ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика и механика
Том 91, 1997

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques et Mecanique
Tome 91, 1997

SINGULARLY PERTURBED DELAYED DIFFERENTIAL INCLUSIONS

TZANKO D. DONCHEV, IORDAN I. SLAVOV

The paper deals with singularly perturbed differential inclusions with time lag. The limit behaviour of the solution set when the singular parameter tends to zero is investigated. The limits of the fast solutions are considered as Radon probability measures. Then the upper semicontinuity of the solution set with respect to uniform convergence of the slow motions and to weak probability convergence of the fast motions is examined.

Keywords: singular perturbation, delayed differential inclusions, Radon measures
1991/95 Math. Subject Classification: main 34A60, 34K99, secondary 34D15

1. INTRODUCTION

The paper deals with singularly perturbed differential inclusions with time lag, having the form

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{y}(t) \end{pmatrix} \in H(t, x_t, y_t), \quad x_0 = \varphi, y_0 = \psi, \tag{1}$$

where $x \in \mathbf{R}^n, y \in \mathbf{R}^m, t \in I \overset{\text{def}}{=} [0,1]$ and $\varepsilon > 0$ represents the singular perturbation. For any $z : [-\tau, 1] \to \mathbf{R}^k$ and $t \in [0,1]$ we let $z_t : [-\tau, 0] \to \mathbf{R}^k$ be defined by $z_t(s) = z(t+s)$, $-\tau \le s \le 0$. Here $\tau > 0$, H is a set-valued map from $I \times C([-\tau,0], \mathbf{R}^n) \times L^1([-\tau,0], \mathbf{R}^m)$ into $\mathbf{R}^{n+m}$ and $\varphi \in C([-\tau,0], \mathbf{R}^n), \psi \in C([-\tau,0], \mathbf{R}^m)$, where C and L^p, $1 \le p < \infty$, are the usual spaces of respectively continuous (equipped with the uniform norm) and p-integrable functions.

The limit behavior of the solution set when the small parameter ε tends to zero is investigated here. In the literature there are mainly three ways to deal with the problem.

1. *Reduction.* In this case we consider solution set $Z(\varepsilon)$, $\varepsilon > 0$, of (1) consisting of all AC (absolutely continuous) functions (x, y) satisfying (1) for a.e. $t \in I$. For $\varepsilon = 0^+$ it is natural to mean by $Z(0)$ the set of all pairs (x, y), with x-AC and y-integrable on I, satisfying for a.e. $t \in I$ the "degenerate" inclusion

$$\begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix} \in H(t, x_t, y_t), \quad x_0 = \varphi, \; y_0 = \psi. \tag{2}$$

The connection between the inclusions (1) and (2) has been investigated in many papers when they are *ordinary* — [4, 7, 8, 13, 15, 16]. The LSC (lower semicontinuity) is proved first in [15] in the ordinary differential case and afterwards for more general systems in [5, 6]. The topology considered is $C \times L^2$. However, to prove the USC (upper semicontinuity) in this topology, one has to "expand" in some sense the set $Z(0)$, but then the LSC will be no longer valid. It is easy to prove USC in the weaker $C \times (L^2\!-\!weak)$ topology but under restrictive conditions. It was done in [4], where the first result concerning "reduction" technique for nonlinear differential inclusions is published.

Considering more general functional-differential inclusion than (1), namely

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in H(t, x, y, x_t, y_t), \quad x_0 = \varphi, \; y_0 = \psi, \tag{3}$$

we proved in [5] under one-sided Lipschitz condition the USC of $Z(\varepsilon)$ at $\varepsilon = 0^+$ in $C \times (L^2\!-\!weak)$ topology. However, generally we do not have LSC. Making restrictive assumptions concerning the dependence of the right-hand side of (3) on y, we get in [5, 6] LSC in some partial cases.

2. *Averaging.* This approach is used mainly for systems in the form

$$\begin{aligned} &\dot{x}(t) \in F(t, x, y, u(t)), \quad x(0) = x^0, \\ &\varepsilon\dot{y}(t) \in G(x, y, u(t)), \quad y(0) = y^0. \end{aligned} \tag{4}$$

Here $u(t) \in U$ (U — compact subset), and $u(\cdot)$ plays the role of a control.

