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A CATEGORICAL FRAMEWORK FOR CODE EVALUATION METHOD*

JORDAN ZASHEV

In the middle of the seventies Skordev proposed to consider in general the so-called fixpoint complete partially ordered algebras, introduced in [3]. The code evaluation method is an universal method for establishing a fixpoint completeness of such algebras. Its principal result — the code evaluation theorem (or the coding theorem, as it was called before) — implies easily all basic results of algebraic recursion theory. In the present work we give a categorical analysis of code evaluation proofs for operative spaces. Thus we obtain an algebraic formulation of the fundamentals of recursion theory which can be considered as an abstract recursion theory of higher level — by one level higher, compared with the usual theory of operative spaces [1]; and it may be otherwise considered as a generalization of the last theory in a new categorical direction, in which the role of multiplication in partially ordered semigroups is played by some kind of weak tensor product in partially ordered (weak) premonoidal categories.

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1. CATEGORICAL PRELIMINARIES

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. By $\mathcal{C}_F$ we denote the category of *F-algebras in* $\mathcal{C}$; objects of $\mathcal{C}_F$ are the arrows $\varphi : F(X) \to X$ in $\mathcal{C}$, and arrows between two objects $\varphi : F(X) \to X$ and $\psi : F(Y) \to Y$ of $\mathcal{C}$ are the arrows $f : X \to Y$ in $\mathcal{C}$ such that $f \circ \varphi = \psi \circ F(f)$. The least fixed point of

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F is an initial object $\underline{\mathrm{m}} : F(M) \to M$ in $\mathcal{C}_F$. For every object $\varphi : F(X) \to X$ of $\mathcal{C}_F$ there is an unique $h : M \to X$ in $\mathcal{C}$ such that $h \circ \underline{\mathrm{m}} = \varphi \circ F(h)$. The last equality can be considered as an abstract definition of the *evaluator* h by primitive recursive iteration. To treat the more general case with primitive recursion, we use the following concept: Let Δ and F' be endofunctors in $\mathcal{C}$, and let $\vartheta : \Delta \circ F \dot{\to} F' \circ \Delta$ be a natural transformation. Then we say that a *parameterized evaluation* holds for the least fixed point $\underline{\mathrm{m}} : F(M) \to M$ of F with respect to Δ, F' and ϑ iff for every arrow $f : F'(X) \to X$ in $\mathcal{C}$ there is an unique $\mathcal{C}$-arrow $\xi : \Delta(M) \to X$ such that $\xi \circ \Delta(\underline{\mathrm{m}}) = f \circ F'(\xi) \circ \vartheta$:

$$\begin{array}{ccccc}
\Delta(F(M)) & & \xrightarrow{\quad\Delta(\underline{\mathrm{m}})\quad} & & \Delta(M) \\
\Big\downarrow \vartheta & & & & \Big\downarrow \xi \\
F'(\Delta(M)) & \xrightarrow{F'(\xi)} & F'(X) & \xrightarrow{\quad f \quad} & X
\end{array}$$

Theorem 1.1. *Suppose $\mathcal{C}$ has an initial object O and co-limits of all ω-sequences $X_0 \to X_1 \to \cdots$, and the functors F and Δ commute with those co-limits and $\Delta(O) \cong O$. Then the least fixed point $\underline{\mathrm{m}} : F(M) \to M$ of F exists and a parameterized evaluation holds for it with respect to Δ, F' and ϑ, where F' and ϑ are arbitrary.*

Proof. The least fixed point $\underline{\mathrm{m}} : F(M) \to M$ is obtained from a limiting cone $\bar{\vartheta}_n : F^n(O) \to M$ of the sequence $O \to F(O) \to F^2(O) \to \cdots$ of arrows $\vartheta_n = F^n(\vartheta_0) : F^n(O) \to F^{n+1}(O)$ in a well-known way, namely: since F preserves these co-limits, then $F(\bar{\vartheta}_n) : F^{n+1}(O) \to F(M)$ is a limiting cone for the sequence $F(O) \to F^2(O) \to \cdots$, whence there is an unique $\underline{\mathrm{m}} : F(M) \to M$ such that $\bar{\vartheta}_{n+1} = \underline{\mathrm{m}} \circ F(\bar{\vartheta}_n)$ for all natural n; this $\underline{\mathrm{m}}$ is the least fixed point of F.

Now let $f : F'(X) \to X$ be an $\mathcal{C}$-arrow. Since Δ commutes with co-limits of ω-sequences in $\mathcal{C}$, we have a limiting cone

$$\Delta(\bar{\vartheta}_n) : \Delta(F^n(O)) \to \Delta(M) \tag{1}$$

for the sequence

$$\Delta(\vartheta_n) : \Delta(F^n(O)) \to \Delta(F^{n+1}(O)).$$

Define a sequence of arrows $\xi_n : \Delta(F^n(O)) \to X$ by induction on n: ξ_0 is determined uniquely, since $\Delta(O)$ is an initial object in $\mathcal{C}$, and $\xi_{n+1} = f \circ F'(\xi_n) \circ \vartheta$. Then by induction on n we have

$$\xi_n = \xi_{n+1} \circ \Delta(\vartheta_n). \tag{2}$$

Indeed, for $n = 0$ this is trivial, since $\Delta(O)$ is an initial object in $\mathcal{C}$, and for the induction step:

$$\begin{aligned}\xi_{n+2} \circ \Delta(\vartheta_{n+1}) &= f \circ F'(\xi_{n+1}) \circ \vartheta \circ \Delta(\vartheta_{n+1}) = f \circ F'(\xi_{n+1}) \circ F'(\Delta(\vartheta_n)) \circ \vartheta \\ &= f \circ F'(\xi_{n+1} \circ \Delta(\vartheta_n)) \circ \vartheta = f \circ F'(\xi_n) \circ \vartheta = \xi_{n+1}.\end{aligned}$$

From the limiting cone (1) we obtain an unique arrow $\xi : \Delta(M) \to X$ such that $\xi_n = \xi \circ \Delta(\bar{\vartheta}_n)$ for all n. Next we show that

$$\xi \circ \Delta(\underline{\mathrm{m}}) = f \circ F'(\xi) \circ \vartheta \tag{3}$$

by proving that for all n

$$\xi_n = f \circ F'(\xi) \circ \vartheta \circ \Delta(\underline{\mathrm{m}}^{-1}) \circ \Delta(\bar{\vartheta}_n).$$

For $n = 0$ the last equality is trivial, and for $n > 0$ we have

$$\begin{aligned} f \circ F'(\xi) \circ \vartheta \circ \Delta(\underline{\mathrm{m}}^{-1} \circ \bar{\vartheta}_n) &= f \circ F'(\xi) \circ \vartheta \circ \Delta(F(\bar{\vartheta}_{n-1})) \\ &= f \circ F'(\xi) \circ F'(\Delta(\bar{\vartheta}_{n-1})) \circ \vartheta = f \circ F'(\xi_{n-1}) \circ \vartheta = \xi_n. \end{aligned}$$

Conversely, if $\xi : \Delta(M) \to X$ satisfies (3), then for all n

$$\xi_n = \xi \circ \Delta(\bar{\vartheta}_n), \tag{4}$$

whence it follows that the arrow ξ, satisfying (3), is unique. For $n = 0$ the equality (4) is obvious, and for the other cases we use induction:

$$\begin{aligned} \xi \circ \Delta(\bar{\vartheta}_{n+1}) &= \xi \circ \Delta(\underline{\mathrm{m}} \circ F(\bar{\vartheta}_n)) = f \circ F'(\xi) \circ \vartheta \circ \Delta(F(\bar{\vartheta}_n)) \qquad \text{(by (3))} \\ &= f \circ F'(\xi) \circ F'(\Delta(\bar{\vartheta}_n)) \circ \vartheta = f \circ F'(\xi_n) \circ \vartheta = \xi_{n+1}. \end{aligned}$$

The term 'parameterized evaluation' is motivated by the following example: $\mathcal{C}$ is the category of sets, F is a 'polinomial' $F(X) = \sum_{j=0}^{m} A_j \times X^j$, $\Delta(X) = Y \times X$ and $F'(X) = Y \times F(X)$ for a fixed set Y of 'parameters'. An F-algebra $f : F(X) \to X$ in $\mathcal{C}$ is then an universal algebra with a set (corresponding to A_j) of j-ary operations for all $j \leq m$. For the least fixed point $\underline{\mathrm{m}} : F(M) \to M$ of F, M is the set of terms freely generated by those operations. The equality (3) then may be interpreted as

$$\xi(y, a(t_0, \ldots, t_{j-1})) = f(y, a(\xi(y, t_0), \ldots, \xi(y, t_{j-1}))),$$

where $y \in Y$ is a parameter, a is a j-ary operation from basic ones, and $t_0, \ldots, t_{j-1}$ are terms from M; it is a definition of ξ by some kind of parameterized recursion.

A *partially ordered category* is a category $\mathcal{C}$ with partial order in every hom-set such that a composition of arrows is increasing on both arguments. We denote the partial order with the usual symbol $\leq$, i.e. $f \leq g$ for two arrows in a partially ordered category $\mathcal{C}$ means that f and g have the same domain and co-domain and f precedes g in the sense of the partial order in the corresponding hom-set. The universal example of partially ordered category is the category of posets and increasing mappings, shortly referred to as 'category of posets'. The partial order in the last category is defined in an obvious way: $f \leq g$ means that $f(x) \leq g(x)$ for all x in the domain of f and g.

The notion of increasing functor $F : \mathcal{C} \to \mathcal{D}$ between two partially ordered categories $\mathcal{C}$ and $\mathcal{D}$ is also obvious: F is increasing iff $f \leq g$ implies $F(f) \leq F(g)$ for every pair of arrows f, g in $\mathcal{C}$.

Let $\mathcal{C}$ be a partially ordered category and let $F : \mathcal{C} \to \mathcal{C}$ be an increasing endofunctor in $\mathcal{C}$. Then we shall call an F-algebra $\underline{\mathrm{m}} : F(M) \to M$ in $\mathcal{C}$ a *proper* least fixed point of F iff for every F-algebra $f : F(X) \to X$ in $\mathcal{C}$ there is an arrow $\tilde{f} : M \to X$ in $\mathcal{C}$ such that $\tilde{f} \circ \underline{\mathrm{m}} = f \circ F(\tilde{f})$ and the following two conditions hold moreover:

a) for every $\mathcal{C}$-arrow $\varphi : M \to X$, such that $f \circ F(\varphi) \leq \varphi \circ \underline{\mathrm{m}}$, we have $\tilde{f} \leq \varphi$;

b) for every $\mathcal{C}$-arrow $\varphi : M \to X$, such that $\varphi \circ \underline{\mathrm{m}} \leq f \circ F(\varphi)$, we have $\varphi \leq \tilde{f}$.

Obviously, a proper least fixed point of F is also such one in the usual sense, and $\tilde{f}$ is the corresponding evaluator for any F-algebra f. We shall desist from discussing general criteria of existence of proper least fixed points, restricting ourselves with the remark that this is a natural concept. Typically, the usual least fixed points of increasing endofunctors are proper. For instance: consider the 'polinomial' $F(X) = \sum_{j=0}^{m} A_j \times X^j$ in the category $\mathcal{C}$ of partially ordered sets and increasing mappings. The category $\mathcal{C}$ is partially ordered in an obvious way: $f \leq g$ means that $f(x) \leq g(x)$ for every x in the domain of f and g. The least fixed point $\underline{\mathrm{m}} : F(M) \to M$ exists and the object M is the set of all terms generated by j-ary operations corresponding to the elements of A_j (for all $j \leq m$) with the trivial partial order coinciding with the equality. This least fixed point is proper one and conditions a) and b) express in abstract way the possibility of proving inequalities by induction on complexity of terms (for instance in a) we prove $\tilde{f}(t) \leq \varphi(t)$ by induction on the complexity of the term $t \in M$).

