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SYMMETRIES OF THE HERGLOTZ VARIATIONAL PRINCIPLE IN THE CASE OF ONE INDEPENDENT VARIABLE[1]

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This paper provides a method for calculating the symmetry groups of the functional defined by the generalized variational principle of Herglotz in the case of one independent variable. A variational description is found for several named ordinary differential equations. Variational symmetry groups are calculated for a Liouville's equation and a Lane-Emden equation.

Keywords: variational symmeties, Herglotz variational principle, invariant functional, Herglotz

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1. INTRODUCTION

It is well known that a variational description of a differential equation or a system of such equations is very desirable both from mathematical and from physical point of view. The classical variational principle, although far-reaching and very powerful, can not describe many important differential equations. In 1930 Gustav Herglotz proposed a *generalized variational principle* with one independent variable, which generalizes the classical variational principle by defining the functional, whose extrema are sought, by a certain differential equation, see Herglotz [7] and Guenther et al. [5]. Herglotz variational principle contains the classical variational principle as a special case. His original idea was published in 1979 in

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his collected works, Herglotz [6] and [8]. This variational principle can describe not only all physical processes with one independent variable which the classical variational principle can, but also many others for which the classical variational principle is not applicable. For example, it can give a variational description of nonconservative processes even when the Lagrangian is not dependent on time, something which can not be done with the classical variational principle. It is also related to contact transformations.

The generalized variational principle of Herglotz defines the functional z, whose extrema are sought, by the differential equation

$$\frac{dz}{dt} = L\Big(t, x(t), \frac{dx(t)}{dt}, z\Big) \tag{1.1}$$

where t is the independent variable, and $x(t) \equiv (x_1(t), \ldots, x_n(t))$ stands for the argument functions. In order for the equation (1.1) to define a functional $z = z[x]$ of $x(t)$ equation (1.1) must be solved with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and the solution $z(t)$ must be evaluated at the same fixed final time $t = T$ for all argument functions $x(t)$.

The equations whose solutions produce the extrema of this functional are

$$\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k} = 0\ , \qquad k = 1, \ldots, n\ , \tag{1.2}$$

where $\dot{x}_k$ denotes dx_k/dt. Herglotz called them the *generalized Euler-Lagrange equations.* See Guenther et al. [5] for a derivation of this system. The solutions of these equations, when written in terms of the dependent variables x_k and the associated momenta $p_k = \partial L/\partial \dot{x}_k$, determine a family of *contact transformations.* See Guenther et al. [5].

For equations which can be obtained from Herglotz variational principle as (1.2) one can systematically derive conserved quantities, as shown in Georgieva et al. [2], by applying the first Noether-type theorem formulated and proven in the same paper. For convenience of the reader we state this result here. Consider the one-parameter group of transformations

$$\bar{t} = \phi(t, x, \varepsilon), \tag{1.3}$$

$$\bar{x}_k = \psi_k(t, x, \varepsilon), \qquad k = 1, ..., n$$

where ε is the parameter, $\phi(t, x, 0) = t$, and $\psi_k(t, x, 0) = x_k$, with infinitesimal generator

$$\mathbf{v} = \tau(t, x)\frac{\partial}{\partial t} + \xi_k(t, x)\frac{\partial}{\partial x_k}$$

where

$$\tau(t, x) = \frac{d\phi}{d\varepsilon}\Big|_{\varepsilon=0} \text{and}\ \ \xi_k(t, x) = \frac{d\psi_k}{d\varepsilon}\Big|_{\varepsilon=0}\ . \tag{1.4}$$

Throughout this paper we assume that the summation convention on repeated indices holds and that $\cdot$ denotes differentiation with respect to t.

Theorem 1.1 (First Noether-type theorem for the generalized variational principle)**.** *If the functional* $z = z[x(t)]$ *defined by the differential equation* $\dot{z} = L(t, x, \dot{x}, z)$ *is invariant under the one-parameter group of transformations* (1.3) *then the quantity*

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\right)\left(\left(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\right)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\right) \tag{1.5}$$

is conserved along the solutions of the generalized Euler-Lagrange equations (1.2).

The present paper shows how a group of transformations can be found under which the functional of a given Herglotz variational principle is invariant. The importance of this problem is that once such a group of symmetries is found, conserved quantities for the corresponding system of generalized Euler-Lagrange equations can be written down directly, applying the first Noether-type theorem. The symmetry group generators are obtained from a system of first order partial differential equations as shown in section 3. In section 2 a variational description is found for several named ordinary differential equations. Several examples of calculating variational symmetry groups, and from them the corresponding first integrals, are given in section 4.