Fix x and consider the following *associated* system:

$$\begin{aligned} &x = \text{const}, \\ &\dot{y}(\tau) \in G(x, y(\tau), u(\tau)), \quad y(0) \in Q \subset \mathbf{R}^m, \quad u(\tau) \in U, \quad \tau \geq 0. \end{aligned} \tag{5}$$

For given x and t the Aumann's integral

$$\bar{V}(t, x, S, Q) = \mathrm{cl}\left\{ \frac{1}{S} \int_0^S F(t, x, Y(\tau, x, S, Q), u(\tau))\, d\tau : u(\tau) \in U \right\},$$

where $Y(\tau, x, S, Q)$ is the solution set on the interval $[0, S]$ of (5) and "cl" denotes the closed hull, possesses a limit

$$\bar{V}(t,x) = \lim_{S\to\infty} \bar{V}(t,x,S,Q)$$

when certain conditions are met. Then it can be shown, see, e.g. [10, 11], that the "slow part", i.e. the projection of $Z(\varepsilon)$ on $\mathbf{R}^n$, converges in the C–topology to the solution set of the averaged inclusion

$$\dot{x}(t) \in \bar{V}(t,x), \quad x(0) = x^0, \quad t \in I.$$

Some other averaging results are obtained in [9, 12].

In the forthcoming paper [7] we combine the averaging technique with the notion of generalized solutions (introduced via Radon probability measures over a compact set K containing all "fast" solutions) and obtain that $Z(\varepsilon)$ has a limit at $\varepsilon = 0^+$ in $C \times [L^1(I, C(K))]^*-weak*$ topology.

3. *Invariant measures.* The fundamental theorem of Tikhonov [14] states that for single-valued H depending on (x, y) instead of (x_t, y_t), i.e. $H \equiv H(t, x, y)$, under appropriate conditions the unique solution of (1) converges as $\varepsilon \to 0$ to a special solution of (2) in $C(I, \mathbf{R}^n) \times C([\delta, 1], \mathbf{R}^m)$ for every $0 < \delta < 1$.

Its recent generalizations for systems of ordinary differential equations and control systems are done in [1, 2, 17]. They are based on the identification of the limits of the fast solutions y_ε with invariant measures of the associated system. The convergence in y_ε is in some statistical sense, while the slow part converges to a solution of specially defined "reduced" system.

We finish the introduction with some notations and definitions. For $A \subset \mathbf{R}^{n+m}$, we denote by $\hat{A}$ the projection of A on $\mathbf{R}^n$ and by $\tilde{A}$ the projection of A on $\mathbf{R}^m$. Throughout the paper $\langle \cdot, \cdot \rangle$ is the scalar product, $|\cdot|$ is the norm. For a set A denote by $\sigma(x, A) := \sup_{y\in A} \langle x, y \rangle$ its support function and by $D_H(A, B)$ the Hausdorff distance between the sets A, B.

The multifunction F from the space X into the space Y is said to be U(pper) S(emi) C(ontinuous) (L(ower)S(emi)C(ontinuous)) at $x \in X$ when to every open $V \supset F(x)$ ($V \bigcap F(x) \neq \emptyset$) there exists a neighbourhood $W \ni x$ such that $V \supset F(y)$ ($V \bigcap F(y) \neq \emptyset$) for $y \in W$. All the concepts non-discussed in details in the sequel can be found in [3] or [18].

2. THE RESULTS

Suppose that:

A1. The map H is compact, convex valued, bounded on the bounded sets. Also $H(\cdot, \alpha, \beta)$ is measurable and $H(t, \cdot, \cdot)$ is USC.

A2. There exist constants $a, b, \mu > 0$ such that for every $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and a.e. $t \in I$:

$$\begin{aligned}\sigma(x, \hat{H}(t, \alpha, \beta)) &\le a(1 + |\alpha(0)|^2 + \|\beta\|_C^2), \quad \alpha \in \Omega_1, \; \beta \in C([-\tau, 0], \mathbf{R}^m),\\ \sigma(y, \tilde{H}(t, \alpha, \beta)) &\le b(1 + \|\alpha\|_C^2) - \mu|\beta(0)|^2, \quad \alpha \in C([-\tau, 0], \mathbf{R}^n), \; \beta \in \Omega_2.\end{aligned}$$

Here

$$\Omega_1 = \left\{ \alpha \in C([-\tau, 0], \mathbf{R}^n) : |\alpha(0)| = \|\alpha\|_C = \max_{-\tau \le s \le 0} |\alpha(s)| \right\},$$

$$\Omega_2 = \left\{ \beta \in C([-\tau, 0], \mathbf{R}^m) : |\beta(0)| = \|\beta\|_C = \max_{-\tau \le s \le 0} |\beta(s)| \right\}$$

and $\alpha(0) = x, \beta(0) = y$.