2. NORMAL EVALUATION IN STRUCTURED RING-CATEGORIES

A structured ring-category (shortly, SRC) is by definition a 5-tuple $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$, where: $\mathcal{C}$ is a category with finite co-products and co-limits of ω-sequences $X_0 \to X_1 \to \cdots$; $\odot$ is a bi-endofunctor $\mathcal{C}^2 \to \mathcal{C}$ in $\mathcal{C}$ such that for any fixed object Y of $\mathcal{C}$ the functor $Y \odot \_$ preserves those co-products and co-limits; $\bar{a} : X \odot (Y \odot Z) \to (X \odot Y) \odot Z$ is a natural transformation (not necessarily isomorphism) satisfying Mac Lane pentagonal $\bar{a} \circ \bar{a} = (\bar{a} \odot 1) \circ \bar{a} \circ (1 \odot \bar{a})$; R is an endofunctor in $\mathcal{C}$ and $\vartheta_R : X \odot R(Y) \to R(X \odot Y)$ is a natural in $X, Y \in \mathcal{C}$ transformation satisfying *R-coherence*: $R(\bar{a}) \circ \vartheta_R \circ (1 \odot \vartheta_R) = \vartheta_R \circ \bar{a}$, i.e. the following commutative diagram:

$$\begin{array}{ccccc}
X \odot (Y \odot R(Z)) & \xrightarrow{(1 \odot \vartheta_R)} & X \odot R(Y \odot Z) & \xrightarrow{\vartheta_R} & R(X \odot (Y \odot Z)) \\
\Big\downarrow \bar{a} & & & & \Big\downarrow R(\bar{a}) \\
(X \odot Y) \odot R(Z) & & \xrightarrow{\qquad\qquad \vartheta_R \qquad\qquad} & & R((X \odot Y) \odot Z)
\end{array}$$

We fix a SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ and we shall write shortly $\mathcal{C}$ for the last 5-tuple. To ensure existence of some least fixed points and applicability of Theorem 1.1 in some cases below, we shall suppose that the endofunctors R, $F_1(X) = X \odot X$ and $F_2(X) = X \odot B$ in $\mathcal{C}$ commute with co-limits of ω-sequences $X_0 \to X_1 \to \cdots$ for any fixed object B of $\mathcal{C}$. Binary co-products in $\mathcal{C}$ will be denoted by $+$, and we write I_0 and I_1 for the canonical monics $X_i \to X_0 + X_1$ of the co-product $X_0 + X_1$ $(i = 0, 1)$. Thus I_0 and I_1 are natural in X_0, X_1 transformations, and we shall use short notations for their compositions, for instance: $I_{01} = I_0 \circ I_1$, $I_{101} = I_1 \circ I_0 \circ I_1$, etc. We also write $[f_0, f_1]$ for the unique arrow $X_0 + X_1 \to Y$ such that $[f_0, f_1] \circ I_i = f_i : X_i \to Y$ $(i = 0, 1)$. Since the functor $Y \odot \_$ preserves binary co-products, there is an isomorphism

$$\delta_\odot : Y \odot (X_0 + X_1) \to (Y \odot X_0) + (Y \odot X_1),$$

natural in Y, X_0, X_1, such that $\delta_\odot^{-1} = [1 \odot I_0, 1 \odot I_1]$. This means that $\delta_\odot \circ (1 \odot I_i) = I_i$ for both $i = 0, 1$, and for every pair of arrows $\varphi, \psi : Y \odot (X_0 + X_1) \to A$ in $\mathcal{C}$, such that $\varphi \circ (1 \odot I_i) = \psi \circ (1 \odot I_i)$ for both $i = 0, 1$, we have $\varphi = \psi$. A proof of an equality $\varphi = \psi$ of this kind based on the last principle will be called below a 'proof of $\varphi = \psi$ by considering cases'.

Algebraic structures concerning the present paper are represented in this context by *standard $\mathcal{C}$-algebras*, i.e. arrows $a : (X \odot X) + R(X) \to X$ in $\mathcal{C}$, satisfying the following two equalities:

$$a_0 \circ (1 \odot a_0) = a_0 \circ (a_0 \odot 1) \circ \bar{a}, \tag{5}$$

$$a_0 \circ (1 \odot a_1) = a_1 \circ R(a_0) \circ \vartheta_R, \tag{6}$$

where $a_i = a \circ I_i$, $i = 0, 1$. The equality (5) means that $a_0 : X \circ X \to X$ is a 'premonoid' in $\mathcal{C}$ (note that $\mathcal{C}$ is not supposed to be premonoidal category with respect to $\odot$, since the associativity transformation $\bar{a}$ may not be an isomorphism). Equality (6) corresponds to the equality $(\varphi, \psi)\chi = (\varphi\chi, \psi\chi)$ in operative spaces in notations of Ivanov [1]. Thus operative spaces are standard $\mathcal{C}$-algebras in the SRC of sets, i.e. the SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$, where $\mathcal{C}$ is the category of sets, $\odot$ is the usual Cartesian product, $R(X) = X \times X$, $\bar{a}$ is the usual associativity isomorphism, and $\vartheta_R : X \times (Y \times Y) \to (X \times Y) \times (X \times Y)$ is the natural transformation defined by

$$\vartheta_R((x, (y, y'))) = ((x, y), (x, y')) \qquad (x \in X,\ y, y' \in Y).$$

The forgetful functor $P : \mathbf{SA}(\mathcal{C}) \to \mathcal{C}$ from the category $\mathbf{SA}(\mathcal{C})$ of standard $\mathcal{C}$-algebras to $\mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathbf{SA}(\mathcal{C})$, which in the case of the SRC of sets assigns to each set X the free standard $\mathcal{C}$-algebra $L(X)$ generated by X. In the present section we shall give an explicit construction of this adjunction in terms of the least fixed points, and this construction is essential for the categorical axiomatization of code evaluation which we give in the next two sections 3 and 4. The reader will probably notice the analogy between sections 2 and 3.

Now consider two bi-endofunctors S and $\dot{S}$ in the fixed SRC $\mathcal{C}$, defined by

$$S(B, X) = B + ((X \odot X) + R(X))$$

and

$$\dot{S}(B, X) = B + ((X \odot B) + R(X))$$

for objects B, X of $\mathcal{C}$ and arrows as well. We shall fix the object B of $\mathcal{C}$ and write shortly $S(X)$ and $\dot{S}(X)$ for $S(B, X)$ and $\dot{S}(B, X)$, respectively (for arrows $S(f)$ means $S(B, f) = S(1_B, f)$ and, similarly, for $\dot{S}$). We have a functor $N : \mathcal{C}_S \to \mathcal{C}_{\dot{S}}$ from the category of S-algebras in $\mathcal{C}$ to that one of $\dot{S}$-algebras, defined by

$$N(f) = [f_0, [f_{10} \circ (1 \odot f_0), f_{11}]]$$

for objects $f : S(X) \to X$ of $\mathcal{C}_S$, i.e. S-algebras in $\mathcal{C}$, and $N(\varphi) = \varphi$ for arrows φ in $\mathcal{C}_S$, where $f_0 = f \circ I_0$, $f_{10} = f \circ I_{10}$, and $f_{11} = f \circ I_{11}$, as we shall write shortly below for any suitable arrow f in $\mathcal{C}$. N is indeed a functor, since for an arrow $\varphi : f \to g$ between two algebras $f : S(X) \to X$ and $g : (Y) \to Y$ in $\mathcal{C}_S$, i.e. an arrow $\varphi : X \to Y$ in $\mathcal{C}$ such that $\varphi \circ f = g \circ S(\varphi)$, we have

$$\begin{aligned}
\varphi \circ N(f) &= [\varphi \circ f_0, [\varphi \circ f_{10} \circ (1 \odot f_0), \varphi \circ f_{11}]] \\
&= [g \circ S(\varphi) \circ I_0, [g \circ S(\varphi) \circ I_{10} \circ (1 \odot f_0), g \circ S(\varphi) \circ I_{11}]] \\
&= [g_0, [g_{10} \circ (\varphi \odot \varphi) \circ (1 \odot f_0), g_{11} \circ R(\varphi)]] \\
&= [g_0, [g_{10} \circ (\varphi \odot g \circ S(\varphi) \circ I_0), g_{11} \circ R(\varphi)]] \\
&= [g_0, [g_{10} \circ (\varphi \odot g_0), g_{11} \circ R(\varphi)]] \\
&= [g_0, [g_{10} \circ (1 \odot g_0), g_{11}]] \circ (1_B + ((\varphi \odot 1_B) + R(\varphi))) \\
&= [g_0, [g_{10} \circ (1 \odot g_0), g_{11}]] \circ \dot{S}(\varphi) = N(g) \circ \dot{S}(\varphi),
\end{aligned}$$

i.e. $\varphi : N(f) \to N(g)$ in $\mathcal{C}_{\dot{S}}$.

Let $\tau : S(T) \to T$ and $\dot{\tau} : \dot{S}(\dot{T}) \to \dot{T}$ be the least fixed points of S and $\dot{S}$, respectively, in $\mathcal{C}$. (Actually, T and $\dot{T}$ are endofunctors in $\mathcal{C}$ and $\tau(B) : S(B, T(B)) \to T(B)$ and $\dot{\tau}(B) : \dot{S}(B, \dot{T}(B)) \to \dot{T}(B)$ are natural in B isomorphisms.) In the SRC of sets T is the set of terms generated from elements of B by means of two binary operations — $\tau_{10} = \tau \circ I_{10}$ and $\tau_{11} = \tau \circ I_{11}$, and $\dot{T}$ is the set of *normal* terms.

Denote by $\mathcal{D}$ the full subcategory of $\mathcal{C}_S$, consisting of those S-algebras $f : S(X) \to X$ for which $f_1 = f \circ I_1 : (X \odot X) + R(X) \to X$ is a standard $\mathcal{C}$-algebra. We are looking for an algebra $f : S(X) \to X$ in $\mathcal{D}$ such that $N(f) = \dot{\tau}$.

To find such an algebra, consider the natural in $X, Y \in \mathcal{C}$ transformation

$$\dot{\vartheta} : Y \odot \dot{S}(X) \to \dot{S}(Y) + \dot{S}(Y \odot X),$$

defined by

$$\dot{\vartheta} = (I_{10} + I_1 \circ (\bar{a} + \vartheta_R) \circ \delta_{\odot}) \circ \delta_{\odot}.$$

This definition is equivalent to the following three equalities:

$$\dot{\vartheta} \circ (1 \odot I_0) = I_{010}, \tag{7}$$

$$\dot{\vartheta} \circ (1 \odot I_{10}) = I_{110} \circ \bar{a}, \tag{7'}$$

$$\dot{\vartheta} \circ (1 \odot I_{11}) = I_{111} \circ \vartheta_R. \tag{7''}$$

Proposition 2.1. *Every algebra $f : S(X) \to X$ from $\mathcal{D}$ satisfies the equality*

$$f_{10} \circ (1 \odot N(f)) = N(f) \circ [1, \dot{S}(f_{10})] \circ \dot{\vartheta}, \tag{8}$$

$$\begin{array}{ccccc}
X \odot \dot{S}(X) & & \xrightarrow{\quad 1 \odot N(f) \quad} & & X \odot X \\
\Big\downarrow \dot{\vartheta} & & & & \Big\downarrow f_{10} \\
\dot{S}(X) + \dot{S}(X \odot X) & \xrightarrow{[1, \dot{S}(f_{10})]} & \dot{S}(X) & \xrightarrow{N(f)} & X
\end{array}$$

Proof. By considering cases. Denoting by φ and ψ the left- and right-hand sides of (8), respectively, we shall conclude $\varphi = \psi$ by showing that $\varphi \circ (1 \odot I_0) = \psi \circ (1 \odot I_0)$, $\varphi \circ (1 \odot I_{10}) = \psi \circ (1 \odot I_{10})$ and $\varphi \circ (1 \odot I_{11}) = \psi \circ (1 \odot I_{11})$. Consider, for instance, the second of the last three equalities, leaving the other ones to the reader:

$$\varphi \circ (1 \odot I_{10}) = f_{10} \circ (1 \odot N(f) \circ I_{10}) = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)),$$

and

$$\begin{aligned}\psi \circ (1 \odot I_{10}) &= N(f) \circ [1, \dot{S}(f_{10})] \circ I_{110} \circ \bar{a} = N(f) \circ \dot{S}(f_{10}) \circ I_{10} \circ \bar{a} \\ &= N(f) \circ I_{10} \circ (f_{10} \odot 1) \circ \bar{a} = f_{10} \circ (1 \odot f_0) \circ (f_{10} \odot 1) \circ \bar{a} \\ &= f_{10} \circ (f_{10} \odot f_0) \circ \bar{a} = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)),\end{aligned}$$

the last equality being the equality (5) for the algebra $f \in \mathcal{D}$. In the case with $(1 \odot I_{11})$ the equality (6) is used in a similar way.