The interested reader can find a generalization of the Herglotz variational principle to one with several independent variables in Georgieva et al. [3]. There a theorem of Noether-type is formulated and proven, for the case of finite-dimensional symmetry groups of the functional, and applications are given. That paper also contains a proposition characterizing the variational symmetry groups of differential equations describing physical fields.

Historically, the question of calculating the symmetries of a given Lagrangian functional was answered by W. Killing [9] in 1892 in the context of describing the motions of a n-dimensional manifold of fundamental form given by

$$L = \frac{1}{2}g_{kl}\dot{x}^k\dot{x}^l$$

(see Eisenhart [1] and Logan [10]). In the case of a classical variational functional, some authors refer to the system of partial differential equations for the unknown symmetry group generators as the *generalized Killing equations.* For the derivation of these equations in the case of the classical variational principle, see Logan [10].

2. VARIATIONAL DESCRIPTION VIA HERGLOTZ VARIATIONAL PRINCIPLE

In this section we use the generalized variational principle of Herglotz to give a variational description of several named ordinary differential equations.

First we show that the class of ordinary differential equations

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} + h(x) = 0 \tag{2.1}$$

for the function $x = x(t)$ can be given a variational description via the Herglotz variational principle, by letting L in the defining equation (1.1) be

$$L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z - U(x) \tag{2.2}$$

where $U(x)$ is any solution of the ODE

$$\frac{dU(x)}{dx} + 2f(x)U(x) = h(x).$$

Indeed,

$$\begin{aligned}
&\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} = \\
&\quad = -\frac{dU}{dx} - 2\dot{x}\frac{df}{dx}z - \frac{d}{dt}\Big(\dot{x} - 2fz\Big) - (2f\dot{x} + g)(\dot{x} - 2fz) \\
&\quad = -\frac{dU}{dx} - \ddot{x} + 2f\Big(\frac{1}{2}\dot{x}^2 - 2f\dot{x}z - gz - U\Big) - 2f\dot{x}^2 + 4f^2\dot{x}z - \dot{x}g + 2fgz \\
&\quad = -\ddot{x} - f\dot{x}^2 - g\dot{x} - 2fU - \frac{dU}{dx} = -(\ddot{x} + f\dot{x}^2 + g\dot{x} + h).
\end{aligned}$$

Equation (2.1) contains several well known named equations as special cases:
a. When $h(x) = kx$, with $k = constant$, $f(x) = 0$ and $g(t) = a = constant$, (2.1) is the equation of the damped harmonic oscillator

$$\ddot{x} + a\dot{x} + kx = 0. \tag{2.3}$$

The corresponding Lagrangian is

$$L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az. \tag{2.4}$$

b. In the special case when $h(x) = kx$, $k = constant$ and $f(x) = 0$, equation (2.1) becomes the Lienard's equation

$$\ddot{x} + g(t)\dot{x} + kx = 0. \tag{2.5}$$

The corresponding Lagrangian is

$$L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - g(t)z. \tag{2.6}$$

c. In the case when $h(x) = x^n$, $f(x) = 0$ and $g(t) = 2/t$, equation (2.1) becomes the Lane-Emden equation

$$\ddot{x} + \frac{2}{t}\dot{x} + x^n = 0 \,, \quad n \neq -1. \tag{2.7}$$

In that case the Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 - \frac{x^{n+1}}{n+1} - \frac{2}{t}z. \tag{2.8}$$

d. As a final example consider the special case when $h(x) = 0$. Then equation (2.1) is the Liouville's equation

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0 \ . \tag{2.9}$$

The Lagrangian for it is

$$L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z. \tag{2.10}$$

3. THE FUNDAMENTAL INVARIANCE IDENTITY AND THE GENERALIZED KILLING EQUATIONS

Let the functional z be defined by the ordinary differential equation (1.1). Consider the one-parameter group of transformations (1.3) with the coefficients of its generator given in (1.4).

Lemma 3.1. *The following identity holds*

$$\frac{d}{d\varepsilon}\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0} = \frac{d\xi_k}{dt} - \dot{x}_k\frac{d\tau}{dt} \ , \tag{3.1}$$

provided $\bar{x}_k$ *and* $\bar{t}$ *are defined by the transformation*

$$\bar{t} = t + \tau(t,x)\,\varepsilon$$

$$\bar{x}_k = x_k + \xi_k(t,x)\,\varepsilon \ , \tag{3.2}$$

corresponding to (1.3) *and* (1.4).