First, we prove the following lemma:

Lemma 1. *There exist constants $N_x, N_y, L > 0$ such that*

$$\|x^\varepsilon\|_C \le N_x, \quad \|y^\varepsilon\|_C \le N_y, \quad |H(t, x_t^\varepsilon, y_t^\varepsilon)| \le L$$

for every $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon), \varepsilon > 0$ and $t \in I$.

Proof. Let $\varepsilon > 0$ be given and let $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$. Denote

$$p(t) = \max_{-\tau \le s \le 0} |x^\varepsilon(t+s)|^2, \quad q(t) = \max_{-\tau \le s \le 0} |y^\varepsilon(t+s)|^2.$$

From A2 it follows that

$$\begin{aligned}\langle x^\varepsilon(t), \dot{x}^\varepsilon(t)\rangle &\le \sigma(x^\varepsilon(t), \hat{H}(t, x_t^\varepsilon, y_t^\varepsilon))\\ &\le a(1 + |x^\varepsilon(t)|^2 + \|y_t^\varepsilon\|_C^2)\end{aligned}$$

when $|x^\varepsilon(t)| = \|x_t^\varepsilon\|_C := \max_{-\tau \le s \le 0} |x^\varepsilon(t+s)|$, and

$$\begin{aligned}\langle y^\varepsilon(t), \varepsilon\dot{y}^\varepsilon(t)\rangle &\le \sigma(y^\varepsilon(t), \tilde{H}(t, x_t^\varepsilon, y_t^\varepsilon))\\ &\le b(1 + \|x_t^\varepsilon\|_C^2) - \mu|y^\varepsilon(t)|^2\end{aligned}$$

when $|y^\varepsilon(t)| = \|y_t^\varepsilon\|_C := \max_{-\tau \le s \le 0} |y^\varepsilon(t+s)|$.

Obviously, $p(\cdot)$ and $q(\cdot)$ are absolutely continuous functions, hence a.e. differentiable. Then we have the following two possibilities for $p(t)$ and $q(t)$, respectively:

$$\begin{aligned}\dot{p}(t) &\le 2a(1 + p(t) + q(t)) \;\text{ or }\; \dot{p}(t) \le 0,\\ \varepsilon\dot{q}(t) &\le 2b(1 + p(t)) - 2\mu q(t) \;\text{ or }\; \dot{q}(t) \le 0,\end{aligned}$$

reasoning like in the proof of [5, Lemma 2.1]. It is not difficult to see that $p(t) \le u(t)$, $q(t) \le v(t)$, where

$$\begin{aligned}\dot{u}(t) &= 2a(1 + u(t) + v(t)), \quad u(0) = \max\left\{\|\varphi\|_C, \frac{\mu}{b}\|\psi\|_C\right\},\\ \varepsilon\dot{v}(t) &= 2b(1 + u(t)) - 2\mu v(t), \quad v(0) = \psi(0).\end{aligned}$$

By the first equation $\dot{u}(t) \geq 0$, $t \in I$, so $b(1+u(t))/\mu$ is increasing function. Then, since $v(0) \leq b(1+u(0))/\mu$, we have $v(t) \leq b(1+u(t))/\mu$, $t \in I$. Suppose the opposite, i.e. that there are $t_0 \in (0,1)$ and $\delta > 0$ such that $v(t_0) = b(1+u(t_0))/\mu$ and $v(t) > b(1+u(t))/\mu$, $t \in (t_0, t_0+\delta)$. Therefore, by the second equation of the above system, $\dot{v}(t) < 0$, $t \in (t_0, t_0+\delta)$, thus $v(t)$ decreases and

$$v(t) < v(t_0) = \frac{b}{\mu}(1+u(t_0)) \leq \frac{b}{\mu}(1+u(t)) \ \text{ for } \ t \in (t_0, t_0+\delta).$$

This is a contradiction.