Now suppose an algebra $f : S(X) \to X$ from $\mathcal{D}$ satisfies $N(f) = \dot{\tau}$. Then, obviously, $X = \dot{T}$, and composing the equality $N(f) = \dot{\tau}$ from right by I_0 and I_{11}, we obtain $f_0 = \dot{\tau}_0 = \dot{\tau} \circ I_0$ and $f_{11} = \dot{\tau}_{11} = \dot{\tau} \circ I_{11}$, whence f should be of the form $[\dot{\tau}_0,\ [\mu, \dot{\tau}_{11}]]$ for some arrow $\mu : \dot{T} \odot \dot{T} \to \dot{T}$ in $\mathcal{C}$. Then the equality $N(f) = \dot{\tau}$ is equivalent to $\mu \circ (1 \odot \dot{\tau}_0) = \dot{\tau}_{10} = \dot{\tau} \circ I_{10}$. If $f \in \mathcal{D}$, then by (8) we obtain

$$\mu \circ (1 \odot \dot{\tau}) = \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta}, \tag{9}$$

$$\begin{array}{ccccc} \dot{T} \odot \dot{S}(\dot{T}) & \xrightarrow{\ 1 \odot \dot{\tau}\ } & \dot{T} \odot \dot{T} & \xrightarrow{\ \mu\ } & \dot{T} \\ \downarrow{\scriptstyle \dot{\vartheta}} & & & & \uparrow{\scriptstyle \dot{\tau}} \\ \dot{S}(\dot{T}) + \dot{S}(\dot{T} \odot \dot{T}) & & \xrightarrow{\ [1, \dot{S}(\mu)]\ } & & \dot{S}(\dot{T}) \end{array}$$

The last equality determines μ uniquely by the principle of the parameterized evaluation, i.e. by Theorem 1.1 (with the functors $\dot{S}$ for F, $\dot{T} \odot X$ for $\Delta(X)$ and $\dot{S}(\dot{T}) + F(X)$ for $F'(X)$). This suggests to *define* μ by (9). Then the arrow $f = [\dot{\tau}_0,\ [\mu, \dot{\tau}_{11}]]$ satisfies $N(f) = \dot{\tau}$, because a composition of (9) from right by $1 \odot I_0$ yields

$$\mu \circ (1 \odot \dot{\tau}_0) = \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_0) = \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{010} = \dot{\tau}_{10}.$$

The arrow f will be denoted below by τ^N, i.e.

$$\tau^N = [\dot{\tau}_0,\ [\mu, \dot{\tau}_{11}]],$$

where μ satisfies (9). Thus we see that $\tau^N : S(\dot{T}) \to \dot{T}$ is the unique S-algebra f satisfying (8), such that $N(f) = \dot{\tau}$.

Proposition 2.2. *For every S-algebra $f : S(X) \to X$ in $\mathcal{C}$ satisfying* (8) *there is an unique arrow $h : \tau^N \to f$ in $\mathcal{C}_S$, i.e. an unique $h : \dot{T} \to X$ in $\mathcal{C}$, such that*

$$h \circ \tau^N = f \circ S(h), \tag{10}$$

and the arrow h is the unique one $h : \dot{\tau} \to N(f)$ in $\mathcal{C}_{\dot{S}}$, i.e.

$$h \circ \dot{\tau} = N(f) \circ \dot{S}(h). \tag{11}$$

Proof. If $h : \tau^N \to f$ is an arrow in $\mathcal{C}_S$, then $h = N(h) : \dot{\tau} \to N(f)$ is such one in $\mathcal{C}_{\dot{S}}$, but the arrow $h : \dot{\tau} \to N(f)$ in $\mathcal{C}_{\dot{S}}$ is unique, since $\dot{\tau}$ is the least fixed point of $\dot{S}$. Therefore the arrow $h : \dot{T} \to X$, satisfying (10), can be only the unique arrow $h : \dot{\tau} \to N(f)$ in $\mathcal{C}_{\dot{S}}$. To show that the last arrow satisfies (10), we consider the cases:

$$h \circ \tau^N \circ I_0 = h \circ \dot{\tau}_0 = N(f) \circ \dot{S}(h) \circ I_0 = N(f) \circ I_0 = f_0 = f \circ I_0 = f \circ S(h) \circ I_0;$$

$$h \circ \tau^N \circ I_{11} = h \circ \dot{\tau}_{11} = N(f) \circ \dot{S}(h) \circ I_{11} = N(f) \circ I_{11} \circ R(h) = f_{11} \circ R(h) = f \circ S(h) \circ I_{11};$$

but $h \circ \tau^N \circ I_{10} = h \circ \mu$ and $f \circ S(h) \circ I_{10} = f_{10} \circ (h \odot h)$. Therefore it remains to show that

$$h \circ \mu = f_{10} \circ (h \odot h). \tag{12}$$

We shall do this by the principle of the parameterized evaluation. For that define $\varphi_0 = h \circ \mu$, $\varphi_1 = f_{10} \circ (h \odot h)$, and

$$\eta = N(f) \circ [\dot{S}(h), 1] : \dot{S}(\dot{T}) + \dot{S}(X) \to X.$$

By the principle of the parameterized evaluation (Theorem 1.1) there is an unique $\mathcal{C}$-arrow $\varphi : \dot{T} \odot \dot{T} \to X$ such that

$$\varphi \circ (1 \odot \dot{\tau}) = \eta \circ (1 + \dot{S}(\varphi)) \circ \dot{\vartheta}. \tag{13}$$

We shall show that both φ_0 and φ_1 satisfy (13) with respect to φ, whence it will follow (12) and the proof will be completed. For φ_0 this can be done without using (8):

$$\begin{aligned}\varphi_0 \circ (1 \odot \dot{\tau}) &= h \circ \mu \circ (1 \odot \dot{\tau}) = h \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} = N(f) \circ \dot{S}(h) \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \\ &= N(f) \circ [\dot{S}(h), \dot{S}(\varphi_0)] \circ \dot{\vartheta} = N(f) \circ [\dot{S}(h), 1] \circ (1 + \dot{S}(\varphi_0)) \circ \dot{\vartheta} \\ &= \eta \circ (1 + \dot{S}(\varphi_0)) \circ \dot{\vartheta}.\end{aligned}$$

For φ_1 the equality (8) is used:

$$\begin{aligned}\varphi_1 \circ (1 \odot \dot{\tau}) &= f_{10} \circ (h \odot h \circ \dot{\tau}) = f_{10} \circ (h \odot N(f) \circ \dot{S}(h)) \\ &= f_{10} \circ (1 \odot N(f)) \circ (h \odot \dot{S}(h)) = N(f) \circ [1, \dot{S}(f_{10})] \circ \dot{\vartheta} \circ (h \odot \dot{S}(h)) \\ &= N(f) \circ [1, \dot{S}(f_{10})] \circ (\dot{S}(h) + \dot{S}(h \odot h)) \circ \dot{\vartheta} \\ &= N(f) \circ [\dot{S}(h), \dot{S}(\varphi_1)] \circ \dot{\vartheta} = \eta \circ (1 + \dot{S}(\varphi_1)) \circ \dot{\vartheta}.\end{aligned}$$

Since $\tau : S(T) \to T$ and $\dot{\tau} : \dot{S}(\dot{T}) \to \dot{T}$ are least fixed points, there are unique arrows $\dot{\nu} : T \to \dot{T}$ and $\iota : \dot{T} \to T$ such that

$$\dot{\nu} \circ \tau = \tau^N \circ S(\dot{\nu}) \tag{14}$$

and

$$\iota \circ \dot{\tau} = N(\tau) \circ \dot{S}(\iota), \tag{15}$$

respectively.

Proposition 2.3. *$\dot{\nu}$ is a retraction with inverse ι, i.e.*

$$\dot{\nu} \circ \iota = 1 = 1_{\dot{T}}.$$

Proof. It is enough to show that $\dot{\nu} \circ \iota \circ \dot{\tau} = \dot{\tau} \circ \dot{S}(\dot{\nu} \circ \iota)$, since the arrow $1 : \dot{\tau} \to \dot{\tau}$ in $\mathcal{C}_{\dot{T}}$ is unique. But

$$\begin{aligned}\dot{\nu} \circ \iota \circ \dot{\tau} &= \dot{\nu} \circ N(\tau) \circ \dot{S}(\iota) = \dot{\nu} \circ [\tau_0, [\tau_{10} \circ (1 \odot \tau_0), \tau_{11}]] \circ \dot{S}(\iota) \\ &= [\dot{\nu} \circ \tau_0, [\dot{\nu} \circ \tau_{10} \circ (1 \odot \tau_0), \dot{\nu} \circ \tau_{11}]] \circ \dot{S}(\iota)\end{aligned}$$

$$= [\tau^N \circ S(\dot\nu) \circ I_0, [\tau^N \circ S(\dot\nu) \circ I_{10} \circ (1 \odot \tau_0), \tau^N \circ S(\dot\nu) \circ I_{11}]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\mu \circ (\dot\nu \odot \dot\nu) \circ (1 \odot \tau_0), \dot\tau_{11} \circ R(\dot\nu)]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\mu \circ (\dot\nu \odot \dot\tau_0), \dot\tau_{11} \circ R(\dot\nu)]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\dot\tau_{10} \circ (\dot\nu \odot 1), \dot\tau_{11} \circ R(\dot\nu)]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\dot\tau_{10}, \dot\tau_{11}]] \circ (1 + ((\dot\nu \odot 1) + R(\dot\nu))) \circ \dot S(\iota)$$
$$= \dot\tau \circ \dot S(\dot\nu) \circ \dot S(\iota) = \dot\tau \circ \dot S(\dot\nu \circ \iota).$$

Denote by ν the morphism $\iota \circ \dot\nu : T \to T$. This is the 'normalizing' morphism, in the SRC of sets ν assigns to each term its normal form. For any S-algebra $f : S(X) \to (X)$ in $\mathcal{C}$ denote by $\tilde f$ the evaluator of f with respect to τ, i.e. the unique arrow $\tilde f : T \to X$ such that $\tilde f \circ \tau = f \circ S(\tilde f)$. (In the case of SRC of sets $\tilde f$ assigns to each term in T its value in the algebra X.)

Corollary 2.1. *For any S-algebra $f : S(X) \to X$ in $\mathcal{C}$ the following conditions are equivalent*:

(a) $\tilde f \circ \nu = \tilde f$;

(b) *there is a morphism $h : \tau^N \to f$ in $\mathcal{C}_S$*;

(b′) *there is an unique morphism $h : \tau^N \to f$ in $\mathcal{C}_S$*;

(c) *there is a morphism $h : \dot T \to X$ in $\mathcal{C}$ such that $h \circ \dot\nu = \tilde f$*;

(c′) *there is an unique morphism $h : \dot T \to X$ in $\mathcal{C}$ such that $h \circ \dot\nu = \tilde f$*;

and when they hold, the unique arrows h in (b′) *and* (c′) *are the same as the evaluator of $N(f)$ with respect to $\dot\tau$ or the unique morphism $h : \dot\tau \to N(f)$ in $\mathcal{C}_{\dot S}$.*

Proof. (a) ⇒ (b) Let $\tilde f \circ \nu = \tilde f$ and $h = \tilde f \circ \iota$. Then

$$\begin{aligned} h \circ \tau^N &= \tilde f \circ \iota \circ \tau^N \circ S(\dot\nu \circ \iota) && \text{(by Proposition 2.3)} \\ &= \tilde f \circ \iota \circ \dot\nu \circ \tau \circ S(\iota) && \text{(by (14))} \\ &= \tilde f \circ \nu \circ \tau \circ S(\iota) = \tilde f \circ \tau \circ S(\iota) = f \circ S(\tilde f) \circ S(\iota) = f \circ S(h). \end{aligned}$$

(b) ⇒ (c) & (b′) Let $h : \dot T \to X$ and $h \circ \tau^N = f \circ S(h)$. Then

$$h \circ \dot\nu \circ \tau = h \circ \tau^N \circ S(\dot\nu) = f \circ S(h) \circ S(\dot\nu) = f \circ S(h \circ \dot\nu),$$

and by the uniqueness of the evaluator $\tilde f$ we have $\tilde f = h \circ \dot\nu$. Thence, also $\tilde f \circ \iota = h \circ \dot\nu \circ \iota = h$ and therefore the morphism $h : \tau^N \to f$ in $\mathcal{C}_S$ is unique.

(c) ⇒ (a) Let $h : \dot T \to X$ and $h \circ \dot\nu = \tilde f$. Then

$$\tilde f \circ \nu = h \circ \dot\nu \circ \nu = h \circ \dot\nu \circ \iota \circ \dot\nu = h \circ \dot\nu = \tilde f.$$

(c) ⇒ (c′) Because $h \circ \dot\nu = \tilde f$ implies $h = h \circ \dot\nu \circ \iota = \tilde f \circ \iota$.

The implications (b′) ⇒ (b) and (c′) ⇒ (c) are trivial. The equivalence of (a)–(c′) is proved. If they hold, then for the morphism $h : \tau^N \to f$ in (b′) we have $h = N(h) : \dot\tau = N(\tau^N) \to N(f)$ in $\mathcal{C}_{\dot S}$. From the proof of (b) ⇒ (c) it is clear also that the morphism satisfying (b′) coincides with the unique morphism in (c′).

The morphism h from this corollary, when (a)–(c′) hold, will be called below a *normal evaluator* of f. The next proposition is partially a reverse one to Proposition 2.1.

