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EFFECTIVE ENUMERATIONS OF FAMILIES OF RECURSIVE FUNCTIONS*

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In this paper necessary and sufficient conditions for a family of finite functions and a family of totally recursive functions to have a universal partially recursive function are given.

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The problem of finding necessary and sufficient conditions for a given family of partially recursive (p.r.) functions (recursively enumerable (r.e.) sets) to have a universal p.r. function (r.e. set) is one of the interesting problems in the Recursion theory. For example, if we want to find a recursive model for a given recursively enumerable theory in some cases, we have to know if a given family of recursive functions has a universal recursive function or not. It is well-known that the family of all p.r. functions (r.e. sets) has a universal p.r. function (r.e. set), while the family of all recursive functions (totally defined on some $\mathbb{N}^n$) has no universal recursive function. On the other hand, in the works [2, 3, 5] a related problem is considered. Some necessary and some sufficient conditions for the family of all recursive functions and some finite initial functions to have a universal r.e. set are obtained.

In [1, 4] Ishmuhametov and Selivanov have obtained sufficient conditions for a special class of families of r.e. sets. In [6] the author has characterized the families

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of finite sets which admit a universal recursively enumerable set. But until now such a necessary and sufficient condition is not found for arbitrary families of r.e. sets (p.r. functions).

In this paper we notice that the characterization of the families of finite functions which admit a universal partially recursive function is the same (in some sense) as in the case of families of finite sets of natural numbers [6] and we give necessary and sufficient conditions for a given family of recursive functions to have a universal recursive function. We want to stress the analogy between both cases.

Here we use $\mathbb{N}$ to denote the set of all natural numbers $\{0,1,2,\ldots\}$ and N_n to denote the initial segment $\{k \mid k \in \mathbb{N} \,\&\, k < n\}$ of the set $\mathbb{N}$. We suppose that there is some fixed effective coding $\langle .,. \rangle$ of the pairs of natural numbers and $\lambda x.(x)_0$, $\lambda x.(x)_1$ are such recursive functions that $(\langle x_0, x_1 \rangle)_0 = x_0$ and $(\langle x_0, x_1 \rangle)_1 = x_1$. If f is a partial function, we use $Dom(f)$ to denote the domain, and $Ran(f)$ to denote the range of values of the function f. In the case when $Dom(f) \subseteq \mathbb{N}^k$ and $Ran(f) \subseteq \mathbb{N}$, we shall write $f : \mathbb{N}^k \multimap \mathbb{N}$, and if $Dom(f) = \mathbb{N}^k$ and $Ran(f) \subseteq \mathbb{N}$, we shall write $f : \mathbb{N}^k \to \mathbb{N}$. For the sake of simplicity, we use mainly unary functions defined on a subset of $\mathbb{N}$. If f and g are functions, we say that f is a subfunction of g (and write $f \subseteq g$) iff $\forall x(x \in Dom(f) \Rightarrow x \in Dom(g) \,\&\, f(x) = g(x))$. We say in such a case that g is an extension of the function f, as well. Denote by θ_v the finite function with cannonical index v. For example, if θ is such a function that $Dom(\theta)$ is finite and $Dom(\theta) = \{x_1, \ldots, x_k\}$, $x_1 < \ldots < x_k$, then we can consider $v = p_0^k . p_1^{2^{x_1}.3^{\theta(x_1)}} \ldots p_k^{2^{x_k}.3^{\theta(x_k)}}$. Here $p_0, p_1, \ldots$ is the increasing sequence of all prime numbers. If $Dom(\theta) = N_k$ for some natural k, we say that θ is defined on an initial segment. By φ_e we denote the e-th partially recursive function in the standard enumeration of the partially recursive functions.

Let $\Psi : \mathbb{N}^2 \multimap \mathbb{N}$ and $\mathfrak{F}$ be a family of partial functions defined on $\mathbb{N}$. The function Ψ is said to be *universal for the family* $\mathfrak{F}$ iff for any n the function $\lambda x.\Psi(n,x)$ is in the family $\mathfrak{F}$, and, conversely, for any function $f \in \mathfrak{F}$ there exists such n that $f = \lambda x.\Psi(n,x)$. If $\Psi : \mathbb{N}^2 \to \mathbb{N}$, then Ψ_n denotes the unary function $\lambda x.\Psi(n,x)$.

