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COMPLETE SYSTEMS OF BESSEL AND INVERSED BESSEL POLYNOMIALS IN SPACES OF HOLOMORPHIC FUNCTIONS*

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Let $B_n(z)$, $n = 0, 1, \dots$, be the Bessel polynomials generated by

$$(1-4zw)^{-1/2}\exp\left\{\frac{1-(1-4zw)^{1/2}}{2z}\right\} = \sum_{n=0}^{\infty} B_n(z)w^n, \quad |4zw| < 1$$

and the functions $\widetilde{B}_n(z)$ be defined by the relations

$$\widetilde{B}_n(z) = 4^{-n}z^n B_n(1/z)\exp(-z/2).$$

Let $K = \{k_n\}_{n=0}^{\infty}$ be an increasing sequence of non-negative integers.

Sufficient conditions for the completeness of the systems $\{B_{k_n}(z)\}_{n=0}^{\infty}$ and $\left\{\widetilde{B}_{k_n}(z)\right\}_{n=0}^{\infty}$ in spaces of holomorphic functions are given in terms of the density of the sequence K.

Keywords: holomorphic functions, complete systems, Bessel polynomials.
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1. INTRODUCTION

Let G be an arbitrary region in the complex plane $\mathbb{C}$ and $H(G)$ be the space of the complex functions holomorphic in G. As usual, we consider $H(G)$ with the topology of uniform convergence on compact subsets of G. A system $\{\varphi_n(z)\}_{n=0}^{\infty} \subset$

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$H(G)$ is called complete in $H(G)$ if for every $f \in H(G)$, every compact set $K \subset G$ and every $\varepsilon > 0$ there exists a linear combination

$$P(z) = \sum_{n=0}^{N} c_n \varphi_n(z), \quad c_n \in \mathbb{C}; \quad n = 0, 1, 2, \dots, N,$$

such that $|f(z) - P(z)| < \varepsilon$ whenever $z \in K$. For example, if $G \subset \mathbb{C}$ is simply connected, the system $\{z^n\}_{n=0}^{\infty}$ is complete in $H(G)$ and this assertion is nothing but a particular case of the Runge's approximation theorem [1, (2.1), p. 176].

Let γ be a Jordan curve in $\mathbb{C}$ and C_γ be the closure of its outside with respect to the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By H_γ we denote the (vector) space of all complex functions, holomorphic in an open set containing C_γ and vanishing at infinity. The next statement is a criterion for completeness in the space $H(G)$ [2, Theorem 17, p. 211].

(CC) A system $\{\varphi_n(z)\}_{n=0}^{\infty}$ of complex functions holomorphic in a simply connected region $G \subset \mathbb{C}$ is complete in the space $H(G)$ iff for every rectifiable Jordan curve $\gamma \subset G$ and every function $F \in H$ the equalities

$$\int_\gamma F(z)\varphi_n(z)\, dz = 0, \quad n = 0, 1, 2, \dots,$$

imply $F \equiv 0$.

Completeness of systems of special functions in spaces of holomorphic functions has been considered also by Kazmin [3], Leontiev [4, Ch. 3], Rusev [5–9].

2. BESSEL AND INVERSED BESSEL POLYNOMIALS

Let us define the function $\Phi(z, w)$ as

$$\Phi(z, w) = (1 - 4zw)^{-1/2} \exp\left\{\frac{1 - (1 - 4zw)^{1/2}}{2z}\right\}, \quad |4zw| < 1. \tag{2.1}$$

Note that the identity

$$\frac{1 - (1 - 4zw)^{1/2}}{2z} = \frac{2w}{1 + (1 - 4zw)^{1/2}} \tag{2.2}$$

implies that the point $z = 0$ is a removable singularity of this function for every fixed w.