Proposition 2.4. *If the S-algebra $f : S(X) \to X$ satisfies* (8) *and the arrow $1_X \odot (1_X \odot \tilde{f})$ is an epic, then $f \in \mathcal{D}$.*

Proof. We have to prove that

$$f_{10} \circ (1 \odot f_{10}) = f_{10} \circ (f_{10} \odot 1) \circ \bar{a} \tag{16}$$

and

$$f_{10} \circ (1 \odot f_{11}) = f_{11} \circ R(f_{10}) \circ \vartheta_R. \tag{17}$$

The equality (17) follows easily from (8) by a composition from right with $1 \odot I_{11}$. Similarny, a composition with $1 \odot I_{10}$ yields the equality

$$f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) = f_{10} \circ (f_{10} \odot f_0) \circ \bar{a}, \tag{18}$$

which is weaker than (16). To prove the last one, we shall show that

$$f_{10} \circ (1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota)) = f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \circ \bar{a}. \tag{19}$$

From (19) the equality (16) will follow immediately by canceling from right the arrow $1 \odot (1 \odot \tilde{f} \circ \iota)$. This can be done because $1 \odot (1 \odot \tilde{f})$ is right-cancelable by suppositions of Proposition 2.4, and $\tilde{f} = \tilde{f} \circ \nu = \tilde{f} \circ \iota \circ \dot{\nu}$, whence $1 \odot (1 \odot \tilde{f} \circ \iota)$ is also right cancelable. Therefore it remains to prove (19). For that we shall use the principle of the parameterized evaluation. Consider the $\mathcal{C}$-arrow

$$\psi = [f_{10} \circ (1 \odot N(f)), N(f)] : (X \odot \dot{S}(X)) + \dot{S}(X) \to X$$

and the natural in $X, Y \in \mathcal{C}$ transformation

$$\vartheta_1 : X \odot (X \odot \dot{S}(Y)) \to (X \odot \dot{S}(X)) + \dot{S}(X \odot (X \odot Y))$$

defined by

$$\vartheta_1 = ((1 \odot I_{10}) + I_1 \circ (\bar{a} \circ (1 \odot \bar{a}) + \vartheta_R \circ (1 \odot \vartheta_R))) \circ (1 + \delta_\odot) \circ \delta_\odot \circ (1 \odot (1 + \delta_\odot) \circ \delta_\odot),$$

which is equivalent to the following three equalities:

$$\vartheta_1 \circ (1 \odot (1 \odot I_0)) = I_0 \circ (1 \odot I_{10}); \tag{20}$$

$$\vartheta_1 \circ (1 \odot (1 \odot I_{10})) = I_{110} \circ \bar{a} \circ (1 \odot \bar{a}); \tag{20'}$$

$$\vartheta_1 \circ (1 \odot (1 \odot I_{11})) = I_{111} \circ \vartheta_R \circ (1 \odot \vartheta_R). \tag{20''}$$

By Theorem 1.1, applied to functors $F(Y) = \dot{S}(Y)$, $\Delta(Y) = X \odot (X \odot Y)$ and $F'(Y) = (X \odot \dot{S}(X)) + F(Y)$, there is an unique $\mathcal{C}$-arrow $\varphi : X \odot (X \odot \dot{T}) \to X$ such that

$$\varphi \circ (1 \odot (1 \odot \dot{\tau})) = \psi \circ (1 + \dot{S}(\varphi)) \circ \vartheta_1. \tag{21}$$

Therefore, to complete the proof, it is enough to show that both sides of (19) satisfy (21) with respect to φ. Denote the left- and right-hand sides of (19) by φ_0 and φ_1, respectively. To prove

$$\varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) = \psi \circ (1 + \dot{S}(\varphi_0)) \circ \vartheta_1, \tag{22}$$

we use the following form of the principle of considering cases: composed from right by $1 \odot (1 \odot I_0)$, $1 \odot (1 \odot I_{10})$ and $1 \odot (1 \odot I_{11})$, the two sides of (22) become equal, hence we shall conclude (22). We shall show this for $1 \odot (1 \odot I_{10})$, leaving the other two cases for the reader (they are similar or simpler). First, notice that by Proposition 2.2 and Corollary 2.1 we have $\tilde{f} = h \circ \dot{\nu}$, where h is the normal evaluator of f, and therefore $\tilde{f} \circ \iota = h$ (by Proposition 2.3), whence

$$\tilde{f} \circ \iota \circ \dot{\tau} = N(f) \circ \dot{S}(\tilde{f} \circ \iota), \tag{23}$$

and

$$\varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) = f_{10} \circ (1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota \circ \dot{\tau})) = f_{10} \circ (1 \odot f_{10} \circ (1 \odot N(f) \circ \dot{S}(\tilde{f} \circ \iota))).$$

Then

$$\begin{aligned}\varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})) &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot N(f) \circ \dot{S}(\tilde{f} \circ \iota) \circ I_{10})) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0))).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\psi \circ (1 + \dot{S}(\varphi_0)) \circ \vartheta_1 \circ (1 \odot (1 \odot I_{10})) = \psi \circ (1 + \dot{S}(\varphi_0)) \circ I_{110} \circ \bar{a} \circ (1 \odot \bar{a}) \\ &\quad = \psi \circ I_1 \circ \dot{S}(\varphi_0) \circ I_{10} \circ \bar{a} \circ (1 \odot \bar{a}) = N(f) \circ I_{10} \circ (\varphi_0 \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &\quad = f_{10} \circ (\varphi_0 \odot f_0) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &\quad = f_{10} \circ (f_{10} \circ (1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota)) \odot f_0) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &\quad = f_{10} \circ (f_{10} \odot f_0) \circ ((1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota)) \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &\quad = f_{10} \circ (f_{10} \odot f_0) \circ \bar{a} \circ (1 \odot (f_{10} \circ (1 \odot \tilde{f} \circ \iota) \odot 1)) \circ (1 \odot \bar{a}) \\ &\quad = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (1 \odot (f_{10} \circ (1 \odot \tilde{f} \circ \iota) \odot 1)) \circ (1 \odot \bar{a}) \\ &\quad = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (1 \odot (f_{10} \odot 1)) \circ (1 \odot ((1 \odot \tilde{f} \circ \iota) \odot 1) \circ \bar{a}) \\ &\quad = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (1 \odot (f_{10} \odot 1)) \circ (1 \odot \bar{a} \circ (1 \odot (\tilde{f} \circ \iota \odot 1))) \\ &\quad = f_{10} \circ (1 \odot f_{10} \circ (f_{10} \odot f_0)) \circ (1 \odot \bar{a}) \circ (1 \odot (1 \odot (\tilde{f} \circ \iota \odot 1))) \\ &\quad = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0))) \circ (1 \odot (1 \odot (\tilde{f} \circ \iota \odot 1))) \\ &\quad = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0))) = \varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})).\end{aligned}$$

To prove

$$\varphi_1 \circ (1 \odot (1 \odot \dot{\tau})) = \psi \circ (1 + \dot{S}(\varphi_1)) \circ \vartheta_1, \tag{24}$$

consider the cases as in the proof of (22). Again, we shall consider the case with $1 \odot (1 \odot I_{10})$ only, leaving the other ones to the reader (note that in the case with $1 \odot (1 \odot I_{11})$ the R-coherence is used in the same way in which the Mac Lane pentagonal diagram for $\bar{a}$ is used in the case with $1 \odot (1 \odot I_{10})$). We have, using (23) as before,

$$\begin{aligned}\varphi_1 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})) &= f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \circ \bar{a} \circ (1 \odot (1 \odot \dot{\tau}_{10})) \\ &= f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota \circ \dot{\tau} \circ I_{10}) \circ \bar{a} = f_{10} \circ (f_{10} \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0)) \circ \bar{a}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi \circ (1 + \dot{S}(\varphi_1)) \circ \vartheta_1 \circ (1 \odot (1 \odot I_{10})) &= \psi \circ (1 + \dot{S}(\varphi_1)) \circ I_{110} \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= N(f) \circ \dot{S}(\varphi_1) \circ I_{10} \circ \bar{a} \circ (1 \odot \bar{a}) = N(f) \circ I_{10} \circ (\varphi_1 \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_0) \circ (\varphi_1 \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_0) \circ (f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \circ \bar{a} \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_0) \circ (f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \odot 1) \circ \bar{a} \circ \bar{a}\end{aligned}$$

$$= f_{10} \circ (f_{10} \odot f_0) \circ \bar{a} \circ (f_{10} \odot (\tilde{f} \circ \iota \odot 1)) \circ \bar{a}$$
$$= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (f_{10} \odot (\tilde{f} \circ \iota \odot 1)) \circ \bar{a}$$
$$= f_{10} \circ (f_{10} \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0)) \circ \bar{a} = \varphi_1 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})).$$

Corollary 2.2. *The algebra $\tau^N : S(\dot{T}) \to \dot{T}$ belongs to $\mathcal{D}$, and therefore it is an initial object of $\mathcal{D}$.*

Proof. Since the evaluator $\dot{\nu} : T \to \dot{T}$ of τ^N is right-invertible by Proposition 2.3, the arrow $1 \odot (1 \odot \dot{\nu})$ is an epic. Then by Proposition 2.4 the algebra τ^N belongs to $\mathcal{D}$ and by Proposition 2.2 it is an initial object of $\mathcal{D}$.

3. MINIMAL EVALUATION IN PARTIALLY ORDERED SRC

A *partially ordered* SRC is a SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ such that $\mathcal{C}$ is in the same time a partially ordered category and all involved functors (i.e. $\odot$, R and $+$) are increasing with respect to the partial order in $\mathcal{C}$ on every argument. In the present section we shall fix a partially ordered SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ satisfying all conditions from the previous section, and we shall suppose moreover that the least fixed point

$$\tau(B) : S(T(B)) \to T(B)$$

is a proper one with respect to the partial order in $\mathcal{C}$. The bi-endofunctors S and $\dot{S}$, defined as in the previous section, are increasing. Consider the bi-endofunctor $S^+(B, X) = \dot{S}(B, X) + X$ which also is increasing. As in the previous section, we shall write shortly $S(X)$, $\dot{S}(X)$ and $S^+(X)$ for $S(B, X)$, $\dot{S}(B, X)$ and $S^+(B, X)$, respectively. We have the functor $\dot{N} : \mathcal{C}_{\dot{S}} \to \mathcal{C}_{S^+}$ defined by $\dot{N}(f) = [f, 1]$ for objects $f : \dot{S}(X) \to X$ of $\mathcal{C}_{\dot{S}}$ and $\dot{N}(\varphi) = \varphi$ for morphisms φ. The composition $\dot{N} \circ N : \mathcal{C}_S \to \mathcal{C}_{S^+}$ preserves morphisms and we shall write shortly f^+ for the value $\dot{N}(N(f))$ of $\dot{N} \circ N$ for objects, i.e.

$$f^+ = [[f_0, [f_{10} \circ (1 \odot f_0), f_{11}], 1].$$

Now, for the fixed object B of $\mathcal{C}$ we shall suppose that $B = B_0 + B_1$, where B_0 and B_1 are two fixed objects of $\mathcal{C}$. As usual, we denote the canonical monics $B_0 \to B$ and $B_1 \to B$ by I_0 and I_1, respectively. Intuitively, the object B_0 will be considered as the object of 'parameters', and B_1 — as the object of 'variables', which is just the case for the SRC of sets. We use the short notations $S_0(X)$, $\dot{S}_0(X)$ and $S_0^+(X)$ for $S(B_0, X)$, $\dot{S}(B_0, X)$ and $S^+(B_0, X)$, respectively. (Thus we define endofunctors S_0, $\dot{S}_0$ and S_0^+ in $\mathcal{C}$, for instance, $S_0(f) = S(B_0, f) = S(1_{B_0}, f)$ for an arrow f in $\mathcal{C}$, etc.) We have a functor $P : \mathcal{C}_S \to \mathcal{C}_{S_0}$ defined by $P(f) = f \circ S(I_0, 1)$ for objects f, and $P(\varphi) = \varphi$ for arrows φ of $\mathcal{C}_S$, where I_0 is here the canonical monic $I_0 : B_0 \to B$ of the co-product $B_0 + B_1$. Intuitively, the functor P simply ignores interpretation of variables. We have also another functor $Q : \mathcal{B} \to \mathcal{C}_S$ which is in some sense inverse to P. Here $\mathcal{B}$ is the category, defined as follows: objects of $\mathcal{B}$ are pairs (x, f), where $x : B_1 \to X$ and $f : S_0(X) \to X$ are $\mathcal{C}$-arrows with the same co-domain X, and morphisms $\varphi : (x, f) \to (y, g)$ in $\mathcal{B}$, where $y : B_1 \to Y$ and

$g : S_0(Y) \to Y$, are the $\mathcal{C}$-arrows $\varphi : X \to Y$ which are simultaneously morphisms in the comma category $(B_1 \downarrow \mathcal{C})$ and in $\mathcal{C}_{S_0}$, i.e. $y = \varphi \circ x$ and $\varphi \circ f = g \circ S_0(\varphi)$. For objects $(x, f) \in \mathcal{B}$ the functor Q is defined by

$$Q(x, f) = [[f_0, x], f_1] : S(X) \to X,$$

and for arrows φ in $\mathcal{B}$ the functor Q is defined trivially: $Q(\varphi) = \varphi$. The reader can easily check that Q is indeed a functor and $Q(f_{01}, P(f)) = f$ for any object $f : S(X) \to X$ of $\mathcal{C}_S$, where, as usual, $f_{01} = f_0 \circ I_1 : B_1 \to X$ and also $P(Q(x, f)) = f$ for all $(x, f) \in \mathcal{B}$.