It is well-known [cf. 7, p. 38] that if $\Psi : \mathbb{N}^2 \to \mathbb{N}$ is a recursive function which is universal for the family $\mathfrak{F}$, then there exists a recursive function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for all n $\theta_{f(n,0)} \subseteq \theta_{f(n,1)}, \ldots$ and $\lim_s \theta_{f(n,s)}(x) = \Psi(n,x)$, i.e. $\forall n \forall x \exists s_0 \forall s \geq s_0(\theta_{f(n,s)}(x) = \Psi(n,x))$.

The following theorems and their proofs are analogous to the ones in [6].

Theorem 1. *Let $\mathfrak{F}$ be a nonempty family of finite fuctions defined on $\mathbb{N}$. Then the family $\mathfrak{F}$ has a universal partial recursive function iff the following three conditions hold*:

(i) *The set $V = \{v \mid \exists f(f \in \mathfrak{F} \,\&\, \theta_v \subseteq f)\}$ is recursively enumerable*;

(ii) *The set $I = \{v \mid \theta_v \in \mathfrak{F}\}$ is Σ_2^0 (in the arithmetical hierarchy)*;

(iii) *There exists such a partial recursive function h that*

$$V \subseteq Dom(h) \ \text{ and } \ \forall v(v \in V \Rightarrow \theta_v \subseteq \varphi_{h(v)} \in \mathfrak{F}).$$

Theorem 2. *Let $\mathfrak{F}$ be a nonempty family of finite functions defined on $\mathbb{N}$ such that for every $f \in \mathfrak{F}$ at most finitely many functions g exist in $\mathfrak{F}$ such that*

$f \subseteq g$. *Then the family has a universal partial recursive function iff the following two conditions hold*:

(i) *The set* $V = \{v \mid \exists f(f \in \mathfrak{F} \,\&\, \theta_v \subseteq f\}$ *is recursively enumerable*;

(ii) *The set* $I = \{v \mid \theta_v \in \mathfrak{F}\}$ *is in the class* Σ^0_2.

Theorem 3. *Let* $\mathfrak{F}$ *be a nonempty family of finite functions defined on* $\mathbb{N}$ *such that the set* $I = \{v \mid \theta_v \in \mathfrak{F}\}$ *is in the class* Π^0_1. *Then the family has a universal partial recursive function iff the following two conditions hold*:

(i) *The set* $V = \{v \mid \exists f(f \in \mathfrak{F} \,\&\, \theta_v \subseteq f)\}$ *is recursively enumerable*;

(ii) *There exists a partial recursive function* h *of two variables such that the following three conditions hold*:

(a) $\forall v(v \in V \Rightarrow \lambda n.h(v,n)$ *is totally defined*);

(b) $\forall v \in V \forall n_1 \forall n_2 (n_1 < n_2 \Rightarrow \theta_v \subseteq \theta_{h(v,n_1)} \subseteq \theta_{h(v,n_2)})$;

(c) $\forall v \in V \exists n(\theta_{h(v,n)} \in \mathfrak{F})$.

Now we shall consider the case of families with recursive functions.

Theorem 4. *Let* $\mathfrak{F}$ *be a nonempty family of unary recursive functions. Then the family* $\mathfrak{F}$ *has a universal recursive function iff the following two conditions hold*:

(i) *The set* $V = \{v \mid \exists f(f \in \mathfrak{F} \,\&\, \theta_v \subseteq f)\}$ *is recursively enumerable*;

(ii) *There exist a family* $\mathfrak{G}$ *such that* $\mathfrak{F} \subseteq \mathfrak{G}$ *and a recursive function* Ψ, *which is universal for the family* $\mathfrak{G}$, *such that the following two conditions are satisfied*:

a) *the set* $I = \{n \mid \Psi_n \in \mathfrak{F}\}$ *is* Σ^0_2 (*in the arithmetical hierarchy*);

b) *there exists such a partial recursive function* h *that*

$$V \subseteq Dom(h) \;\; and \;\; \forall v(v \in V \Rightarrow \theta_v \subseteq \Psi_{h(v)} \in \mathfrak{F}).$$