Now, we get

$$\dot{u}(t) \leq 2a\left(1+u(t)+\frac{b}{\mu}(1+u(t))\right) = M(1+u(t)),$$

where $M = 2a(1+b/\mu)$. By virtue of Gronwall inequality one obtains

$$\begin{aligned} u(t) &\leq (M+u(0))\exp(M) = N_x^2, \\ v(t) &\leq \frac{b}{\mu}(1+u) \leq \frac{b}{\mu}(1+(M+u(0))\exp(M)) = \frac{b}{\mu}\left(1+N_x^2\right). \quad \square \end{aligned}$$

Remark 1. Obviously, we have that

$$N_x^2 = \exp(M)(M+u(0)), \quad N_y^2 = \frac{b}{\mu}\left(1+N_x^2\right),$$

where M and $u(0)$ are defined in the proof above. Furthermore, the boundedness for $\varepsilon = 0$ can be easily proven using Gronwall lemma.

Remark 2. We use A2 only to prove Lemma 1, so we could replace A2 by the requirement of boundedness of all solutions of (1), uniformly in $\varepsilon \geq 0$ and $t \in I$. Or we could assume A2 only locally — over the closed ball (in $\mathbf{R}^{n+m}$) with radius $\left(N_x^2+N_y^2\right)^{1/2}$ and centered at zero, which the solutions of (1) could not abandon.

We give a simple example where A2 is satisfied.

Example. Consider the following control system:

$$\begin{aligned} \dot{x}(t) &\in x_t + y_t + w(t), \quad x_0 \equiv 0, \\ \varepsilon\dot{y}(t) &\in x_t - 2f(y)\max_{-\tau \leq s \leq 0}|y(t+s)| + w(t), \quad y_0 \equiv 0, \end{aligned}$$

where $w(\cdot)$ is measurable, $w(t) \in [-1,1]$ a.e. in I, $f(0) = 0$ and $f(y) = y/|y|$, $y \neq 0$. Then, using the simple inequality $cd \leq (c^2+d^2)/2$, we get for α and β such that $\alpha(0) = x$, $\beta(0) = y$:

$$\langle x, \hat{H}(t,\alpha,\beta)\rangle = \langle\alpha(0),\alpha(\cdot)\rangle + \langle\alpha(0),\beta(\cdot)\rangle\langle\alpha(0),w(t)\rangle$$

$$
\begin{aligned}
&\leq \frac{3|\alpha(0)|^2}{2} + \frac{|\alpha(\cdot)|^2}{2} + \frac{|\beta(\cdot)|^2}{2} + \frac{|w(\cdot)|}{2} \\
&\leq 2(1 + |\alpha(0)|^2 + \|\beta\|_C^2), \\
\langle y, \tilde{H}(t,\alpha,\beta)\rangle &= \langle \beta(0), \alpha(\cdot)\rangle - 2\langle \beta(0), f(\beta(0))|\beta(0)|\rangle + \langle \beta(0), w(t)\rangle \\
&\leq 1 + \|\alpha\|_C^2 - |\beta(0)|^2 \quad \text{for} \quad \beta \in \Omega_2.
\end{aligned}
$$

Then $a = 2$, $b = \mu = 1$.

Theorem 1. *Let* A1, A2 *hold. Suppose in addition*

A3. For every $r \in \mathbf{R}^{n+m}$, $\alpha^i \to \alpha^0$ in $C([-\tau,0],\mathbf{R}^n)$, and $\beta^i \to \beta^0$ in $L^1([-\tau,0],\mathbf{R}^m)$-*weak*

$$
\limsup_{i\to\infty} \sigma\left(r, H(t,\alpha^i,\beta^i)\right) \leq \sigma\left(r, H(t,\alpha^0,\beta^0)\right).
$$

Then the map $\varepsilon \to Z(\varepsilon)$ *is upper semicontinuous at* $\varepsilon = 0^+$ *in* $C(I,\mathbf{R}^n) \times (L^1(I,\mathbf{R}^m)$-*weak*).

Proof. Suppose $\varepsilon_i \to 0$ and $(x^i, y^i) \in Z(\varepsilon_i)$ for $i = 1, 2, \ldots$ By Lemma 1 all sets $Z(\varepsilon)$, $\varepsilon \geq 0$, are contained in a $C(I,\mathbf{R}^n) \times L^1(I,\mathbf{R}^m)$—bounded set, so it is sufficient to prove that every cluster point of $\{(x^i,y^i)\}_{i=1}^{\infty}$ in $C(I,\mathbf{R}^n) \times (L^1(I,\mathbf{R}^m)$-*weak*) belongs to $Z(0)$. We denote where necessary a given sequence and its subsequences in the same way to simplify the notations.