Let $B_n(z)$, $n = 0, 1, \dots$, be the Bessel polynomials defined by [10, (11.2), VII]

$$\Phi(z, w) = \sum_{n=0}^{\infty} B_n(z) w^n, \quad |4zw| < 1. \tag{2.3}$$

The polynomials $y_n(x; a, b)$ [11, 6] are defined by

$$(1 - 2xt)^{-1/2} \exp\left(\frac{1}{2} - \frac{1}{2}(1 - 2xt)^{1/2}\right)^{2-a} \exp\left(\frac{b}{2x}\left(1 - (1 - 2xt)^{1/2}\right)\right)$$
$$= \sum_{n=0}^{\infty} \left(\frac{b}{2}\right)^n y_n(x; a, b) t^n (n!)^{-1}. \tag{2.4}$$

Their explicit form

$$y_n(x;a,b) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k+a-2}{k} k! \left(\frac{x}{b}\right)^k \tag{2.5}$$

is given in [12, 19.7, (19)]. The substitution of x, t, a and b, respectively with $2z$, w, 2 and 2 in (2.4) and (2.5), gives the equality

$$\Phi(z,w) = \sum_{n=0}^{\infty} y_n(2z;2,2)w^n(n!)^{-1},$$

i.e.

$$B_n(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} z^k. \tag{2.6}$$

The polynomials $(-1)^n n! B_n(-z)$, which are also called Bessel polynomials, are considered in [13].

Denote

$$\widetilde{B}_n(z) = 4^{-n} z^n B_n\left(\frac{1}{z}\right) \exp\left(-\frac{z}{2}\right). \tag{2.7}$$

Having in mind (2.6), we get

$$\widetilde{B}_n(z) = \frac{\exp(-z/2)}{n! 4^n} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} z^{n-k}. \tag{2.8}$$

Let

$$\widetilde{\Phi}(z,w) = (1-w)^{-1/2} \exp\left\{-\frac{z}{2}(1-w)^{1/2}\right\}, \quad z \in \mathbb{C},\ w \in \mathbb{C} \setminus [1,\infty). \tag{2.9}$$

Lemma 2.1. *If* $|w| < 1$ *and* $z \in \mathbb{C}$, *then*

$$\widetilde{\Phi}(z,w) = \sum_{n=0}^{\infty} \widetilde{B}_n(z) w^n. \tag{2.10}$$

Proof. The substitutions $z = \zeta^{-1}$ and $w = \zeta\omega/4$ applied consecutively in (2.1), (2.3) give

$$\Phi(\zeta^{-1},w) = (1-4w\zeta^{-1})^{-1/2} \exp\left\{\frac{1-(1-4zw)^{1/2}}{2}\zeta\right\} = \sum_{n=0}^{\infty} B_n(\zeta^{-1})w^n,$$

$$\Phi(\zeta^{-1},\zeta\omega/4) = (1-\omega)^{-1/2} \exp\left\{\frac{1-(1-\omega)^{1/2}}{2}\zeta\right\} = \sum_{n=0}^{\infty} 4^{-n}\zeta^n B_n(\zeta^{-1})\omega^n.$$

After multiplication of the last equality by $\exp(-\zeta/2)$ we obtain

$$\exp(-\zeta/2)\Phi(\zeta^{-1},\zeta\omega/4) = (1-\omega)^{-1/2} \exp\left\{-\frac{(1-\omega)^{1/2}}{2}\zeta\right\} = \sum_{n=0}^{\infty} \widetilde{B}_n(\zeta)\omega^n,$$

and since $|4zw| < |\omega| < 1$, the lemma is proved.

Denote

$$A_\alpha = \{z : z \in \mathbb{C}^*,\ |\arg z| \le \alpha\pi\}, \quad \mathbb{C}^* = \mathbb{C}\setminus\{0\}. \tag{3.1}$$

Lemma 3.1. *Let $G \subset A_\alpha$, $0 < \alpha < 1$, be a simply connected region, $\gamma \subset G$ be a rectifiable Jordan curve, $F \in H_\gamma$, $F \not\equiv 0$, and $\inf\limits_{z\in\gamma} |z| = r$. Let $|w| < 1/(4r)$ and*

$$f(w) = \int\limits_\gamma F(z)\Phi(z,w)\,dz. \tag{3.2}$$

Then the following expansion holds:

$$f(w) = \sum_{n=0}^{\infty} A_n(F)w^n \tag{3.3}$$

with the coefficients

$$A_n(F) = \int\limits_\gamma F(z)B_n(z)\,dz. \tag{3.4}$$

Moreover, the radius of convergence of the series (3.3) *is finite.*

Proof. It follows from (2.3) that $B_n(z) = \dfrac{1}{n!}\left\{\dfrac{\partial^n\Phi(z,w)}{\partial w^n}\right\}_{w=0}$. Since $f(w)$ is holomorphic for $|w| < 1/(4r)$, then $f(w)$ can be expanded in a Taylor series

$$f(w) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int\limits_\gamma F(z)\left\{\frac{\partial^n\Phi(z,w)}{\partial w^n}\right\}_{w=0} dz\right)w^n = \sum_{n=0}^{\infty}\left(\int\limits_\gamma F(z)B_n(z)\,dz\right)w^n,$$

which yield (3.3), if the notations (3.4) are taken into account.