Proof. Suppose first that the family $\mathfrak{F}$ has a universal recursive function Ψ. Then the set

$$\begin{aligned} V &= \{v \mid \exists f(f \in \mathfrak{F} \,\&\, \theta_v \subseteq f)\} \\ &= \{v \mid \exists n \exists k \exists x_1 \ldots \exists x_k (v = p_0^k . p_1^{2^{x_1}.3^{\theta_v(x_1)}} \ldots p_k^{2^{x_k}.3^{\theta_v(x_k)}} \\ &\qquad \& \,\Psi(n,x_1) \cong \theta_v(x_1) \,\&\, \ldots \,\&\, \Psi(n,x_k) \cong \theta_v(x_k))\} \end{aligned}$$

is recursively enumerable.

Fix $\mathfrak{G}$=F. It is obvious that the set $I = \{n \mid \Psi_n \in \mathfrak{F}\} = \mathbb{N}$, so the condition a) from (ii) is satisfied.

Let us define the function h as follows:

$$\begin{aligned} h(v) \cong \mu n[\exists k \exists x_1 \ldots \exists x_k \exists y_1 \ldots \exists y_k (v = p_0^k . p_1^{2^{x_1}.3^{y_1}} \ldots p_k^{2^{x_k}.3^{y_k}} \\ \& \,\Psi(n,x_1) \cong y_1 \,\&\, \ldots \,\&\, \Psi(n,x_k) \cong y_k)]. \end{aligned}$$

It is clear that h is a p.r. function and satisfies b) from (ii).

Conversly, let the conditions (i) – (ii) hold and F be a unary recursive function such that $F(0) = 1$ and $Ran(F) = V$.

In addition, let exist a family $\mathfrak{G}$ such that $\mathfrak{F} \subseteq \mathfrak{G}$ and a recursive function Ψ, which is universal for the family $\mathfrak{G}$, and $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be a unary recursive function such that for all n

$$\theta_{\alpha(n,0)} \subseteq \theta_{\alpha(n,1)} \subseteq \cdots \subseteq \theta_{\alpha(n,s)} \subseteq \cdots,$$

and for all n and x $\lim_s \theta_{\alpha(n,s)}(x) = \Psi(n,x)$. For the sake of simplicity, we suppose that $Dom(\theta_{\alpha(n,s)})$ is an initial segment for all n and s. At the end, let G be a ternary total recursive function such that

$$n \in I \iff \exists x \forall y [G(n,x,y) = 0]$$

and h be a partial recursive function such that $V \subseteq Dom(h)$ and $\forall v (v \in V \Rightarrow \theta_v \subseteq \Psi_{h(v)} \in \mathfrak{F})$.

We construct the so-called strong recursive sequence of finite functions by steps. On step s for any (n,x) we construct a finite function $\theta_{g(n,x,s)}$ for a recursive function g and at the end we will fix

$$\Theta(\langle n,x\rangle, z) = \lim_s \theta_{g(n,x,s)}(z).$$

Let us describe the construction of the recursive sequence of finite functions.

Step $s = 0$. Take $g(n,x,s) = 1$, i.e. $\theta_{g(n,x,s)} = \varnothing$.

Step $s > 0$. We consider two cases:

Case I. $\forall y [y \leq s \iff G(n,x,y) = 0]$.

Take $g(n,x,s) = F(t_{n,x,s})$, where $t_{n,x,s} \leq s$ is such that $Dom(\theta_{F(t_{n,x,s})})$ is a maximal initial segment in the set $\{Dom(\theta_{F(0)}), \ldots, Dom(\theta_{F(s)})\}$ such that $\theta_{F(t_{n,x,s})}$ is a subfunction of the function Ψ_n.

Case II. $\exists y [y \leq s \,\&\, G(n,x,y) \neq 0]$.