Let (x^i, y^i) and (x_t^i, y_t^i), $i = 1, 2, \ldots$, be subsequences, converging to (x^0, y^0), respectively (x_t^0, y_t^0) in $C(I,\mathbf{R}^n)$ $\times(L^1(I,\mathbf{R}^m)$-*weak*). Obviously, $\dot{x}^i(\cdot) \to \dot{x}^0(\cdot)$ in $L^1(I,\mathbf{R}^m)$-*weak*.

Let $r \in \mathbf{R}^n$ be arbitrary. Then by A3 we have

$$
\limsup_{k\to\infty} \sigma\left(r, H(t,x_t^i,y_t^i)\right) \leq \sigma\left(r, H(t,x_t^0,y_t^0)\right) \quad \text{for a.e.} \quad t \in I
$$

and with standard arguments (see [5]) one can show that (2) is fulfilled. □

Remark 3. We note that A3 is satisfied, for example, if for fixed (t,α) the map $H(t,\alpha,\cdot)$ has convex graph.

Reformulated Theorem 1 states that if $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ is a generalized sequence of solutions of (1), then it has a subsequence converging in $C \times (L^1$-*weak*) to (x^0, y^0), where x^0 is AC, y^0 is in L^1 and

$$
\begin{pmatrix} \dot{x}^0(t) \\ 0 \end{pmatrix} \in H(t, x_t^0, y_t^0), \quad x_0^0 = \varphi,\ y_0^0 = \psi, \tag{6}
$$

for a.e. $t \in I$.

If A3 does not hold, we will not be able to claim the above result. But we will derive a close result considering the "fast" y-parts of $Z(\varepsilon)$ as measures over the *compact* set $K = \{y \in \mathbf{R}^m : |y| \leq N_y\}$ containing all y-solutions (N_y is the constant found in Lemma 1).

To this end let $\Re(K)$ be the set of all Radon probability measures on K and define the set of functions

$$\wp := \{\nu : I \to \Re(K) \mid \nu(\cdot) \text{ is measurable}\}.$$

If every point $y \in K$ is considered as the Dirac measure δ_y concentrated at the point y (i.e. $\delta_y(\{y\}) = 1$), we can represent every measurable function $y : I \to K$ as $\bar{\nu}(\cdot) = \delta_{y(\cdot)}$, which is an element of $\wp$.

Let E be the space of all Caratheodory functions $f(\cdot,\cdot)$ on $I \times K$ with values in $\mathbf{R}^m$, i.e. $f(\cdot, y)$ is measurable, $f(t,\cdot)$ is continuous and integraly bounded. Then E is isometrically isomorphic to $L^1(I, C(K, \mathbf{R}^m))$ (see [18, Theorem I.5.25]). Moreover, from Dunford-Pettis theorem [18, Theorem IV.1.8], we know that $\wp$ with the weak norm topology is isomorphic to the space $[L^1(I, C(K, \mathbf{R}^m))]^*$ equipped with the weak$*$ topology. Then $\nu^i \to \nu$ for ν^i, $\nu \in \wp$ and $i = 1, 2, \dots$ if and only if

$$\int_I \Big(\int_K f(t,y)\nu^i(t)\,(dy) \Big) dt \to \int_I \Big(\int_K f(t,y)\nu(t)\,(dy) \Big) dt \quad \text{for every } f \in E,$$

which means that $y^i(\cdot) \in L^1(I, \mathbf{R}^m)$ converges to ν in $(L^1(I, C(K, \mathbf{R}^m))^*$-weak* if and only if

$$\lim_{i\to\infty} \int_I f\left(t, y^i(t)\right)\, dt = \int_I \Big(\int_K f(t,y)\nu(t)\,(dy) \Big) dt$$

for every $f \in E$.

Theorem 2. *Let* A1 *and* A2 *be fulfilled and let* $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ *be a generalized sequence of solutions of* (1) *with* $\varepsilon \to 0$. *Then there exists a subsequence* $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ *(denoted in the same way) such that* $x^\varepsilon \to x^0$ *in* C *and* $y^\varepsilon \to \nu$ *in the weak* topology of* $[L^1(I, C(K))]^*$ *as* $\varepsilon \to 0$.

