithms is also presented.

Three generic model problems are considered. The spectral solutions exhibit a fifth-order algebraic convergence except for the case of the coupled system pertinent to the convection in a vertical slot, where the rate is of fourth order for one of the functions. The fourth or fifth order means that although algebraic, the convergence is fast enough for all practical purposes. The theoretical and numerical findings are illustrated graphically.

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