Take $\theta_{g(n,x,s)} = \theta_{\alpha(h(g(n,x,s_0-1)),s)}$, where $s_0 \cong \mu s[G(n,x,s) \neq 0]$.

Thus the construction is completed.

Obviously, the construction is effective, so the function g is recursive.

First of all, we shall see that for all fixed n, x and z the limit $\lim_s \theta_{g(n,x,s)}(z)$ exists and belongs to the family $\mathfrak{F}$. We consider two cases:

Case I. $\forall y [G(n,x,y) = 0]$. Then $\Psi_n \in \mathfrak{F}$ and for all s $g(n,x,s) = F(t_{n,x,s})$, where $t_{n,x,s}$ is such that $Dom(\theta_{F(t_{n,x,s})})$ is a maximal initial segment in the set $\{Dom(\theta_{F(0)}), \ldots, Dom(\theta_{F(s)})\}$ such that $\theta_{F(t_{n,x,s})}$ is a subfunction of the function Ψ_n. Thus, $\theta_{g(n,x,0)} \subseteq \theta_{g(n,x,1)} \subseteq \ldots$ and the limit exists and it is $\Psi_n(z)$, because for all k there exists s such that $\{0, \ldots, k\} \subseteq Dom(\theta_{g(n,x,s)})$.

Case II. $\exists y [G(n,x,y) \neq 0]$. Let $s_0 \cong \mu s[G(n,x,s) \neq 0]$. Then $g(n,x,s) = \alpha(h(g(n,x,s_0-1)),s)$ and for all $s \geq s_0$ $\theta_{g(n,x,s)} = \theta_{\alpha(h(g(n,x,s_0-1)),s)}$. Therefore the limit $\lim_s \theta_{g(n,x,s)}(z)$ exists and it is $\Psi_{h(g(n,x,s_0-1))}(z)$.

Now let $f \in \mathfrak{F}$. Then $f = \Psi_n$ for some $n \in I$. Therefore a natural x exists such that $\forall y [G(n,x,y) = 0]$. It is clear now that $f(z) = \Psi_n(z) = \lim_s \theta_{g(n,x,s)}(z)$.

Define the function Θ as follows:

$$\Theta(k,z) = \begin{cases} \lim_s \theta_{g(n,x,s)}(z), & \text{if } k = \langle n,x\rangle, \\ f_0(z), & \text{otherwise,} \end{cases}$$

where f_0 is a fixed element of the family $\mathfrak{F}$.

It is clear that Θ is total, recursive and universal for the family $\mathfrak{F}$.

The following examples show that none of the conditions (i) – (ii) can be skipped.

Example 1. Let A be a nonrecursively enumerable set. Define the family of recursive finctions by the following equality:

$$f_n(x) = \begin{cases} 0, & \text{if } x = n, \\ 1, & \text{otherwise.} \end{cases}$$

Then let $\mathfrak{F} = \{f_n \mid n \in A\}$. It is easy to see that the family $\mathfrak{F}$ does not have a universal recursive function but $\mathfrak{F}$ satisfies the condition (ii).

Example 2 a). Let $\mathfrak{F}$ be the family of all total recursive unary functions. It is well-known that the family $\mathfrak{F}$ has not a universal recursive function. On the other hand, it is obvious that the conditions (i) and a) from (ii) are fulfilled.

Example 2 b). Let A be the set of cannonical codes of finite sets such that A is not in Σ^0_2. Define the family $\mathfrak{G} = \{f_v\}_{v\in\mathbb{N}} \cup \{g_v\}_{v\in\mathbb{N}}$ of recursive functions by the following equalities:

$$f_v(x) = \begin{cases} 0, & \text{if } x \in E_v, \\ 1, & \text{otherwise,} \end{cases}$$

$$g_v(x) = \begin{cases} 1, & \text{if } x \in E_v, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathfrak{F} = \{f_v \mid v \in A\}$. It is easy to see that the family $\mathfrak{F}$ has not a universal recursive function, but $\mathfrak{F}$ satisfies the conditions (i) and b) from (ii).