Proof. Suppose $\varepsilon \to 0$ and $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$ for every $\varepsilon > 0$. The net $\{x^\varepsilon(\cdot)\}_{\varepsilon>0}$ is $C(I, \mathbf{R}^n)$ precompact due to Lemma 1 and to Arzela-Ascoli theorem. We know that $\{y^\varepsilon(\cdot)\}_{\varepsilon>0}$ is $[L^1(I, C(K, R^m))]^*-weak^*$ precompact [18, Theorem IV.2.1]. Therefore passing to subsequences if necessary, $(x^\varepsilon, y^\varepsilon)$ converges to (x^0, ν) in considered topology, where $\nu \in \wp$. □

Obviously we have $x_t^\varepsilon \to x_t^0$ in $C([-\tau, 0], \mathbf{R}^n)$ and $y_t^\varepsilon \to \nu_t$ in $L^1([-\tau, 0], \mathcal{L})$ for every $t \in I$, where $\mathcal{L} = [L^1(I, C(K, \mathbf{R}^m))]^*-weak^*$. But more important question is to define an inclusion corresponding to (6) which is satisfied by x^0 and ν (like in [7], where ordinary differential inclusions are considered). In some partial cases it is possible.

Consider first a functional-differential inclusion with constant time lag $\tau > 0$:

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in H(t, x_t, y, y(t-\tau)), \quad x_0 = \varphi,\ y(s) = \psi(s),\ s \in [-\tau, 0]. \qquad (7)$$

Theorem 3. *Suppose the following is true*:

A1′. The map H is compact, convex valued, bounded on the bounded sets. Also H is almost continuous, i.e. for every $\delta > 0$ there exists $I_\delta \subset I$ with measure greater than $1 - \delta$ such that H is continuous on $I_\delta \times \mathbf{R}^{m+2n}$.

A2′. There exist constants $a, b, \mu > 0$ such that for every $x \in \mathbf{R}^n$ and $y, v \in \mathbf{R}^m$

$$\begin{aligned} \sigma(x, \hat{H}(t,\alpha,y,v)) &\le a(1+|\alpha(0)|^2+|y|^2+|v|^2), \quad \alpha \in \Omega_1, \\ \sigma(y, \tilde{H}(t,\alpha,y,v)) &\le b(1+\|\alpha\|_C^2) - \mu|y|^2, \quad \alpha \in C([-\tau,0],\mathbf{R}^n), \end{aligned}$$

for a.e. $t \in I$. Here $\alpha(0) = x$, $v(t) = y(t-\tau)$.

Then to every generalized sequence $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ of solutions of (1) there exists a subsequence (denoted in the same way) such that $x^\varepsilon \to x^0$ and $y^\varepsilon \to \nu$ in the same topologies as in Theorem 2 and

$$\begin{pmatrix} \dot{x}^0(t) \\ 0 \end{pmatrix} \in \int\limits_{K\times K} H(t, x_t^0, z)\mu(t)\,(dz), \quad x_0 = \varphi, \tag{8}$$

where $\mu(t)$ is the measure product $\nu(t) \otimes \nu(t-\tau)$. Here $\nu(s) = \delta_{\psi(s)}$, $s \in [-\tau, 0]$.

Proof. Substitute $z(t) = (y(t),\ y(t-\tau))$. Then like in the proof of Theorem 2 we have $\varepsilon_i \to 0$ and $(x^i, y^i) \in Z(\varepsilon_i)$ for every $i = 1, 2, \ldots$ such that (passing to subsequences if necessary) (x^i, z^i) converges to (x^0, μ) in considered topologies and $(\dot{x}^i(\cdot), \varepsilon_i \dot{y}^i(\cdot))$ converges to $(\dot{x}_0(\cdot), 0)$ in $L^1(I, \mathbf{R}^{n+m})$-*weak*.

Let $r \in \mathbf{R}^{n+m}$ be arbitrary and let $[s,t] \subset I$. For every i one has

$$\big\langle r, (x^i(t) - x^i(s), \varepsilon_i(y^i(t) - y^i(s)))\big\rangle \le \int_s^t \sigma(r, H(\tau, x^i(\tau), z^i(\tau)))\,d\tau.$$

Due to [18, Theorem IV.2.9],

$$\lim_{i\to\infty} \int_s^t \sigma(r, H(\tau, x^i(\tau), z^i(\tau)))\,d\tau = \int_s^t \Big\{ \int\limits_{K\times K} \sigma(r, H(\tau, x_0(\tau), z))\mu_0(\tau)\,(dz)\Big\}\,d\tau.$$