Proof. We have

$$\frac{d\bar{x}_k}{dt} \equiv \frac{\partial\bar{x}_k}{\partial t} + \frac{\partial\bar{x}_k}{\partial x_h}\dot{x}_h = \frac{d\bar{x}_k}{d\bar{t}}\frac{d\bar{t}}{dt} \equiv \frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{\partial\bar{t}}{\partial t} + \frac{\partial\bar{t}}{\partial x_h}\dot{x}_h\Big). \tag{3.3}$$

Setting $\varepsilon = 0$ yields

$$\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0} = \delta^k_h\dot{x}_h = \dot{x}_k. \tag{3.4}$$

Differentiate (3.3) with respect to ε and expand both sides

$$\frac{d}{d\varepsilon}\frac{\partial\bar{x}_k}{\partial t} + \dot{x}_h\frac{d}{d\varepsilon}\frac{\partial\bar{x}_k}{\partial x_h} = \frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{d}{d\varepsilon}\frac{\partial\bar{t}}{\partial t} + \dot{x}_h\frac{d}{d\varepsilon}\frac{\partial\bar{t}}{\partial x_h}\Big) + \Big(\frac{\partial\bar{t}}{\partial t} + \frac{\partial\bar{t}}{\partial x_h}\dot{x}_h\Big)\frac{d}{d\varepsilon}\frac{d\bar{x}_k}{d\bar{t}}. \tag{3.5}$$

We set $\varepsilon = 0$ in this equation, substitute in it (3.4) and use the following relations:

$$\frac{d}{d\varepsilon}\frac{\partial \bar{x}_k}{\partial t}\Big|_{\varepsilon=0} = \frac{\partial \xi_k}{\partial t}\,, \quad \frac{d}{d\varepsilon}\frac{\partial \bar{x}_k}{\partial x_h}\Big|_{\varepsilon=0} = \frac{\partial \xi_k}{\partial x_h}\,, \quad \frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial t}\,, \quad \frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial x_h}\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial x_h}$$

$$\frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0} = 1\,, \qquad \frac{\partial \bar{t}}{\partial x_h}\Big|_{\varepsilon=0} = 0.$$

Then equation (3.5) becomes

$$\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial x_h}\dot{x}_h = \dot{x}_k\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x_h}\dot{x}_h\Big) + \frac{d}{d\varepsilon}\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0}.$$

Observe that the total derivatives of ξ_k and τ, with respect to t, appear in the last equation. Solving for the last term in it yields the statement of the lemma. □

The following theorem gives a method for finding symmetry groups of the Herglotz functional.

Theorem 3.2. *The coefficients $\tau(t,x)$ and $\xi_k(t,x)$ of the infinitesimal generator of a one-parameter group of transformations which preserve the value of the functional $z = z[x(t)]$, defined by the differential equation* (1.1)*, are solutions of the system of partial differential equations obtained from the identity*

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial x_j}\dot{x}_j - \dot{x}_k\frac{\partial \tau}{\partial t} - \dot{x}_k\dot{x}_j\frac{\partial \tau}{\partial x_j}\Big) + L\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x_j}\dot{x}_j\Big) = 0 \quad (3.6)$$

by equating to zero the coefficients of the powers of z and $\dot{x}_k$ and of products of such powers.

In analogy with the classical case, we call this identity the *fundamental invariance identity* and the resulting partial differential equations for the coefficients of the infinitesimal generator of the variational symmetry group the *generalized Killing equations.*

Proof. Apply the transformation (3.2)

$$\bar{t} = t + \tau(t,x)\,\varepsilon$$

$$\bar{x}_k = x_k + \xi_k(t,x)\,\varepsilon\,,$$

corresponding to (1.3) and (1.4), to the differential equation (1.1). The result is the defining equation

$$\frac{d\bar{z}}{d\bar{t}} = L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big)$$

for the transformed functional $\bar{z} = \bar{z}[\bar{x}(\bar{t})]$. Since $d\bar{z}/d\bar{t} = (d\bar{z}/dt)\,(dt/d\bar{t}\,)$, we have

$$\frac{d\bar{z}}{dt} = L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big)\frac{d\bar{t}}{dt}. \quad (3.7)$$