Having in mind the identity (2.2), we get

$$\begin{aligned}\Phi(z,w) &= (1-4zw)^{-1/2}\exp\frac{2w}{1+(1-4zw)^{1/2}}\\ &= (1-4zw)^{-1/2}\exp\left\{-\frac{-2w}{1+(1-4zw)^{1/2}}\right\}.\end{aligned} \tag{3.5}$$

Suppose that the radius of convergence of (3.3) is infinite. This means that (3.3) defines an entire function. Let us evaluate the order of $f(w)$. Using (2.1) and (3.2), we get consecutively

$$\begin{aligned}|f(w)| &\le \int\limits_\gamma \left|F(z)(1-4zw)^{-1/2}\exp\left\{\frac{1-(1-4zw)^{1/2}}{2z}\right\}\right| ds\\ &\le \int\limits_\gamma |F(z)|\,|1-4zw|^{-1/2}\exp\left\{|z|^{-1/2}|w|^{1/2}\left|\frac{w^{-1/2}-(w^{-1}-4z)^{1/2}}{2z^{1/2}}\right|\right\} ds.\end{aligned}$$

As $\lim_{|w|\to\infty} \left|w^{-1/2} - \left(w^{-1} - 4z\right)^{1/2}\right| = 2|z|^{1/2}$ and $\lim_{|w|\to\infty} (1-4zw)^{-1/2} = 0$, then the following inequalities hold:

$$\left|\frac{w^{-1/2} - \left(w^{-1} - 4z\right)^{1/2}}{2z^{1/2}}\right| < 2, \quad |1-4zw|^{-1/2} < 1,$$

for sufficiently large $|w|$. Denoting

$$m = \sup_{z\in\gamma} |F(z)|, \quad \mu(\gamma) = L, \quad M = mL, \tag{3.6}$$

we conclude that there exists a constant $B > 0$ such that the inequalities

$$|f(w)| \leq M \exp\left(2|z|^{-1/2}|w|^{1/2}\right) \leq M \exp\left(r^{-1/2}|w|^{1/2}\right)$$

hold for every $|w| > B$. Therefore, the order of the function f is $\rho \leq 1/2$.

Further we apply the Phragmen–Lindelof theorem [14, p. 206] for $f(w)$. To this end, consider first $f(-u)$, $u \geq 0$, and use $\Phi(z, -u)$ as given in (3.5). Since $\gamma \subset A_\alpha$, then $|\arg(1+4zu)| < \alpha\pi$ and $\left|\arg(1+4zu)^{1/2}\right| < \alpha\pi/2$. Therefore $\left|\arg\left(1+(1+4zu)^{1/2}\right)\right| < \alpha\pi/2$. Then $\left|\arg\dfrac{2u}{1+(1+4zu)^{1/2}}\right| < \alpha\pi/2$, i.e. $\operatorname{Re}\dfrac{2u}{1+(1+4zu)^{1/2}} > 0$. Further, using the notations $r_1 \equiv \inf_{z\in\gamma} \operatorname{Re} z$ and (3.6), we get

$$\begin{aligned}
|f(-u)| &\leq m \int_\gamma \left|(1+4zu)^{-1/2}\right| \left|\exp\left(-\frac{2u}{1+(1+4zu)^{1/2}}\right)\right| ds \\
&\leq m(1+4r_1u)^{-1/2} \int_\gamma \exp\left(\operatorname{Re}\frac{-2u}{1+(1+4zu)^{1/2}}\right) ds \\
&\leq M(1+4r_1u)^{-1/2}.
\end{aligned} \tag{3.7}$$