Combining the above two inequalities, we obtain

$$\big\langle r, (x^0(t) - x^0(s), 0)\big\rangle \le \int_s^t \Big\{ \int\limits_{K\times K} \sigma(r, H(\tau, x^0(\tau), z))\mu_0(\tau)\,(dz)\Big\}\,d\tau \tag{9}$$

for every $t \ge s \in I$. Consequently, $x_0(0) = x^0$ and x^0, μ satisfy (8). □

Take now a functional-differential inclusion with two variable time lags:

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in H(t, x_t, y, y(t-h_1(t)), y(t-h_2(t))),$$

$$x_0 = \varphi, \quad y(s) = \psi(s), \quad s \in [-\tau, 0], \tag{10}$$

where $h_1(t), h_2(t) \in [0,\tau]$ and h_1, h_2 are Borel measurable functions on I. The measurability of $h_i(\cdot)$ is required to assure the existence of solutions of (10). We can formulate the same result for (10) like in Theorem 3.

Theorem 4. *Suppose that the following conditions are satisfied:*

A1″. The map H is compact, convex valued, bounded on the bounded sets. Also H is almost continuous, i.e. for every $\delta > 0$ there exists $I_\delta \subset I$ with measure greater than $1-\delta$ such that H is continuous on $I_\delta \times \mathbf{R}^{m+3n}$.

A2″. There exist constants $a, b, \mu > 0$ such that for every $t \in I$, $(x(t), y(t)) \in \mathbf{R}^{n+m}$

$$\begin{aligned}&\sigma(x(t), \hat{H}(t, x_t, y, y(t-h_1(t)), y(t-h_2(t))))\\&\qquad \le a(1+|x(t)|^2+|y(t)|^2+|y(t-h_1(t))|^2+|y(t-h_2(t))|^2),\\&\sigma(y(t), \tilde{H}(t, x_t, y, y(t-h_1(t)), y(t-h_2(t))))\\&\qquad \le b(1+|x(t)|^2+|y(t-h_1(t))|^2+|y(t-h_2(t))|^2)-\mu|y(t)|^2.\end{aligned}$$

A3′. If $\inf_{t\in I}\{h_1(t), h_2(t)\} = 0$, then $\mu > 2b$.

Then to every generalized sequence $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ of solutions of (10) there exists a subsequence (denoted in the same way) such that $x^\varepsilon \to x^0$ and $y^\varepsilon \to \nu$ in the same topologies as in Theorem 2 and

$$\begin{pmatrix}\dot{x}^0(t)\\0\end{pmatrix} \in \int\limits_{K^3} H(t, x_t^0, z)\mu(t)\,(dz), \quad x_0 = \varphi, \tag{11}$$

where $\mu(t) = \nu(t)\otimes\nu(t-h_1(t))\otimes\nu(t-h_2(t))$ and $\nu(s) = \delta_{\psi(s)}$, $s \in [-\tau, 0]$.

Proof. Using A2″ and A3′, we can prove a result analogous to Lemma 1, see, e.g. [5]. Then substituting $z(t) = (y(t), y(t-h_1(t)), y(t-h_2(t)))$ again, like in the previous proof, we have $\varepsilon_i \to 0$ and $(x^i, y^i) \in Z(\varepsilon_i)$, $i = 1, 2, \dots$ such that (passing to subsequences if necessary) (x^i, z^i) converges to (x^0, μ) in considered topologies and $(\dot{x}^i(\cdot), \varepsilon_i\dot{y}^i(\cdot))$ converges to $(\dot{x}_0(\cdot), 0)$ in $L^1(I, \mathbf{R}^{n+m})$-*weak*.

Now, we will show that (x^0, μ) satisfies (11). The proof is very similar to the previous one — we just have to prove (9) (with K^3 in the limits of the second integral instead of $K \times K$) for any $r \in \mathbf{R}^{n+m}$ and $[s,t] \subset I$.