Next we define a natural in $X, Y \in \mathcal{C}$ transformation

$$\vartheta^+ = \vartheta^+_S : X \odot S^+(Y) \to S^+(X) + S^+(X \odot Y),$$

similar to the transformation $\dot{\vartheta}$ in the previous section, namely,

$$\vartheta^+ = [(I_0 + I_0) \circ \dot{\vartheta}, I_{11}] \circ \delta_\odot,$$

which is equivalent to the pair of equalities

$$\vartheta^+ \circ (1 \odot I_0) = (I_0 + I_0) \circ \dot{\vartheta}$$

and

$$\vartheta^+ \circ (1 \odot I_1) = I_{11}.$$

The transformation $\vartheta^+_{S_0}$, defined in the same way for the functor S_0 instead of S, will be denoted shortly by ϑ^+_0.

Proposition 3.1. *For any algebra $f : S(X) \to X$ in $\mathcal{C}_S$ the equalities* (8) *and*

$$f_{10} \circ (1 \odot f^+) = f^+ \circ [1, S^+(f_{10})] \circ \vartheta^+ \tag{25}$$

are equivalent, and therefore (25) *holds for every object f of $\mathcal{D}$.*

Proof. An easy consequence of definitions.

Now consider a morphism $\sigma : B_1 \to \dot{T} = \dot{T}(B)$. In the case of SRC of posets and the trivial order (coinciding with the equality) in B_1, σ assigns to each variable $v \in B_1$ a normal term $\sigma(v) \in \dot{T}$ which may contain any variable from B_1. Thus σ determines a system of inequalities $\{\sigma(v) \leq v \mid v \in B_1\}$. A solution of the last system in an S_0-algebra $f : S_0(X) \to X$ in this SRC is a function $x : B_1 \to X$ such that the evaluator $h : \dot{T} \to X$ of the algebra $N(Q(x, f)) : \dot{S}(X) \to X$ with respect to the least fixed point $\dot{\tau}$ satisfies the inequality

$$h \circ \sigma \leq h \circ \dot{\tau}_{01}, \tag{26}$$

where $\dot{\tau}_{01} = \dot{\tau} \circ I_0 \circ I_1 : B_1 \to \dot{T}$ is the mapping, which assigns to each variable in B_1 the same one considered as a normal term from $\dot{T}$. When $Q(x, f) \in \mathcal{D}$, i.e. f_1 is a standard algebra in the SRC of posets, the mapping $h : \dot{T} \to X$ is a morphism $h : \tau^N \to Q(x, f)$ in $\mathcal{C}_S$ (by Propositions 2.1 and 2.2), and therefore $P(h) = h : P(\tau^N) \to f$ is such one in $\mathcal{C}_{S_0}$, i.e.

$$h \circ \tau^N \circ S(I_0, 1) = f \circ S_0(h). \tag{27}$$

And the solution x can be restored from h, namely, $x = h \circ \dot{\tau}_{01}$, which follows from the equality $h \circ \dot{\tau} = N(Q(x, f)) \circ \dot{S}(h)$ by a composition from right with I_{01}.

In the general case, when we have an arbitrary partially ordered **SRC** $\mathcal{C}$ and an S_0-algebra $f : S_0(X) \to X$ in it, such that $f_1 \in \mathbf{SA}(\mathcal{C})$, the above consideration suggests to treat an arbitrary arrow $\sigma : B_1 \to \dot{T}$ as a system of inequalities and morphisms $h : P(\tau^N) \to f$ of S_0-algebras in $\mathcal{C}$, satisfying (26) as *solutions* of the system σ. The next proposition will give us a more convenient form of (26).

First we define an arrow $\alpha : \dot{T} \to S_0^+(\dot{T})$, called *analyzer* of the system σ, by the following equality:

$$\alpha = [(I_0 + \sigma),\ [(I_{10} + \mu \circ (1 \odot \sigma)) \circ \delta_\odot,\ I_{011}]] \circ \dot{\tau}^{-1}.$$

An equivalent and perhaps more clear form of this equality is the following definition 'by cases':

$$\alpha \circ \dot{\tau}_{00} = I_{00}; \tag{28.1}$$

$$\alpha \circ \dot{\tau}_{01} = I_1 \circ \sigma; \tag{28.2}$$

$$\alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) = I_{010}; \tag{28.3}$$

$$\alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) = I_1 \circ \mu \circ (1 \odot \sigma); \tag{28.4}$$

$$\alpha \circ \dot{\tau}_{11} = I_{011}. \tag{28.5}$$

Proposition 3.2. *For any S_0-algebra $f : S_0(X) \to X$ in $\mathcal{C}$ and any morphism $h : P(\tau^N) \to f$ of such algebras, i.e. any $\mathcal{C}$-arrow $h : \dot{T} \to X$ for which* (27) *holds, we have the equivalence*

$$h \circ \sigma \leq h \circ \dot{\tau}_{01} \ \Leftrightarrow\ f^+ \circ S_0^+(h) \circ \alpha \leq h.$$

Proof. The reverse direction ($\Leftarrow$) of the last equivalence is easy to be proved and does not use (27):

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{01} = f^+ \circ S_0^+(h) \circ I_1 \circ \sigma = f^+ \circ I_1 \circ h \circ \sigma = h \circ \sigma. \tag{29}$$

To prove that $h \circ \sigma \leq h \circ \dot{\tau}_{01}$ implies $f^+ \circ S_0^+(h) \circ \alpha \leq h$, suppose $h \circ \sigma \leq h \circ \dot{\tau}_{01}$. Since $\dot{\tau}$ is an isomorphism, it is enough to prove the inequality

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau} \leq h \circ \dot{\tau}. \tag{30}$$

We shall do this by considering cases as in the definition of α. We have, using (28.1) and (27),

$$\begin{aligned} f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \circ I_0 &= f^+ \circ S_0^+(h) \circ I_{00} = f^+ \circ I_0 \circ \dot{S}_0(h) \circ I_0 \\ &= N(f) \circ I_0 = f_0 = f \circ S_0(h) \circ I_0 = h \circ \tau^N \circ S(I_0, 1) \circ I_0 \\ &= h \circ \tau^N \circ I_{00} = h \circ \dot{\tau}_0 \circ I_0. \end{aligned} \tag{31}$$

On the other hand, by (29) and the supposition $h \circ \sigma \leq h \circ \dot{\tau}_{01}$ we have

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \circ I_1 \leq h \circ \dot{\tau}_0 \circ I_1,$$

whence (using the supposition that the functor $+$ is increasing on both arguments) we conclude that

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \leq h \circ \dot{\tau}_0.$$

In this way it would be enough to show that

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot\tau_{1i} \le h \circ \dot\tau_{1i}$$

for both $i = 0, 1$. The case with $i = 1$ is easier:

$$\begin{aligned} f^+ \circ S_0^+(h) \circ \alpha \circ \dot\tau_{11} &= f^+ \circ S_0^+(h) \circ I_{011} = N(f) \circ \dot S_0(h) \circ I_{11} = N(f) \circ I_{11} \circ R(h) \\ &= f_{11} \circ R(h) = f \circ S_0(h) \circ I_{11} = h \circ \tau^N \circ S(I_0, 1) \circ I_{11} = h \circ \dot\tau_{11}. \end{aligned}$$

For the case $i = 0$ we again consider cases, as follows:

$$\begin{aligned} f^+ \circ S_0^+(h) \circ \alpha \circ \dot\tau_{10} \circ (1 \odot I_0) &= f^+ \circ S_0^+(h) \circ I_{010} = N(f) \circ \dot S_0(h) \circ I_{10} \\ &= N(f) \circ I_{10} \circ (h \odot 1) = f_{10} \circ (1 \odot f_0) \circ (h \odot 1), \end{aligned}$$

and using the chain of equalities (31) and the equality (27), we have

$$\begin{aligned} f^+ \circ S_0^+(h) \circ \alpha \circ \dot\tau_{10} \circ (1 \odot I_0) &= f_{10} \circ (h \odot h \circ \dot\tau_{00}) = f \circ S_0(h) \circ I_{10} \circ (1 \odot \dot\tau_{00}) \\ &= f \circ S_0(h) \circ I_{10} \circ (1 \odot \dot\tau_{00}) = h \circ \tau^N \circ S(I_0, 1) \circ I_{10} \circ (1 \odot \dot\tau_{00}) \\ &= h \circ \mu \circ (1 \odot \dot\tau_0) \circ (1 \odot I_0) = h \circ \dot\tau_{10} \circ (1 \odot I_0); \end{aligned}$$

on the other hand (using twice (27) and the supposition $h \circ \sigma \le h \circ \dot\tau_{01}$),

$$\begin{aligned} f^+ \circ S_0^+(h) \circ \alpha \circ \dot\tau_{10} \circ (1 \odot I_1) &= f^+ \circ S_0^+(h) \circ I_1 \circ \mu \circ (1 \odot \sigma) \\ &= f^+ \circ I_1 \circ h \circ \mu \circ (1 \odot \sigma) = h \circ \mu \circ (1 \odot \sigma) = h \circ \tau^N \circ S(I_0, 1) \circ I_{10} \circ (1 \odot \sigma) \\ &= f \circ S_0(h) \circ I_{10} \circ (1 \odot \sigma) = f_{10} \circ (h \odot h \circ \sigma) \\ &\le f_{10} \circ (h \odot h \circ \dot\tau_{01}) = f \circ S_0(h) \circ I_{10} \circ (1 \odot \dot\tau_{01}) \\ &= h \circ \tau^N \circ S(I_0, 1) \circ I_{10} \circ (1 \odot \dot\tau_{01}) = h \circ \mu \circ (1 \odot \dot\tau_{01}) \\ &= h \circ \mu \circ (1 \odot \dot\tau_0) \circ (1 \odot I_1) = h \circ \dot\tau_{10} \circ (1 \odot I_1). \end{aligned}$$

From the last two chains of equalities and inequalities we conclude

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot\tau_{10} \le h \circ \dot\tau_{10},$$

using again the fact that the composition $\circ$ and the co-product functor $+$ are increasing with respect to $\le$.

Proposition 3.3. *We have the equality*

$$\alpha \circ \mu = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha). \tag{32}$$

$$\begin{array}{ccccc} \dot T \odot \dot T & \xrightarrow{\ \mu\ } & \dot T & \xrightarrow{\ \alpha\ } & S_0^+(\dot T) \\ \Big\downarrow {\scriptstyle 1 \odot \alpha} & & & & \Big\uparrow {\scriptstyle [1, S_0^+(\mu)]} \\ \dot T \odot S_0^+(\dot T) & & \xrightarrow{\ \vartheta_0^+\ } & & S_0^+(\dot T) + S_0^+(\dot T \odot \dot T) \end{array}$$

Remark. Note the analogy between (32) and the equality (9) written in the form $\dot\tau^{-1} \circ \mu = [1, \dot S(\mu)] \circ \dot\vartheta \circ (1 \odot \dot\tau^{-1})$.