Example 3. Let A be the set of cannonical codes of finite sets such that the family $\mathfrak{F}$ does not satisfy the condition (ii) from Theorem 1 (see the Example in [6]). We define the family of recursive finctions by the following equality:

$$f_v(x) = \begin{cases} 0, & \text{if } x \in E_v, \\ 1, & \text{otherwise.} \end{cases}$$

Then let $\mathfrak{F} = \{f_v \mid v \in A\}$. It is not difficult to see that the family $\mathfrak{F}$ has not a universal recursive function, but $\mathfrak{F}$ satisfies the conditions (i) and a) from (ii).

For the next theorem we need to recall a definition. The total function h is said to be a *boundary function* for the family $\mathfrak{F}$ iff for every finite subfunction θ of h there exists a function $f \in \mathfrak{F}$ such that θ is a subfunction of f. If $\mathfrak{F}$ is a family of unary recursive functions and a function h exists such that h is a boundary, we say that $\mathfrak{F}$ has a boundary function, otherwise we say that $\mathfrak{F}$ does not have a boundary function.

Theorem 5. *Let $\mathfrak{F}$ be a nonempty family of unary recursive functions which has not a boundary function. Then the family $\mathfrak{F}$ has a universal recursive function iff the following two conditions hold*:

(i) *The set $V = \{v \mid \exists f(f \in \mathfrak{F} \,\&\, \theta_v \subseteq f)\}$ is recursively enumerable*;

(ii) *There exist a family $\mathfrak{G}$ such that $\mathfrak{F} \subseteq \mathfrak{G}$ and a recursive function Ψ, which is a universal for the family $\mathfrak{G}$, and the set $I = \{n \mid \Psi_n \in \mathfrak{F}\}$ is Σ^0_2 (in the arithmetical hierarchy)*.

Proof. The first part follows from Theorem 4. Let F be a unary recursive function such that $F(0) = 1$ and $Ran(F) = V$. Let in addition a family $\mathfrak{G}$ exist such that $\mathfrak{F} \subseteq \mathfrak{G}$ and a recursive function Ψ exists, which is a universal for the family $\mathfrak{G}$, and $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be a unary recursive function such that for all n $\theta_{\alpha(n,0)} \subseteq \theta_{\alpha(n,1)} \subseteq \cdots \subseteq \theta_{\alpha(n,s)} \subseteq \cdots$ (again for the sake of simplicity we consider that $Dom(\theta_{\alpha(n,s)})$ is an initial segment for all n and s), and for all n and x $\lim_s \theta_{\alpha(n,s)}(x) = \Psi(n,x)$ and G be a ternary total recursive function such that

$$n \in I \iff \exists x \forall y [G(n,x,y) = 0].$$

We construct again a strong recursive sequence of finite functions by steps, analogously to Theorem 4. Let us describe the construction.

Step $s = 0$. Take $g(n,x,s) = 1$, i.e. $\theta_{g(n,x,s)} = \varnothing$.

Step $s > 0$. We consider two cases:

Case I. $\forall y [y \leq s \iff G(n,x,y) = 0]$.

Take $g(n,x,s) = F(t_{n,x,s})$, where $t_{n,x,s} \leq s$ is such that $Dom(\theta_{F(t_{n,x,s})})$ is the maximal initial segment in the set $\{Dom(\theta_{F(0)}), \ldots, Dom(\theta_{F(s)})\}$ such that $\theta_{F(t_{n,x,s})}$ is a subfunction of the function Ψ_n.

Case II. $\exists y [y \leq s \,\&\, G(n,x,y) \neq 0]$. Take $g(n,x,s) = F(t_{n,x,s})$, where $t_{n,x,s}$ is such that $Dom(\theta_{F(t_{n,x,s})})$ is an initial segment, $\theta_{g(n,x,s-1)} \subseteq \theta_{F(t_{n,x,s})}$ and $\theta_{F(t_{n,x,s})}$ is the maximal element of the set $\{\theta_{F(0)}, \ldots, \theta_{F(s)}\}$.

Thus the construction is completed. It is effective, so the function g is recursive.