Differentiate (3.7) with respect to ε and set $\varepsilon = 0$ to obtain

$$\frac{d}{d\varepsilon}\Big(\frac{d\bar{z}}{dt}\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\Big(\frac{d\bar{z}}{d\varepsilon}\Big)\Big|_{\varepsilon=0} = \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0}\,\frac{d\bar{t}}{dt}\Big|_{\varepsilon=0} + L\frac{d}{d\varepsilon}\Big(\frac{d\bar{t}}{dt}\Big)\Big|_{\varepsilon=0}. \tag{3.8}$$

From $\phi(t, x, 0) = t$ it follows that

$$\frac{d\bar{t}}{dt}\Big|_{\varepsilon=0} = 1.$$

Similarly, we have

$$\frac{d}{d\varepsilon}\Big(\frac{d\bar{t}}{dt}\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\Big(\frac{d}{d\varepsilon}\phi(t, x, \varepsilon)\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\tau(t, x).$$

Denote by $\zeta = \zeta(t)$ the total variation of the functional $z = z[x]$ produced by the transformation (3.2), i.e.

$$\zeta(t) = \frac{d}{d\varepsilon}z[x;\varepsilon]\Big|_{\varepsilon=0}.$$

Thus, equation (3.8) becomes

$$\frac{d\zeta}{dt} = \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0} + L\frac{d\tau}{dt}.$$

Expanding the derivative $dL/d\varepsilon$ and setting $\varepsilon = 0$, we obtain

$$\frac{d\zeta}{dt} = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big)\Big|_{\varepsilon=0} + \frac{\partial L}{\partial z}\zeta + L\frac{d\tau}{dt}. \tag{3.9}$$

We now use the assertion of lemma 3.1 and insert expression (3.1) in equation (3.9) to obtain the linear differential equation

$$\frac{d\zeta}{dt} = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{d\xi_k}{dt} - \dot{x}_k\frac{d\tau}{dt}\Big) + L\frac{d\tau}{dt} + \frac{\partial L}{\partial z}\zeta \tag{3.10}$$

for the variation ζ of the functional z. For clarity we denote by $A(s)$ the expression

$$A(s) = \frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{d\xi_k}{ds} - \dot{x}_k\frac{d\tau}{ds}\Big) + L\frac{d\tau}{ds}.$$

The solution $\zeta(t)$ of (3.10) is given by

$$\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big)\zeta - \zeta(0) = \int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\Big)A(s)\;ds.$$

Notice that $\zeta(0) = 0$. Indeed, as explained earlier, in order to have a well-defined functional z of $x(t)$, we must evaluate the solution $z(t)$ of the equation (1.1) with the same fixed initial condition $z(0)$, independently of the function $x(t)$. Then $z(0)$ is independent of ε. Hence, the variation of z evaluated at $t = 0$ is zero.

Since by hypothesis the one-parameter group of transformations (1.3) leaves the functional $z = z[x(t)]$ stationary, we have $\zeta(t) = 0$. Thus, it follows that

$$\int_0^t \exp\left(-\int_0^s \frac{\partial L}{\partial z}\, d\theta\right) A(s)\ ds = 0. \tag{3.11}$$

Taking in consideration the fact that equation (3.11) is valid for all values of t, and that the exponent is always positive, we obtain the identity $A(t) = 0$ which, after writing the total derivatives explicitly, becomes (3.6).

Equation (3.6) is an identity in (t, x_k) for arbitrary directional arguments $\dot{x}_k$. Therefore, we can regard this identity as a set of partial differential equations in the unknowns τ and ξ_k . Due to the arbitrariness of $\dot{x}_k$ and the fact that z depends on $\dot{x}_k$, we can further reduce (3.6) to obtain a system of first order partial differential equations in τ and ξ_k, by equating to zero the coefficients of the powers of $\dot{x}_k$, the powers of z, as well as the coefficients of products of such powers. The solution of this system, if it exists, determines a group of transformations under which the functional defined by equation (1.1) is invariant. □

4. APPLICATIONS

In this section we calculate variational symmetries of several ordinary differential equations and use the first Noether-type theorem 1.1 to find the corresponding conserved quantities.