Proof. Since $\dot\tau$ is an isomorphism, it is enough to show

$$\alpha \circ \mu \circ (1 \odot \dot\tau) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot\tau),$$

which according to (9) is equivalent to

$$\alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}).$$

To prove the last equality, denote its left- and right-hand sides with φ and ψ, respectively, and consider cases as in the definition of α. Indeed, using definitions of $\dot{\vartheta}$, ϑ_0^+ and α, we have

$$\varphi \circ (1 \odot I_{00}) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{00}) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{010} \circ (1 \odot I_0)$$
$$= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) = I_{010},$$

and

$$\psi \circ (1 \odot I_{00}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{00}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_{00})$$
$$= [1, S_0^+(\mu)] \circ (I_0 + I_0) \circ \dot{\vartheta}_0 \circ (1 \odot I_0) = [I_0, S_0^+(\mu) \circ I_0] \circ I_{010}$$
$$= [I_0, I_0 \circ \dot{S}_0(\mu)] \circ I_{010} = I_{010} = \varphi \circ (1 \odot I_{00}),$$

where $\dot{\vartheta}_0$ is the natural transformation $\dot{\vartheta}$ for the functor S_0 instead of S. In a similar way,

$$\psi \circ (1 \odot I_{01}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{01}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_1 \circ \sigma)$$
$$= [1, S_0^+(\mu)] \circ I_{11} \circ (1 \odot \sigma) = S_0^+(\mu) \circ I_1 \circ (1 \odot \sigma) = I_1 \circ \mu \circ (1 \odot \sigma)$$
$$= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{010} \circ (1 \odot I_1)$$
$$= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{01}) = \varphi \circ (1 \odot I_{01}),$$

whence by considering cases we conclude

$$\varphi \circ (1 \odot I_0) = \psi \circ (1 \odot I_0).$$

Next we have

$$\psi \circ (1 \odot I_{10} \circ (1 \odot I_0)) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0))$$
$$= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_{010}) = [1, S_0^+(\mu)] \circ (I_0 + I_0) \circ \dot{\vartheta}_0 \circ (1 \odot I_{10})$$
$$= [I_0, I_0 \circ \dot{S}_0(\mu)] \circ \dot{\vartheta}_0 \circ (1 \odot I_{10}) = [I_0, I_0 \circ \dot{S}_0(\mu)] \circ I_{110} \circ \bar{a}$$
$$= I_0 \circ \dot{S}_0(\mu) \circ I_{10} \circ \bar{a} = I_{010} \circ (\mu \odot 1) \circ \bar{a},$$

and

$$\varphi \circ (1 \odot I_{10} \circ (1 \odot I_0)) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{10} \circ (1 \odot I_0))$$
$$= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{110} \circ \bar{a} \circ (1 \odot (1 \odot I_0)) = \alpha \circ \dot{\tau} \circ \dot{S}(\mu) \circ I_{10} \circ \bar{a} \circ (1 \odot (1 \odot I_0))$$
$$= \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ \bar{a} \circ (1 \odot (1 \odot I_0)) = \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ (1 \odot I_0) \circ \bar{a}$$
$$= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) \circ (\mu \odot 1) \circ \bar{a} = I_{010} \circ (\mu \odot 1) \circ \bar{a} = \psi \circ (1 \odot I_{10} \circ (1 \odot I_0)).$$

Furthermore,

$$\psi \circ (1 \odot I_{10} \circ (1 \odot I_1)) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1))$$
$$= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_1 \circ \mu \circ (1 \odot \sigma)) = [1, S_0^+(\mu)] \circ I_{11} \circ (1 \odot \mu \circ (1 \odot \sigma))$$
$$= S_0^+(\mu) \circ I_1 \circ (1 \odot \mu \circ (1 \odot \sigma)) = I_1 \circ \mu \circ (1 \odot \mu \circ (1 \odot \sigma)),$$

and using also Corollary 2.2,

$$
\begin{aligned}
&\varphi \circ (1 \odot I_{10} \circ (1 \odot I_1)) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{10} \circ (1 \odot I_1)) \\
&= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{110} \circ \bar{a} \circ (1 \odot (1 \odot I_1)) = \alpha \circ \dot{\tau} \circ \dot{S}(\mu) \circ I_{10} \circ \bar{a} \circ (1 \odot (1 \odot I_1)) \\
&= \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ \bar{a} \circ (1 \odot (1 \odot I_1)) = \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ (1 \odot I_1) \circ \bar{a} \\
&= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) \circ (\mu \odot 1) \circ \bar{a} = I_1 \circ \mu \circ (1 \odot \sigma) \circ (\mu \odot 1) \circ \bar{a} \\
&= I_1 \circ \mu \circ (\mu \odot 1) \circ \bar{a} \circ (1 \odot (1 \odot \sigma)) = I_1 \circ \mu \circ (1 \odot \mu \circ (1 \odot \sigma)) \\
&= \psi \circ (1 \odot I_{10} \circ (1 \odot I_1)),
\end{aligned}
$$

whence we obtain

$$\varphi \circ (1 \odot I_{10}) = \psi \circ (1 \odot I_{10}).$$

Finally,

$$
\begin{aligned}
\psi \circ (1 \odot I_{11}) &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{11}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_{011}) \\
&= [1, S_0^+(\mu)] \circ (I_0 + I_0) \circ \dot{\vartheta}_0 \circ (1 \odot I_{11}) \\
&= [I_0, I_0 \circ \dot{S}_0(\mu)] \circ I_{111} \circ \vartheta_R = I_0 \circ \dot{S}_0(\mu) \circ I_{11} \circ \vartheta_R \\
&= I_{011} \circ R(\mu) \circ \vartheta_R
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi \circ (1 \odot I_{11}) &= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{111} \circ \vartheta_R = \alpha \circ \dot{\tau} \circ \dot{S}(\mu) \circ I_{11} \circ \vartheta_R \\
&= \alpha \circ \dot{\tau}_{11} \circ R(\mu) \circ \vartheta_R = I_{011} \circ R(\mu) \circ \vartheta_R = \psi \circ (1 \odot I_{11}).
\end{aligned}
$$

Definition. Given an S_0-algebra $f : S_0(X) \to X$ in $\mathcal{C}$ and an $\mathcal{C}$-arrow $\alpha : \dot{T} \to S_0^+(\dot{T})$, another $\mathcal{C}$-arrow $h : \dot{T} \to X$ will be called an *α-minimal evaluator* of f iff h is the least solution of the inequality

$$f^+ \circ S_0^+(\eta) \circ \alpha \leq \eta \tag{33}$$

with respect to η in $\mathcal{C}(\dot{T}, X)$ and for all $\chi \in \mathcal{C}(\dot{T}, X)$ and $\psi \in \mathcal{C}(\dot{T} \odot \dot{T}, X)$ satisfies an additional condition, written symbolically as follows:

$$\forall \eta \in \mathcal{C}(\dot{T}, X)(\varphi \circ (1 \odot \eta) \leq \psi \Rightarrow \varphi \circ (1 \odot f^+ \circ S_0^+(\eta) \circ \alpha) \leq \psi) \Rightarrow \varphi \circ (1 \odot h) \leq \psi, \quad (*)$$

where $\varphi = f_{10} \circ (\chi \odot 1) : \dot{T} \odot X \to X$.

Lemma 3.1. *Let $\sigma : B_1 \to \dot{T}$ be a system with analyzer $\alpha : \dot{T} \to S_0^+(\dot{T})$, let $f : S_0(X) \to X$ be an S_0-algebra in $\mathcal{C}$ such that $f_1 \in \mathbf{SA}(\mathcal{C})$, and let $h : \dot{T} \to X$ be an α-minimal evaluator of f. Then for every $\mathcal{C}$-arrow $\chi : \dot{T} \to X$ the arrow $f_{10} \circ (\chi \odot h) : \dot{T} \odot \dot{T} \to X$ is the least solution of the inequality*

$$\chi' \circ (1 + S_0^+(\zeta)) \circ \vartheta_0^+ \circ (1 \odot \alpha) \leq \zeta \tag{34}$$

with respect to ζ in $\mathcal{C}(\dot{T} \odot \dot{T}, X)$, where $\chi' = f^+ \circ [S_0^+(\chi), 1] : S_0^+(\dot{T}) + S_0^+(X) \to X$.

Remark. Note the analogy of (34) with (13).

Proof. The arrow h being a solution of (33), we have

$$
\begin{aligned}
f_{10} \circ (\chi \odot h) &\geq f_{10} \circ (\chi \odot f^+ \circ S_0^+(h) \circ \alpha) \\
&= f_{10} \circ (1 \odot f^+) \circ (\chi \odot S_0^+(h)) \circ (1 \odot \alpha)
\end{aligned}
$$

$$= f^+ \circ [1, S_0^+(f_{10})] \circ \vartheta_0^+ \circ (\chi \odot S_0^+(h)) \circ (1 \odot \alpha) \quad \text{(by Proposition 3.1)}$$
$$= f^+ \circ [1, S_0^+(f_{10})] \circ (S_0^+(\chi) + S_0^+(\chi \odot h)) \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$\text{(since } \vartheta_0^+ \text{ is natural)}$$
$$= f^+ \circ [S_0^+(\chi), S_0^+(f_{10} \circ (\chi \odot h))] \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= \chi' \circ (1 + S_0^+(f_{10} \circ (\chi \odot h))) \circ \vartheta_0^+ \circ (1 \odot \alpha) \qquad \text{(by definition of } \chi'),$$

i.e. $\zeta = f_{10} \circ (\chi \odot h)$ satisfies (34). For an arbitrary solution ζ of (34) in $\mathcal{C}(\dot{T} \odot \dot{T}, X)$ we shall show that $f_{10} \circ (\chi \odot h) \leq \zeta$, using the additional condition (*) in the definition of α-minimal evaluator. For an arbitrary $\eta \in \mathcal{C}(\dot{T}, X)$ suppose $f_{10} \circ (\chi \odot 1) \circ (1 \odot \eta) \leq \zeta$, i.e. $f_{10} \circ (\chi \odot \eta) \leq \zeta$. Then

$$f_{10} \circ (\chi \odot 1) \circ (1 \odot f^+ \circ S_0^+(\eta) \circ \alpha) = f_{10} \circ (1 \odot f^+) \circ (\chi \odot S_0^+(\eta)) \circ (1 \odot \alpha)$$
$$= f^+ \circ [1, S_0^+(f_{10})] \circ \vartheta_0^+ \circ (\chi \odot S_0^+(\eta)) \circ (1 \odot \alpha)$$
$$= f^+ \circ [1, S_0^+(f_{10})] \circ (S_0^+(\chi) + S_0^+(\chi \odot \eta)) \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= f^+ \circ [S_0^+(\chi), S_0^+(f_{10} \circ (\chi \odot \eta))] \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= \chi' \circ (1 + S_0^+(f_{10} \circ (\chi \odot \eta))) \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$\leq \chi' \circ (1 + S_0^+(\zeta)) \circ \vartheta_0^+ \circ (1 \odot \alpha) \leq \zeta \quad \text{(by (34))}.$$

This proves the hypothesis in (*) with $\psi = \zeta$, whence $\varphi \circ (1 \odot h) \leq \zeta$, i.e.

$$f_{10} \circ (\chi \odot h) = f_{10} \circ (\chi \odot 1) \circ (1 \odot h) = \varphi \circ (1 \odot h) \leq \zeta.$$

Theorem 3.1. *Let $\sigma : B_1 \to \dot{T}$ be a system with analyzer $\alpha : \dot{T} \to S_0^+(\dot{T})$, let $f : S_0(X) \to X$ be an S_0-algebra in $\mathcal{C}$ such that $f_1 \in \mathbf{SA}(\mathcal{C})$, and let $h : \dot{T} \to X$ be an α-minimal evaluator of f. Then $h : P(\tau^N) \to f$ is a morphism in $\mathcal{C}_{S_0}$, i.e.* (27) *holds.*

Proof. By Lemma 3.1 $f_{10} \circ (h \odot h)$ is the least solution of

$$h' \circ (1 + S_0^+(\zeta)) \circ \vartheta_0^+ \circ (1 \odot \alpha) \leq \zeta \tag{35}$$

with respect to $\zeta \in \mathcal{C}(\dot{T} \odot \dot{T}, X)$, where $h' = f^+ \circ [S_0^+(h), 1]$. But the arrow $h \circ \mu : \dot{T} \odot \dot{T} \to X$ satisfies (35) because

$$h' \circ (1 + S_0^+(h \circ \mu)) \circ \vartheta_0^+ \circ (1 \odot \alpha) = f^+ \circ [S_0^+(h), S_0^+(h \circ \mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= f^+ \circ S_0^+(h) \circ [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha) = f^+ \circ S_0^+(h) \circ \alpha \circ \mu \qquad \text{(by (32))}$$
$$\leq h \circ \mu.$$

Therefore

$$f_{10} \circ (h \odot h) \leq h \circ \mu, \tag{36}$$

which is the same as

$$f \circ S_0(h) \circ I_{10} \leq h \circ \tau^N \circ S(I_0, 1) \circ I_{10}.$$