Analogously, we shall see that for all fixed n, x and z the limit $\lim_s \theta_{g(n,x,s)}(z)$ exists and belongs to the family $\mathfrak{F}$. For fixed n, x and z we have to consider two cases, but the first is the same as in Theorem 4, so we shall consider only the second case.

Case II. $\exists y [G(n,x,y) \neq 0]$. From the construction it is clear that for all s $\theta_{g(n,x,s-1)} \subseteq \theta_{g(n,x,s)}$. Therefore, the limit $\lim_s \theta_{g(n,x,s)}(z)$ exists. Let us suppose that for some $z \in \mathbb{N}$ the limit $\lim_s \theta_{g(n,x,s)}(z)$ is undefined. Then there exists s_0 such that for all $s \geq s_0$ $\theta_{g(n,x,s)} = \theta_{g(n,x,s_0)}$. On the other hand, $\theta_{F(t_{n,x,s_0})} \subseteq f \in \mathfrak{F}$ for some f. Since for all natural k a finite functions $\theta \subseteq f$ exist such that $Dom(\theta) = N_{k+1}$ and $\theta \subseteq f$, then for all such θ there exists s such that $\theta_{F(s)} = \theta$. It is clear now that $\theta_{g(n,x,s)}(z)$ is defined for some s, which contradicts the supposition that the limit $\lim_s \theta_{g(n,x,s)}(z)$ is undefined. Therefore for all $z \in \mathbb{N}$ the limit $\lim_s \theta_{g(n,x,s)}(z)$ is defined.

Assume now that $\lim_s \theta_{g(n,x,s)}$ does not belong to the family $\mathfrak{F}$. Then according to the construction, natural numbers s_1, s_2, $\ldots$ exist such that $\theta_{g(n,x,s_1)} \subset \theta_{g(n,x,s_2)} \subset \ldots$ and $\theta_{g(n,x,s_i)} \subseteq \Psi_{n_i}$ for all i. This means that $\lim_s \theta_{g(n,x,s)}$ is a boundary function for the family $\mathfrak{F}$.

The proof that if $f \in \mathfrak{F}$, then $f = \Psi_n$ for some $n \in I$, is the same as in Theorem 4.

At the end, let the function Θ be defined as in Theorem 4:

$$\Theta(k,z) = \begin{cases} \lim_s \theta_{g(n,x,s)}(z), & \text{if } k = \langle n, x \rangle, \\ f_0(z), & \text{otherwise,} \end{cases}$$

where f_0 is a fixed element of the family $\mathfrak{F}$. The theorem is proved.

Let us note that if a family of (total) recursive functions is finitely generated by some effective operations, then the family has a universal (total) recursive function.

At the end, we shall note the following

Proposition 6. *If the family $\mathfrak{F}$ of unary total recursive functions has a universal recursive function, then there exists a family $\mathfrak{G}$ such that $\mathfrak{F} \subseteq \mathfrak{G}$ and the family $\mathfrak{G}$ is finitely generated by some effective operations.*

Proof. Indeed, let Θ be universal for the family $\mathfrak{F}$. Fix the functions **I**, **S**, **O** and f defined as follows:

$\mathbf{I}(x) = x$, $\mathbf{S}(x) = x + 1$, $\mathbf{O}(x) = 0$ and

$$f(z) = \begin{cases} \Theta(n, x), & \text{if } z = \langle n, x \rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Let us define the binary operation $(.,.)$ between the functions as follows:

$$(f_1, f_2)(x) = \langle f_1(x), f_2(x) \rangle.$$

It is easy to show that the family $\mathfrak{G}$, which is generated from the functions **I**, **S**, **O**, f, $\lambda x.(x)_0$, $\lambda x.(x)_1$ by the operations composition and $(.,.)$, contains the family $\mathfrak{F}$.

Open problem. Given a family $\mathfrak{F}$ of unary total recursive functions, which has a universal recursive function, is it true that the family $\mathfrak{F}$ is generated from finitely many functions belonging to $\mathfrak{F}$ by a finite number of effective operations?

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