Now, let $\max(\alpha, 1-\alpha) < \beta < 1$, $\arg(-w) = (1-\beta)\pi$, $\arg z = \theta$. Then $\arg(-zw) = (1-\beta)\pi + \theta$, and as $-\alpha\pi < \theta < \alpha\pi$, we get consecutively

$$\begin{gathered}
(1-\alpha-\beta)\pi < \arg(-zw) < (1+\alpha-\beta)\pi, \\
(1-\alpha-\beta)\pi < \arg(1-4zw) < (1+\alpha-\beta)\pi, \\
(1-\alpha-\beta)\frac{\pi}{2} < \arg(1-4zw)^{1/2} < (1+\alpha-\beta)\frac{\pi}{2} \cdot
\end{gathered}$$

Denoting $\psi = \arg\left(1+(1-4zw)^{1/2}\right)$, we have

$$(1-\alpha-\beta)\frac{\pi}{2} < \psi < (1+\alpha-\beta\frac{\pi}{2}, \quad \arg\frac{-2w}{1+(1-4zw)^{1/2}} = (1-\beta)\pi - \psi,$$

$$\begin{aligned}
(1-\alpha-\beta)\frac{\pi}{2} &= (1-\beta)\pi - (1+\alpha-\beta)\frac{\pi}{2} \\
&< (1-\beta)\pi - \psi < (1-\beta)\pi - (1-\alpha-\beta)\frac{\pi}{2} = (1+\alpha-\beta)\frac{\pi}{2},
\end{aligned}$$

hence $\left|\arg \frac{-2w}{1+(1-4zw)^{1/2}}\right| < \frac{\pi}{2}$, i.e. $\operatorname{Re}\left(\frac{-2w}{1+(1-4zw)^{1/2}}\right) > 0$. Now, using $\lim_{|w|\to\infty} (1-4zw)^{-1/2} = 0$ and (3.6), we conclude that there exists a constant $P > 0$ such that

$$|f(w)| \le mP \int_\gamma \exp\left\{\operatorname{Re}\left(-\frac{-2w}{1+(1-4zw)^{1/2}}\right)\right\} ds \le MP. \qquad (3.8)$$

The rays $l_1 = \{w : w = -u,\ u > 0\}$ and $l_2 = \{w : \arg(-w) = (1-\beta)\pi\}$ divide the complex plane into two angular domains of sizes $(1 \pm \beta)\pi$. The order of the function is $\rho \le 1/2$. It follows from (3.7) and (3.8) that $|f(w)|$ is bounded along l_1 and l_2. As $1/2 < (1 \pm \beta)^{-1}$, according to the Phragmen–Lindelof theorem $f(w)$ is bounded in both angular domains and therefore in the whole complex plane. Hence $f \equiv \text{const}$. It is seen from (3.7) that $\lim_{u\to\infty} f(-u) = 0$, which means $f \equiv 0$. Since $F \not\equiv 0$ and the system $\{B_n(z)\}_{n=0}^\infty$ is complete in $H(G)$, see Theorem 1, the last equality contradicts the criterion for completeness **(CC)**. Therefore the radius of convergence of the series (3.3) is finite.

Lemma 3.2. *Let $G \subset A_\alpha$, $0 < \alpha < 1/2$, be a simply connected region, $\gamma \subset G$ be a rectifiable curve, $F \in H_\gamma$ and $F \not\equiv 0$. Then there exists a real number $\varphi \in (0, \alpha)$ such that the function f has no singular points outside the set A_φ.*

Proof. The curve γ is a compact set, hence there exists a closed domain A_φ, $0 < \varphi < \alpha$, of the kind (3.1) such that $\gamma \in A_\varphi$ and $\gamma \cap \partial A_\varphi \neq 0$. The values of w, for which $1 - 4zw = 0$, are $w_z = (4z)^{-1}$. Let $z \in \gamma$. Then $w_z \in A_\varphi$ too. Therefore all the points for which $1 - 4zw = 0$ are in the set A_φ and the function $(1-4zw)^{-1/2}$ is a holomorphic function of w outside A_φ. Hence the function (3.2) is holomorphic for $w \in \operatorname{Ext} A_\varphi$ too.