On the other hand,

$$f \circ S_0(h) \circ I_0 = f_0 = N(f) \circ \dot{S}_0(h) \circ I_0 = f^+ \circ S_0^+(h) \circ I_{00} = f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{00}$$
$$\leq h \circ \dot{\tau}_{00} = h \circ \tau^N \circ I_{00} = h \circ \tau^N \circ S(I_0, 1) \circ I_0$$

and

$$f \circ S_0(h) \circ I_{11} = f_{11} \circ R(h) = N(f) \circ \dot{S}_0(h) \circ I_{11} = f^+ \circ S_0^+(h) \circ I_{011}$$
$$= f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{11} \leq h \circ \dot{\tau}_{11} = h \circ \tau^N \circ S(I_0, 1) \circ I_{11},$$

whence by considering cases

$$f \circ S_0(h) \leq h \circ \tau^N \circ S(I_0, 1). \tag{37}$$

To prove the reverse inequality, consider the arrows

$$x = h \circ \dot{\tau}_{01} : B_1 \to X$$

and

$$f_h = Q(x, f) = [[f_0, x], f_1] : S(X) \to X.$$

We shall prove that

$$f_h \circ S(h) \leq h \circ \tau^N. \tag{38}$$

Indeed, since h is the least solution of (33) with respect to η, we have $f^+ \circ S_0^+(h) \circ \alpha = h$, whence

$$x = h \circ \dot{\tau}_{01} = f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{01} = f^+ \circ S_0^+(h) \circ I_1 \circ \sigma = h \circ \sigma$$

and

$$f_h \circ S(h) \circ I_0 = f_h \circ I_0 = [f_0, x] = [f_0, h \circ \sigma]$$
$$= [f^+ \circ S_0^+(h) \circ I_{00}, f^+ \circ S_0^+(h) \circ I_1 \circ \sigma]$$
$$= [f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{00}, f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{01}]$$
$$= f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \circ [I_0, I_1]$$
$$= f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 = h \circ \dot{\tau}_0 = h \circ \tau^N \circ I_0.$$

Again, as before,

$$f_h \circ S(h) \circ I_{11} = f_{11} \circ R(h) = f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{11} = h \circ \dot{\tau}_{11} = h \circ \tau^N \circ I_{11}$$

and the inequality

$$f_h \circ S(h) \circ I_{10} \leq h \circ \tau^N \circ I_{10}$$

is the same as (36). Thus (38) is proved by considering cases. A composition of the last one with $S(\dot{\nu})$ yields

$$f_h \circ S(h \circ \dot{\nu}) \leq h \circ \tau^N \circ S(\dot{\nu}) = h \circ \dot{\nu} \circ \tau$$

(we use (14)), whence follows the inequality

$$\tilde{f}_h \leq h \circ \dot{\nu} \tag{39}$$

for the evaluator $\tilde{f}_h : T \to X$ of the algebra $f_h : S(X) \to X$ with respect to the least fixed point $\tau : S(T) \to T$, using the supposition that the least fixed point τ is proper one. We shall prove the reverse of (39) by showing that

$$h \leq \tilde{f}_h \circ \iota. \tag{40}$$

Indeed, the algebra f_h belongs to $\mathfrak{D}$ and by Propositions 2.1 and 2.2 the normal e-valuator h_1 of f_h exists, and by Corollary 2.1 (c′) $h_1 \circ \dot{\nu} = \tilde{f}_h$, whence by Proposition 2.3 $h_1 = \tilde{f}_h \circ \iota$. So $\tilde{f}_h \circ \iota$ is a morphism $\tau^N \to f_h$ in $\mathfrak{C}_S$, i.e.

$$\tilde{f}_h \circ \iota \circ \tau^N = f_h \circ S(\tilde{f}_h \circ \iota),$$

and composing the last equality from right with $S(I_0, 1)$, we obtain

$$\tilde{f}_h \circ \iota \circ \tau^N \circ S(I_0, 1) = f \circ S_0(\tilde{f}_h \circ \iota).$$

Moreover, using the inequality $\tilde{f}_h \circ \iota \leq h$ which follows from (39), we have

$$\begin{aligned}\tilde{f}_h \circ \iota \circ \sigma \leq h \circ \sigma = x = [f_0, x] \circ I_1 = f_h \circ I_{01} = f_h \circ S(\tilde{f}_h) \circ I_{01} \\ = \tilde{f}_h \circ \tau \circ I_{01} = \tilde{f}_h \circ N(\tau) \circ \dot{S}(\iota) \circ I_{01} = \tilde{f}_h \circ \iota \circ \dot{\tau}_{01}.\end{aligned}$$

Thence by Proposition 3.2 it follows

$$f^+ \circ S_0^+(\tilde{f}_h \circ \iota) \circ \alpha \leq \tilde{f}_h \circ \iota,$$

and since h is the least solution of (33), this implies (40). Using Corollary 2.1(a), from (40) we obtain the reverse inequality of (39):

$$h \circ \dot{\nu} \leq \tilde{f}_h \circ \iota \circ \dot{\nu} = \tilde{f}_h \circ \nu = \tilde{f}_h.$$

Thus we get the equality $h \circ \dot{\nu} = \tilde{f}_h$, whence it follows

$$h \circ \tau^N \circ S(\dot{\nu}) = h \circ \dot{\nu} \circ \tau = f_h \circ S(h \circ \dot{\nu}),$$

and composing this from right with $S(I_0, \iota)$, we obtain

$$h \circ \tau^N \circ S(I_0, 1) = f \circ S_0(h).$$

4. CODING FORMALIZM AND CODE FACTORIZATION OF THE MINIMAL EVALUATOR

The code evaluation method in algebraic recursion theory uses coding to obtain certain simple standard expression for the *minimal evaluator* of a system of inequalities. In the context of the previous Section 3 the last evaluator may be defined as the evaluator with respect to the least fixed point $\tau : S(T) \to T$ of the algebra $Q(x, f)$, where $f : S_0(X) \to X$ is an S_0-algebra in the SRC $\mathcal{C}$ and $x : B_1 \to X$ is the least solution of a 'system' $\sigma : B_1 \to \dot{T}$ in the algebra f. In the present section we propose a conceptual mechanism for treatment of coding on categorical level in the context of Section 3. We give also an interpretation for the case of SRC of posets, which shows how usual coding theorem in operative spaces (in the sense of [1]) can be obtained as a special case.

We assume suppositions and notations of Section 3, especially, we shall have fixed a partially ordered SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$. In the special case with $\mathcal{C}$ — the category of posets, $\odot$ and $\bar{a}$ — the usual product $\times$ in $\mathcal{C}$ and the natural isomorphism of associativity of $\times$, respectively, $R(X)$ — the Cartesian square $X \times X$, and ϑ_R — the natural transformation defined as in the SRC of sets in Section 2, we shall call the 5-tuple $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ the 'SRC of posets'. It is a partially ordered SRC with respect to the obvious order mentioned in Section 1. We fix also an S_0-algebra $f : S_0(X) \to X$.

Consider a set Ξ of natural in $Y \in \mathcal{C}$ transformations $\xi : Y \to X \odot Y$. We shall say for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ that the last one is *linearized* by a natural in $Y \in \mathcal{C}$ transformation $\lambda : X \odot F(Y) \to F(X \odot Y)$ with respect to Ξ iff for all $\xi \in \Xi$ we have

$$F(\xi) = \lambda \circ \xi,$$

which is a short one for the equality

$$F(\xi(Y)) = \lambda(Y) \circ \xi(F(Y)),$$

expressing the commutativity of the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\ \xi\ } & X \odot F(Y) \\ & {\scriptstyle F(\xi)} \searrow & \downarrow \lambda \\ & & F(X \odot Y) \end{array}$$

for all $Y \in \mathcal{C}$.

For example, in the SRC of posets the endofunctor R is linearized by the natural transformation ϑ_R with respect to the set of all natural transformations $\xi : Y \to X \times Y$ of the form $\xi(y) = (x, y)$ for fixed $x \in X$.

In the general case, the set Ξ is partially ordered in a natural way:

$$\xi \leq \xi' \iff \forall Y \in \mathcal{C}(\xi(Y) \leq \xi'(Y)).$$

Given a $\mathcal{C}$-morphism $g : X \to X$ such that $(g \odot 1) \circ \xi \in \Xi$ for all $\xi \in \Xi$, we may consider the inequality

$$(g \odot 1) \circ \xi \leq \xi \tag{41}$$

with an unknown $\xi \in \Xi$.

Definition 4.1. An element $\omega \in \Xi$ will be called Φ-pseudominimal solution of (41) for a subset $\Phi \subseteq \mathcal{C}(X \odot \dot{T}, X)$ iff ω is a solution of (41), i.e. $(g \odot 1) \circ \omega(Y) \leq \omega(Y)$ for all $Y \in \mathcal{C}$, and the following two conditions hold for all $\mathcal{C}$-arrows $\chi : \dot{T} \to X$ and $\psi : \dot{T} \odot \dot{T} \to X$ and all $\varphi \in \Phi$:

$$\forall \xi \in \Xi(\varphi \circ \xi \leq \chi \ \Rightarrow\ \varphi \circ (g \odot 1) \circ \xi \leq \chi) \ \Rightarrow\ \varphi \circ \omega \leq \chi, \tag{$*'$}$$

$$\forall \xi \in \Xi(\varphi' \circ (1 \odot \xi) \leq \psi \ \Rightarrow\ \varphi' \circ (1 \odot (g \odot 1) \circ \xi) \leq \psi) \ \Rightarrow\ \varphi' \circ (1 \odot \omega) \leq \psi, \tag{$*''$}$$

where $\varphi' = f_{10} \circ (\chi \odot \varphi)$.

A remark on notations. As usual, we do not write the arguments in a natural transformation; e.g., writing $\varphi \circ \xi$, we mean this instance of ξ which makes it composable with φ; in this way ξ in $(*')$ and $(*'')$ is $\xi(\dot{T})$ and ω is $\omega(\dot{T})$.

Definition 4.2. Let Ξ be the set of natural transformations $\xi : Y \to X \odot Y$ as above, and let $g : X \to X$ be a $\mathcal{C}$-arrow such that $(g \odot 1) \circ \xi \in \Xi$ for all $\xi \in \Xi$. Let also $\sigma : B_1 \to \dot{T}$ be a 'system', i.e. an $\mathcal{C}$-arrow with analizer $\alpha : \dot{T} \to S_0^+(\dot{T})$. Then by *coding* for the system σ in the algebra $f : S_0(X) \to X$ with respect to Ξ and g we mean a pair $\langle \kappa, \lambda_0^+ \rangle$ consisting of an $\mathcal{C}$-arrow $\kappa : X \odot \dot{T} \to X$ and a natural transformation

$$\lambda_0^+ : X \odot S_0^+(Y) \to S_0^+(X \odot Y),$$

which linearizes S_0^+ with respect to Ξ, such that the following equality holds for the *coding morphism* κ:

$$\kappa \circ (g \odot 1) = f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha). \tag{42}$$

$$\begin{array}{ccccc}
X \odot \dot{T} & \xrightarrow{\quad g \odot 1 \quad} & X \odot \dot{T} & \xrightarrow{\quad \kappa \quad} & X \\
\Big\downarrow {\scriptstyle 1 \odot \alpha} & & & & \Big\uparrow {\scriptstyle f^+} \\
X \odot S_0^+(\dot{T}) & \xrightarrow{\quad \lambda_0^+ \quad} & S_0^+(X \odot \dot{T}) & \xrightarrow{\quad S_0^+(\kappa) \quad} & S_0^+(X)
\end{array}$$

Theorem 4.1. *Let $\langle \kappa, \lambda_0^+ \rangle$ be a coding for $\sigma : B_1 \to \dot{T}$ in $f : S_0(X) \to X$ with respect to Ξ, g be as in the previous definition, and let $\omega \in \Xi$ be a $\{\kappa\}$-pseudominimal solution of (41). Then the morphism $\kappa \circ \omega : \dot{T} \to X$ is an α-minimal evaluator of f, where α is the analizer of σ.*

Proof. The proof is rather straightforward:

$$\begin{aligned}
f^+ \circ S_0^+(\kappa \circ \omega) \circ \alpha &= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ \omega \circ \alpha && \text{(because } \lambda_0^+ \text{ linearizes } S_0^+\text{)} \\
&= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha) \circ \omega && \text{(since } \omega \text{ is natural)} \\
&= \kappa \circ (g \odot 1) \circ \omega && \text{(by (42))} \\
&\le \kappa \circ \omega && \text{(since } \omega \text{ is a solution of (41)),}
\end{aligned}$$

i.e. $\kappa \circ \omega$ is a solution of (33). To show that it is the least solution of (33), suppose $\kappa \circ \xi \le \eta$ for an arbitrary solution $\eta : \dot{T} \to X$ of (33) in $\mathcal{C}(\dot{T}, X)$ and an arbitrary $\xi \in \Xi$. Then

$$\begin{aligned}
\kappa \circ (g \odot 1) \circ \xi &= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha) \circ \xi && \text{(by (42))} \\
&= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ \xi \circ \alpha && \text{(since } \xi \text{ is natural)} \\
&= f^+ \circ S_0^+(\kappa \circ \xi) \circ \alpha && \text{(because } \lambda_0^+ \text{ linearizes } S_0^+\text{)} \\
&\le f^+ \circ S_0^+(\eta) \circ \alpha \le \eta.
\end{aligned}$$