Since H is almost continuous, we have by [18, Theorem IV.2.9]

$$\lim_{i\to\infty}\int_s^t \sigma(r, H(\tau, x^i(\tau), z^i(\tau)))\,d\tau \le \int_s^t\Big\{\int_{K^3}\sigma(r, H(\tau, x_0(\tau), z))\mu_0(\tau)\,(dz)\Big\}\,d\tau.$$

Consequently,

$$\big\langle r, (x^0(t)-x^0(s), 0)\big\rangle \le \int_s^t\Big\{\int_{K^3}\sigma(r, H(\tau, x^0(\tau), z))\mu_0(\tau)\,(dz)\Big\}\,d\tau$$

for every $r \in \mathbf{R}^{n+m}$ and $t \ge s \in I$, which finishes the proof. □

Obviously, we are able to extend the above result for inclusions with finite number of delays

$$\begin{pmatrix}\dot{x}(t)\\ \varepsilon\dot{y}(t)\end{pmatrix} \in H(t, x_t, y, y(t-h_1(t)), \dots, y(t-h_k(t))),\ x_0 = \varphi, y(s) = \psi(s), s \in [-\tau, 0],$$

where $h_j(t) \in [0, \tau]$, $j = 1, \ldots, k$, and h_j are Borel measurable functions on I. But proving the corresponding theorem for the general case (1) is an open question.

ACKNOWLEDGEMENTS. This work is partially supported by the National Foundation for Scientific Research at the Bulgarian Ministry of Science and Education under Grant MM-701/97.

REFERENCES

1. Artstein, A., V. Gaitsgory. Tracking fast trajectories along a slow dynamics: A singular perturbation approach. *SIAM J. Control & Optimization*, **35**, 1997, 1487–1507.
2. Artstein, A., A. Vigodner. Singularly perturbed ordinary differential equations with fast dynamics. *Proceedings of the Royal Society of Edinburgh*, **126A**, 1996, 541–569.
3. Deimling, K. Multivalued Differential Equations. De Gruyter, Berlin, 1992.
4. Dontchev, A., I. Slavov. Upper semicontinuity of solutions of singularly perturbed differential inclusions. In: *System Modelling and Optimization*, eds. H.-J. Sebastian and K. Tammer, *Lecture Notes in Control and Information Sciences*, Vol. 143, Springer-Verlag, Berlin etc., 1991, 273–280.
5. Donchev, T., I. Slavov. Singularly perturbed functional differential inclusions. *Set Valued Analysis*, **3**, 1995, 113–128.
6. Donchev, T., I. Slavov. Tikhonov's theorem for functional-differential inclusions. *Ann. Sof. Univ., Fac. Math. et Inf.*, **89**, 1, 1995.
7. Donchev, T., I. Slavov. Averaging method for one-sided Lipschitz differential inclusions with generalized solutions. *SIAM J. Control & Optimiztion* (submitted).
8. Dontchev, A., V. Veliov. Singular perturbation in Mayer's problem for linear systems. *SIAM J. Control Optimization*, **21**, 1983, 566–581.
9. Filatov, O. Averaging of differential inclusions with control. *Differential Equations*, **33**, 1997, 782–785 (in Russian).
10. Gaitzgory, V. Control of Systems with Fast and Slow Motions. Nauka, Moscow, 1991 (in Russian).
11. Grammel, G. Singularly perturbed differential inclusions: an averaging approach. *Set-Valued Analysis*, **4**, 1996, 361–374.
12. Plotnikov, V. Averaging Methods in Control Problems. Libid Kiev, Odessa, 1992 (in Russian).
13. Quincampoix, M. Singular perturbations for systems of differential inclusions. In: *Geometry in Nonlinear Control and Differential Inclusions*, Banach Center Publications, Warsaw, 1995, 341–348.
14. Tikhonov, A. Systems of differential equations containing a small parameter in the derivatives. *Mat. Sbornik*, **31(73)**, 1952, 575–586 (in Russian).
15. Veliov, V. Differential inclusions with stable subinclusions. *Nonlinear Anal. TMA*, **23**, 1994, 1027–1038.
16. Veliov, V. A generalization of the Tikhonov theorem for singularly perturbed differential inclusions. *J. Dynamical and Control systems*, **3**, 1997, 291–319.

17. Vigodner, A. Limits of singularly perturbed control problems with statistical dynamics of fast motions. *SIAM J. Control & Optimization*, **35**, 1997, 1–28.

18. Warga, R. Optimal Control of Differential and Functional Differential Equations. Academic Press, 1973.

Received February 27, 1998
Revised May 6, 1998

Tzanko Donchev
Department of Mathematics
University of Mining and Geology
BG-1100 Sofia, BULGARIA
E-mail: `donchev@staff.mgu.bg`

Iordan Slavov
Institute of Applied Mathematics
Technical University of Sofia, bl. 2
BG-1000 Sofia, BULGARIA
E-mail: `iis@vmei.acad.bg`