Lemma 3.3. *Let $G \subset A_\alpha$, $0 < \alpha < 1/2$, be a simply connected region, $\gamma \subset G$ be a rectifiable Jordan curve, $F \in H_\gamma$ and $F \not\equiv 0$. Let*

$$\widetilde{f}(w) = \int_\gamma F(z)\widetilde{\Phi}(z, w)\, dz, \quad w \in \mathbb{C} \setminus [1, \infty). \qquad (3.9)$$

Then the following expansion holds:

$$\widetilde{f}(w) = \sum_{n=0}^{\infty} \widetilde{A}_n(F) w^n \qquad (3.10)$$

for $|w| < 1$ with coefficients

$$\widetilde{A}_n(F) = \int_\gamma F(z)\widetilde{B}_n(z)\, dz. \qquad (3.11)$$

Moreover, the radius of convergence of the series of (3.10) is finite and it has no singular points in $\mathbb{C} \setminus [1, \infty)$.

Proof. From (2.10) it follows that $\widetilde{B}_n(z) = \frac{1}{n!}\left\{\frac{\partial^n \widetilde{\Phi}(z,w)}{\partial w^n}\right\}_{w=0}$. As $\widetilde{f}(w)$ is holomorphic for $|w| < 1$, then $\widetilde{f}(w)$ can be expanded in a Taylor series, i.e.:

$$\widetilde{f}(w) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_\gamma F(z) \left\{\frac{\partial^n \widetilde{\Phi}(z,w)}{\partial w^n}\right\}_{n=0} dz\right) w^n = \sum_{n=0}^{\infty}\left(\int_\gamma F(z)\widetilde{B}_n(z)\, dz\right) w^n,$$

which yields (3.10), if the notations (3.11) are taken into account.

Suppose that (3.10) has infinite radius of convergence. This means that (3.10) defines the entire function $\widetilde{f}$. From (2.9) and (3.9) we obtain

$$\left|\widetilde{f}(w)\right| \le \int_\gamma |F(z)|\,|1-w|^{-1/2} \exp\left\{\frac{|z|}{2}\,|w|^{1/2}\left|w^{-1}-1\right|^{1/2}\right\} ds.$$

Since $\lim\limits_{|w|\to\infty} \left|w^{-1}-1\right|^{1/2} = 1$ and $\lim\limits_{|w|\to\infty} |1-w|^{-1/2} = 0$, then the inequalities $\left|w^{-1}-1\right|^{1/2} < 2$, $|1-w|^{-1/2} < 1$ hold for sufficiently large $|w|$. If we denote $R = \sup\limits_{z\in\gamma} |z|$ and use (3.6), we obtain that there exists a constant $D > 0$ such that the inequality $\left|\widetilde{f}(w)\right| \le M \exp\left(R|w|^{1/2}\right)$ holds for every $|w| > D$. This means that $\widetilde{f}$ is of order $\rho \le 1/2$.

Now let us investigate the behaviour of $\widetilde{f}(w)$ along each of the rays $l_1 = \{w : w = -u,\ u > 0\}$ and $l_3 = \{w : \arg(-w) = (1-2\alpha)\pi/2\}$. As $\gamma \subset A_\alpha$, then $\left|\arg\left(\frac{z}{2}(1+u)^{1/2}\right)\right| < \alpha\pi$, i.e. $\mathrm{Re}\left(\frac{z}{2}(1+u)^{1/2}\right) > 0$. Using the notation (3.6), we get

$$\begin{aligned}\left|\widetilde{f}(-u)\right| &\le m(1+u)^{-1/2} \int_\gamma \exp\left\{\mathrm{Re}\left(-\frac{z}{2}(1+u)^{1/2}\right)\right\} ds \\ &\le M(1+u)^{-1/2} \le M. \end{aligned} \tag{3.12}$$

Now let $w \in l_3$. As $-\alpha\pi < \arg z < \alpha\pi$, we have consecutively

$$0 < \arg(1-w) < (1-2\alpha)\frac{\pi}{2},$$

$$0 < \arg(1-w)^{1/2} < (1-2\alpha)\frac{\pi}{4},$$

$$0 < \arg\left(\frac{z}{2}(1-w)^{1/2}\right) < (1+2\alpha)\frac{\pi}{4}, \quad \text{i.e. } \mathrm{Re}\left(\frac{z}{2}(1-w)^{1/2}\right) > 0.$$