This proves the hypothesis in $(*')$ for $\varphi = \kappa$ and $\chi = \eta$, and since ω is a $\{\kappa\}$-pseudominimal solution of (41), we obtain $\kappa \circ \omega \le \eta$. Therefore $\kappa \circ \omega$ is the least solution of (33). To check the condition $(*)$, take arbitrary $\mathcal{C}$-arrows $\chi : \dot{T} \to X$ and $\psi : \dot{T} \odot \dot{T} \to X$, let $\varphi = f_{10} \circ (\chi \odot 1)$ and suppose also that for every $\mathcal{C}$-arrow $\eta : \dot{T} \to X$

$$\varphi \circ (1 \odot \eta) \le \psi \;\Rightarrow\; \varphi \circ (1 \odot f^+ \circ S_0^+(\eta) \circ \alpha) \le \psi. \tag{43}$$

We have then to prove that

$$\varphi \circ (1 \odot \kappa \circ \omega) \le \psi.$$

For an arbitrary $\xi \in \Xi$ suppose $\varphi \circ (1 \odot \kappa) \circ (1 \odot \xi) \le \psi$. Then by (43)

$$\varphi \circ (1 \odot f^+ \circ S_0^+(\kappa \circ \xi) \circ \alpha) \le \psi,$$

and therefore

$$\begin{aligned}
\varphi \circ (1 \odot \kappa) \circ (1 \odot (g \odot 1) \circ \xi) &= \varphi \circ (1 \odot f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha) \circ \xi) \\
&= \varphi \circ (1 \odot f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ \xi \circ \alpha) \\
&= \varphi \circ (1 \odot f^+ \circ S_0^+(\kappa \circ \xi) \circ \alpha) \le \psi,
\end{aligned}$$

which proves the hypothesis in $(*'')$ for $\varphi' = f_{10} \circ (\chi \odot \kappa) = \varphi \circ (1 \odot \kappa)$, and since ω is a $\{\kappa\}$-pseudominimal solution of (41), we conclude that $\varphi \circ (1 \odot \kappa) \circ (1 \odot \omega) \leq \psi$, i.e. $\varphi \circ (1 \odot \kappa \circ \omega) \leq \psi$. Therefore $\kappa \circ \omega$ is an α-minimal evaluator of f.

Now consider the special case with the SRC $\langle \mathcal{C}, \times, \bar{a}, R, \vartheta_R \rangle$ of posets. Take an *operative space* X in the sense of Ivanov [1]. This is, up to notational variations, a partially ordered algebra X with two binary operations — *multiplication* (denoted in the usual way: xy is the result of applying this operation on $x, y \in X$) and *pairing* (notation: $[x, y]$ for the result of applying this operation on $x, y \in X$) and three constants $\boldsymbol{e}$, $\boldsymbol{i}_0$, $\boldsymbol{i}_1$ such that the multiplication is associative with the unit $\boldsymbol{e}$ and the following three equalities hold for all $x, y, y' \in X$:

$$x[y, y'] = [xy, xy']; \qquad [x, y]\boldsymbol{i}_0 = x; \qquad [x, y]\boldsymbol{i}_1 = y.$$

Consider also a set $B_0 \in \mathcal{C}$ with the trivial partial order (coinciding with equality) and a mapping $f_0 : B_0 \to X$. The set B_0 is supposed to contain three different elements regarded as symbols for the constants $\boldsymbol{e}$, $\boldsymbol{i}_0$, $\boldsymbol{i}_1$, and f_0 is supposed to map those symbols on those constants, respectively. The other elements of B_0 are treated as parameters. The mapping f_0 and the space X determine an S_0-algebra $f : S_0(X) \to X$ in $\mathcal{C}$ such that f_{10} and f_{11} are multiplication and pairing in X, respectively. As in Section 3, we consider also a set B_1 with the trivial partial order, the elements of which are treated as variables. The sum (i.e. the disjoint union) $B = B_0 + B_1$ has also the trivial partial order, and such is the order in the objects T and $\dot{T}$ of the least fixed points $\tau : S(T) \to T$ and $\dot{\tau} : \dot{S}(\dot{T}) \to \dot{T}$, respectively, the elements of which are all terms and normal terms (in the sense, for instance, of [5]), respectively.

Take for Ξ the set of all natural in $Y \in \mathcal{C}$ transformations $\xi : Y \to X \times Y$ defined for all $Y \in \mathcal{C}$ and all $y \in Y$ by

$$\xi(y) = (x, y), \tag{44}$$

where $x \in X$. In the category $\mathcal{C}$ of posets Ξ and X are isomorphic — the obvious isomorphism assigns to each $\xi \in \Xi$ the unique $x \in X$ for which (44) holds for every $y \in Y$ and all $Y \in \mathcal{C}$. This isomorphism transforms the inequality (41) into the inequality $g(x) \leq x$ with one unknown $x \in X$ for every $\mathcal{C}$-arrow (i.e. an increasing mapping) $g : X \to X$. The notion of the Φ-pseudominimal solution of (41) is transformed as follows.

An element $\omega \in \Xi$ is a Φ-pseudominimal solution of (41) iff every subset $J \subseteq X$ of one of the following two forms:

$$\begin{cases} \{x \in X \mid \varphi(x, t) \leq \chi(t) \text{ for all } t \in \dot{T}\}, \\ \{x \in X \mid \chi(s)\varphi(x, t) \leq \psi(s, t) \text{ for all } t, s \in \dot{T}\}, \end{cases} \tag{45}$$

where $\varphi \in \Phi$, $\chi : \dot{T} \to X$ and $\psi : \dot{T} \times \dot{T} \to X$ are arbitrary $\mathcal{C}$-mappings, which is invariant with respect to g, i.e. $g(J) \subseteq J$, contains the element $w \in X$ corresponding to ω (i.e. $\omega(y) = (w, y)$ for all $y \in Y$ and $Y \in \mathcal{C}$).

Take for Φ the set of all mappings $\varphi : X \times \dot{T} \to X$ of the form $\varphi(x, t) = xk(t)$, where $k : \dot{T} \to X$ is an arbitrary function (since the order in $\dot{T}$ is trivial, all such functions belong to $\mathcal{C}$); and take for g the mapping $g : X \to X$ defined by

$g(x) = [e, x]r$, where $r \in X$. We shall call an element $w \in X$ an *iteration* of $r \in X$ iff $g(w) = w$ and $w \in J$ for every set of one of the forms (45) such that $g(J) \subseteq J$. The supposition that every $r \in X$ has an iteration is a possible version of the notion of iterativity for the operative space X. (This version differs from the version of Ivanov in [1] and is close to the version in [5]. However, it is a natural version — the examples of iterative spaces in [1] are typically iterative in this sense also.) Therefore, supposing the space X iterative in this sense, we have that for every mapping g of the form $g(x) = [e, x]r$ there is a Φ-pseudominimal solution $\omega \in X$ of (41).

Next define the natural in $Y \in \mathcal{C}$ transformation

$$\lambda_0^+ : X \odot S_0^+(Y) \to S_0^+(X \odot Y)$$

by

$$\lambda_0^+ = ((\pi + (\bar{a} + \vartheta_R) \circ \delta_\odot) \circ \delta_\odot + 1) \circ \delta_\odot,$$

where π is the projection $X \times B_0 \to B_0$. A direct checking shows that λ_0^+ linearizes S_0^+ with respect to Ξ. For the coding morphism $\kappa : X \times \dot{T} \to X$ we have to ask that $\kappa \in \Phi$, i.e. $\kappa = f_{10} \circ (1 \times k)$ for a suitable $k : \dot{T} \to X$, and that the coding equality (42) is satisfied. In terms of elements, the last equality is equivalent to the following five ones:

$$\begin{aligned}
&g(x)k(b) = f_0(b) && \text{for all } b \in B_0,\\
&g(x)k(v) = xk(\sigma(v)) && \text{for all } v \in B_1,\\
&g(x)k(tb) = xk(t)f_0(b) && \text{for all } t \in \dot{T} \text{ and } b \in B_0,\\
&g(x)k(tv) = xk(\mu(t, \sigma(v))) && \text{for all } t \in \dot{T} \text{ and } v \in B_1,\\
&g(x)k([t, s]) = x[k(t), k(s)] && \text{for all } t, s \in \dot{T}.
\end{aligned}$$

Here we use short notations for terms in $\dot{T}$: b for $\dot{\tau}_{00}(b)$, v for $\dot{\tau}_{01}(v)$, tb for $\dot{\tau}_{10}(t, \dot{\tau}_{00}(b))$, tv for $\dot{\tau}_{10}(t, \dot{\tau}_{01}(v))$, and $[t, s]$ for $\dot{\tau}_{11}(t, s)$. The mapping σ represents a system of inequalities: $\sigma(v) \leq v$ $(v \in B_1)$. The last five equalities follow easily from the following ones:

$$\begin{aligned}
&rk(b) = \boldsymbol{i}_0 f_0(b) && \text{for all } b \in B_0,\\
&rk(v) = \boldsymbol{i}_1 k(\sigma(v)) && \text{for all } v \in B_1,\\
&rk(tb) = \boldsymbol{i}_1 k(t) f_0(b) && \text{for all } t \in \dot{T} \text{ and } b \in B_0,\\
&rk(tv) = \boldsymbol{i}_1 k(\mu(t, \sigma(v))) && \text{for all } t \in \dot{T} \text{ and } v \in B_1,\\
&rk([t, s]) = \boldsymbol{i}_1 [k(t), k(s)] && \text{for all } t, s \in \dot{T},
\end{aligned}$$

and when the last ones are fulfilled, we say that k and r provide a coding for the system σ with respect to f_0 (compare with the notion of coding in [5, 6]); they can be satisfied comparatively straightforwardly, using a representation of primitive recursive functions and a weak form of axioms for the translation operation (see [5]). This construction of coding combined with the code evaluation theorem implies easily all basic facts of algebraic recursion theory in operative spaces. The last theorem states that if k and r provide a coding for a system σ and w is iteration of r, then the mapping $x : B_1 \to X$ defined by $x(v) = wk(v)$ is the least solution of the system σ; and it follows from Theorems 4.1 and 3.1. Indeed, by Theorem 4.1 $\kappa \circ \omega = f_{10} \circ (1 \times k) \circ \omega$ is an α-minimal evaluator of f, where $\omega(y) = (w, y)$ for all $y \in Y$ and all $Y \in \mathcal{C}$. Thus, for $t \in \dot{T}$ we have $(\kappa \circ \omega)(t) = f_{10}(w, k(t)) = wk(t)$. By Theorem 3.1 $\kappa \circ \omega$ is a $\mathcal{C}_{S_0}$-morphism, whence by Proposition 3.2 it is the least such morphism $h : P(\tau^N) \to f$ satisfying the inequality

$$h(\sigma(v)) \leq h(v) = h(\dot{\tau}_{01}(v))$$

for all $v \in B_1$. Thence it follows that the mapping $x : B_1 \to X$ defined by $x(v) = (\kappa \circ \omega)(v) = wk(v)$ (i.e. the 'restriction' of h on B_1) is the least solution of the system represented by σ, since every mapping $x : B_1 \to X$ can be uniquely extended to a $\mathcal{C}_{S_0}$-morphism $h : P(\tau^N) \to f$, the mapping $h : \dot{T} \to X$ assigning to each term $t \in \dot{T}$ its value under the evaluation provided by x.

In this sense the theory of operative spaces is a special case of the results of Sections 3 and 4. The natural categorical generality for the last theory being thus reached, various other special cases may be expected to be of interest. Especially, we shall mention one of them, which is connected with an attempt by Petrov and Skordev [4] to generalize Skordev's theory of combinatory spaces for some kind of category-like partial ordered structures in which the role of multiplication is played by a composition of arrows. This special case is obtained by applying the theory of Sections 3 and 4 to an SRC $\langle \mathcal{C}, \odot, \tilde{a}, R, \vartheta_R \rangle$ in which $\mathcal{C}$ is a suitable subcategory of the category of directed graphs and $\odot$ is the product $\times_O$ over a fixed set O of objects in the terminology and notations of [2]. It may be optimistically said that in this way a theory of that kind, which was aimed at by Petrov and Skordev in [4], may be reached in full (in [4] only a part of the desirable results has been reached, especially, the corresponding analogue of the recursion theorem has not been obtained). We are leaving this topic for the possible further publications.

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Department of Mathematics and Computer Science
Sofia University
Blvd. James Bourchier 5
1164 Sofia, Bulgaria
E-mail address: `yzshv@fmi.uni-sofia.bg`