We start with the equation for the damped harmonic oscillator $\ddot{x}+a\dot{x}+kx = 0$, where a and k are constants. The Lagrangian is $L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az$. The fundamental invariance identity (3.6) of theorem 3.2 assumes the form

$$-kx\xi + \dot{x}\Big(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\dot{x} - \dot{x}\frac{\partial \tau}{\partial t} - \dot{x}^2\frac{\partial \tau}{\partial x}\Big) + \Big(\frac{1}{2}\dot{x}^2 - \frac{k}{2}x^2 - az\Big)\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x}\dot{x}\Big) = 0.$$

The system of partial differential equations for τ and ξ is obtained by equating to zero the coefficients in front of z, $\dot{x}z$, and powers of $\dot{x}$. The solution of this system is $\tau =$ *constant* and $\xi = 0$. Without loss of generality, we take $\tau = 1$. Applying the first Noether-type theorem 1.1 we obtain the corresponding conserved quantity

$$Q = -e^{at}\Big(\frac{1}{2}\big(\dot{x}^2 + kx^2\big) + az\Big).$$

It will be nice to express the conserved quantity in terms of x and $\dot{x}$ only, and for this we need to express z in terms of x and $\dot{x}$. The reader can check that $z = \frac{1}{2}x\dot{x}$ satisfies the defining equation for z, namely $\dot{z} = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az$, with x being a solution of the damped harmonic oscillator. Thus, the conserved quantity is

$$e^{at}\Big(\dot{x}^2 + ax\dot{x} + kx^2\Big) = constant. \tag{4.1}$$

This method produces no non-trivial variational symmetries of the Lienard's equation (2.5), except in the case when $g(t) = constant$, which is the case of the damped harmonic oscillator presented above.

As another application we calculate a variational symmetry group of the Liouville's equation $\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0$ with a specific choice of the coefficient functions, namely

$$f(x) = \frac{h}{kx + a} \qquad g(t) = \frac{c}{2kt + b} \tag{4.2}$$

where a, b, c, h, and k are arbitrary constants (if $k = 0$ then a and b must be non-zero). As noted in section 1, this equation can be given a variational description via the Herglotz variational principle if the functional z is defined by the differential equation

$$\dot{z} = \frac{1}{2}\dot{x}^2 - (2f(x)\dot{x} + g(t))z.$$

The fundamental invariance identity (3.6) takes the form

$$-\frac{dg}{dt}z\tau - 2\frac{df}{dx}\dot{x}z\xi + (\dot{x} - 2f(x)z)\Big(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\Big) +$$
$$+\Big(\frac{1}{2}\dot{x}^2 - 2f(x)\dot{x}z - g(t)z\Big)\Big(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\dot{x}\Big) = 0.$$

With the specific choices (4.2) for $f(x)$ and $g(t)$ the system of partial differential equations obtained from this identity after equating to zero the proper coefficients has the solutions $\xi = kx + a$ and $\tau = 2kt + b$. Thus, the variational symmetry of the Liouville's equation produced by this method is

$$\bar{x} = x + (kx + a)\varepsilon\ , \quad \bar{t} = t + (2kt + b)\varepsilon. \tag{4.3}$$

The corresponding conserved quantity of the Liouville's equation is obtained through an application of theorem 1.1, and is

$$Q = \Big(\frac{kx(t) + a}{kx(0) + a}\Big)^{2h/k}\Big(\frac{2kt + b}{b}\Big)^{c/2k}\Big(\dot{x}(kx + a) - (2kt + b)\frac{\dot{x}^2}{2} - (2h + c)z\Big). \tag{4.4}$$

As a last application, we calculate a variational symmetry for the equation

$$\ddot{x} + \frac{2}{t}\dot{x} + \frac{1}{x^3} = 0\ .$$

In this case the functional z is defined by the equation

$$\dot{z} = \frac{1}{2}\dot{x}^2 + \frac{1}{2x^2} - \frac{2}{t}z\ ,$$

and the fundamental invariance identity (3.6) assumes the form

$$2\frac{1}{t^2}z\tau - \frac{1}{x^3}\xi + \dot{x}\Big(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\Big)$$

$$+\Big(\frac{1}{2}\dot{x}^2+\frac{1}{2x^2}-\frac{2}{t}z\Big)\Big(\frac{\partial\tau}{\partial t}+\frac{\partial\tau}{\partial x}\dot{x}\Big)=0.$$

The system of PDE's for the coefficients $\tau(t,x)$ and $\xi(t,x)$ of the infinitesimal generator of the variational symmetry group has the solution $\tau = 2kt\ \xi = kx$, where k is an arbitrary constant. The corresponding conserved quantity is

$$Q=-k\frac{t^2}{e^2}\Big(\Big(\dot{x}^2-\frac{1}{x^2}\Big)t-x\dot{x}+4z\Big). \tag{4.5}$$

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