Using that $\lim\limits_{|w|\to\infty} |1-w|^{-1/2} = 0$ and (3.6), we conclude that there exists a constant $Q > 0$ such that

$$\left|\widetilde{f}(w)\right| \le mQ \int_\gamma \exp\left\{\mathrm{Re}\left(-\frac{z}{2}(1-w)^{1/2}\right)\right\} ds \le MQ. \tag{3.13}$$

The rays l_1 and l_3 divide the complex plane into two angular domains with sizes $(1-2\alpha)\pi/2$ and $(3+2\alpha)\pi/2$. The order of the function is $\rho \leq 1/2$. As seen from (3.12) and (3.13), $\widetilde{f}(w)$ is bounded along l_1 and l_3. Because of $1/2 < 2(1-2\alpha)^{-1}$ and $1/2 < 2(3+2\alpha)^{-1}$, according to the Phragmen-Lindelof theorem $\widetilde{f}(w)$ is bounded in both angular domains and therefore on the whole complex plane. Hence $\widetilde{f} \equiv$ const. From (3.12) it is seen that $\lim\limits_{u\to\infty} \widetilde{f}(-u) = 0$, that is $\widetilde{f} \equiv 0$. Since $F \not\equiv 0$ and the system $\left\{\widetilde{B}_n(z)\right\}_{n=0}^{\infty}$ is complete in $H(G)$, see Theorem 2, the last equality contradicts the criterion **(CC)**. Therefore the series (3.10) has a finite radius of convergence. Finally, let us note that (3.9) has no singular points in $\mathbb{C}\setminus[1,\infty)$.

4. MAIN RESULTS

Theorem 4.1. *Let $G \subset \mathbb{C}$ be a simply connected region. Then:*

i) *The system of the polynomials $\{B_n(z)\}_{n=0}^{\infty}$ is complete in the space $H(G)$;*

ii) *The system of the functions $\left\{\widetilde{B}_n(z)\right\}_{n=0}^{\infty}$ is complete in the space $H(G)$.*

Proof. i) According to (2.6) $\deg B_n = n$, $n = 0, 1, 2, \ldots$, and therefore the system $\{B_n(z)\}_{n=0}^{\infty}$ is linearly independent. Therefore $\{B_n(z)\}_{n=0}^{\infty}$ is a basis in the space of the algebraic polynomials. Hence z^n is a linear combination of $\{B_k(z)\}_{k=0}^{n}$, therefore it can be concluded that $\{B_n(z)\}_{n=0}^{\infty}$ is complete in $H(G)$.

ii) According to (2.8) the coefficients of the polynomials $\exp(z/2)\widetilde{B}_n(z)$ are all different from zero, i.e. $\deg\left(\exp(z/2)\widetilde{B}_n(z)\right) = n$, $n = 0, 1, 2, \ldots$ Therefore the system $\left\{\exp(z/2)\widetilde{B}_n(z)\right\}_{n=0}^{\infty}$ is linearly independent, which means that it is a basis in the space of algebraic polynomials. Then z^n is a linear combination of $\left\{\exp(z/2)\widetilde{B}_k(z)\right\}_{k=0}^{n}$. That is why $\left\{\exp(z/2)\widetilde{B}_n(z)\right\}_{n=0}^{\infty}$ is compete in $H(G)$, and since $\exp(z/2) \neq 0$ for each $z \in \mathbb{C}$, the correctness of the theorem is proved.

Theorem 4.2. *Let $0 < \alpha < 1$ and $\lim\limits_{n\to\infty}(n/k_n) = \delta \geq \alpha$. Then the system of the polynomials*

$$\{B_{k_n}(z)\}_{n=0}^{\infty} \tag{4.1}$$

is complete in the space $H(G)$ for each simply connected region $G \subset A_\alpha$.

Proof. Suppose the statement is not correct. Then there exists a simply connected region $G \subset A_\alpha$ such that the system (4.1) is not complete in $H(G)$. According to the criterion **(CC)** this means that there exist a rectifiable Jordan curve $\gamma \subset G$ and a function $G \in H_\gamma$ such that $F \not\equiv 0$, but

$$\int\limits_{\gamma} F(z) B_{k_n}(z)\, dz = 0, \quad n = 0, 1, 2, \ldots \tag{4.2}$$

Let $r = \inf_{z\in\gamma} |z|$ and $|w| < (4r)^{-1}$. Consider the complex-valued function $f(w)$, defined in (3.2). Let us note that it is not identically zero. Moreover, if $\widetilde{k}_n$ are the indices of the coefficients (3.4) in the power series (3.3), for which $\left\{\widetilde{k}_n\right\}_{n=0}^{\infty} = \{n\}_{n=0}^{\infty} \setminus \{k_n\}_{n=0}^{\infty}$, it follows from (4.2) that

$$f(w) = \sum_{n=0}^{\infty} A_{\tilde{k}_n}(F) w^{\tilde{k}_n}. \tag{4.3}$$

For the density of the sequence $\left\{\widetilde{k}_n\right\}_{n=0}^{\infty}$ we have

$$\Delta = 1 - \delta \leq 1 - \alpha. \tag{4.4}$$

As $F \not\equiv 0$, not all the complex numbers (3.4) are zeroes. Then, according to Lemma 2, there exists a number $\varphi \in (0, \alpha)$ such that all singular points on the circle $|w| = R$ (R is the radius of the convergence of the series (3.3)) lie in the set A_φ, i.e. there is a closed arc with lenght $2\pi(1-\varphi)$, where (3.3) has no singular points. On the other hand, by a Polya theorem [15, Th. 7, p. 625] every closed arc of the circle $|w| = R$ with lenght $2\pi\Delta$ contains at least one singular point of (4.3). Because of (4.4) we have $2\pi\Delta = 2\pi(1-\delta) \leq 2\pi(1-\alpha) < 2\pi(1-\varphi)$ and we come to a contradiction. Therefore the system (4.1) is complete in $H(G)$ for every simply connected region $G \subset A_\alpha$.

Theorem 4.3. *Let $0 < \alpha < 1/2$ and $\lim_{n\to\infty} (n/k_n) = \delta > 0$. Then the system of the functions*

$$\left\{\widetilde{B}_{k_n}(z)\right\}_{n=0}^{\infty} \tag{4.5}$$

is complete in the space $H(G)$ for every simply connected region $G \subset A_\alpha$.

Proof. Let us suppose that the statement is not correct. Then there exists a simply connected region $G \subset A_\alpha$ such that the system (4.5) is not complete in $H(G)$. That means that there exist a rectifiable Jordan curve $\gamma \subset G$ and a function $F \in H_\gamma$ such that $F \not\equiv 0$, but

$$\int_\gamma F(z) \widetilde{B}_{k_n}(z)\, dz = 0, \quad n = 0, 1, 2, \ldots \tag{4.6}$$

Let $|w| < 1$. Consider the complex-valued function $\widetilde{f}(w)$, defined by the equality (3.9). Observe that it is not identically zero. Moreover, if $\widetilde{k}_n$ are the indices of the coefficients (3.11) in the power series (3.10) for which $\left\{\widetilde{k}_n\right\}_{n=0}^{\infty} = \{n\}_{n=0}^{\infty} \setminus \{k_n\}_{n=0}^{\infty}$, it follows from (4.6) that

$$\widetilde{f}(w) = \sum_{n=0}^{\infty} \widetilde{A}_{\tilde{k}_n}(F) w^{\tilde{k}_n}. \tag{4.7}$$

We have

$$\Delta = 1 - \delta < 1 \tag{4.8}$$

for the density of the sequence $\left\{\widetilde{k}_n\right\}_{n=0}^{\infty}$. As $F \not\equiv 0$, not all of the complex numbers (3.11) are equal to zero. Then, according to Lemma 3, the unique singular point of $f(w)$ on the circle $|w| = R$ (R is the radius of convergence of the series (3.10)) is $w = R$. On the other hand, according to a Polya theorem [15], every closed arc of the circle $|w| = R$ with lenght $2\pi\Delta$ contains at least one singular point of (4.7). Because of (4.8) we have $2\pi\Delta = 2\pi(1-\delta) < 2\pi$ and we come to a contradiction. Therefore the system (4.5) is complete in $H(G)$ for every simply connected region $G \subset A_\alpha$.

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