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QUASIMONOGENIC FUNCTIONS ON A COMMUTATIVE ALGEBRA WITH DIVISORS OF ZERO

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Марин Маринов. КВАЗИГОМОГЕННЫЕ ФУНКЦИИ НАД КОММУТАТИВНОЙ АЛГЕБРОЙ С ДЕЛИТЕЛЯМИ НУЛЯ

Рассматриваются некоторые соотношения между комплексным анализом и теорией функций над алгебрами. Получени новие эквивалентные условия для квазимоногенных (в смисле Рошкулеца) функций.

Marin Marinov. QUASIMONOGENIC FUNCTIONS ON A COMMUTATIVE ALGEBRA WITH DIVISORS OF ZERO

Some relations between complex analysis and the function theory on algebras are considered. New equivalent conditions about quasimonogenic (in the sence of Roşculeţ) functions are given.

1. INTRODUCTION

Let $\mathbb{H}$ be the algebra of the real quaternions. Every quaternion $q \in \mathbb{H}$ is represented in the form (cf. [2])

$$q = z_1 + z_2 j, \quad z_1 \in \mathbb{C},\ z_2 \in \mathbb{C},\ j^2 = -1. \tag{1}$$

Nôno and Inenaga [5] have introduced an algebra $\mathbb{H}_* = (\mathbb{H}, *)$ by the multiplication $* : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$:

$$(z_1 + z_2 j) * (w_1 + w_2 j) := z_1 w_1 - z_2 w_2 + (z_1 w_2 + z_2 w_1) j.$$

In this definition the multiplication bj, $b \in \mathbb{C}$, is the quaternionic multiplication of b by j. The algebra $\mathbb{H}_*$ is a commutative and associative R-algebra with divisors of sero. So, denoting

$$\omega_1 := \frac{1+ij}{2}, \quad \omega_2 := \frac{1-ij}{2},$$

we have $\omega_1 * \omega_2 = \omega_2 * \omega_1 = 0$, i.e. ω_1 and ω_2 are divisors of sero. By these elements every quaternion q has an expression

$$q = q_1\omega_1 + q_2\omega_2, \quad q_1 \in \mathbb{C},\ q_2 \in \mathbb{C}. \tag{2}$$

Hence

$$q^n = q_1^n\omega_1 + q_2^n\omega_2, \quad n \in \mathbb{N}. \tag{3}$$

Let us remark

Lemma 1. *An element $q \in \mathbb{H}_*$ is representable in the forms* (1) *and* (2) *if and only if $q_1 = z_1 - z_2 i$, $q_2 = z_1 + z_2 i$.*

The algebra $\mathbb{H}_*$ has an R-isomorphic model in $\mathbb{R}^4$. Let $\{e_0, e_1, e_2, e_3\}$ be a basis in $\mathbb{R}^4$. We shall consider the algebra $\mathbb{R}_* := (\mathbb{R}^4, *) \sim \mathbb{H}_*$ by the composition law

$$e_n * e_m := \sum_{t=0}^{3} a_{n,m}^t e_t, \quad a_{n,m}^t \in \mathbb{R},$$

with a table of multiplications

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	e_3	$-e_0$	$-e_1$
e_3	e_3	$-e_2$	$-e_1$	e_0

Obviously, e_0 is an unite element of $\mathbb{R}_*$ and we may put $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$ as usual.

The algebra $\mathbb{R}_*$ was introduced by Em. Ivanoff [9] in 1905 and, irrespective of him, by Ljush [10] in 1934.

This paper is written by the conceptual viewpoint of the function theory on algebras.

2. REGULARITY CONDITION AND MOISIL–FUETER OPERATOR

In this section we obtain new conditions of regularity for quaternionic functions with values in the algebra $\mathbb{H}_*$, keeping in mind the expression (2).

On $\mathbb{C}^2$ we introduce a quaternionic structure by the mapping $\alpha : \mathbb{H} \to \mathbb{C}^2$, where $\alpha(q) := (z_1, z_2)$ for $q := z_1 + z_2 j \in \mathbb{H}$. The quaternionic norm $\|q\|$ is equal to the complex norm $|z|$, $z = \alpha(q)$, and we may introduce a topology in $\mathbb{H}$ from $\mathbb{C}^2$. Thus, we say that a quaternionic function $f = f_1 + f_2 j : D \to \mathbb{H}_*$ is continuously differentiable on an open set $D \subset \mathbb{H}$ and we write $f \in C^1(D)$ if and only if the functions f_1 and f_2 are continuously differentiable on D (i.e. the complex functions f_m, $m = 1, 2$, are continuously differentiable on $\alpha(D) \subset \mathbb{C}^2$).

Definition 1 ([5]). A function $f : D \to \mathbb{H}_*$ is called regular on an open set $D \subset \mathbb{H}$ if $f \in C^1(D)$ and $\mathbb{D} * f = 0$ on D, where

$$\mathbb{D} := \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2} j.$$

Definition 2 ([5]). A function $f : D \to \mathbb{H}_*$ is called H-regular on D if and only if f is regular on D and $\mathbb{D}_h * f = 0$ on D, where

$$\mathbb{D}_h := \frac{\partial}{\partial \bar{z}_1} - \frac{\partial}{\partial \bar{z}_2} j.$$

Definition 3 ([5]). A function $f : D \to \mathbb{H}_*$ is called S-regular on D if and only if f is H-regular on D and $\mathbb{D}_s * f = 0$, where

$$\mathbb{D}_s := \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} j.$$

The theory of these functions is developed in [5, 4].

Let us remark that the operator $2\mathbb{D}$ is a quaternionic recording of the Moisil–Fueter operator from hypercomplex analysis [2, 3], and on the algebra $\mathbb{R}_*$

$$D = \frac{1}{2} \sum_{m=0}^{3} e_m \frac{\partial}{\partial x_m}.$$

To the end, let D be a domain in $\mathbb{H}$.

Theorem 1. *Let $f : D \to \mathbb{H}_*$ be a continuously differentiable function on D. Then the following conditions are equivalent:*

1) *The function f is H-regular on D;*

2) *The function f has an expression*

$$f(q) = f_1(q_1, q_2)\omega_1 + f_2(q_1, q_2)\omega_2, \quad q = q_1\omega_1 + q_2\omega_2 \in D,$$

where f_m, $m = 1, 2$, are holomorphic functions on $\alpha(D)$.

Proof. By Lemma 1

$$\mathbb{D} * f = \left(\frac{\partial}{\partial \bar{z}_1} - i \frac{\partial}{\partial \bar{z}_2} \right) f_1\omega_1 + \left(\frac{\partial}{\partial \bar{z}_1} + i \frac{\partial}{\partial \bar{z}_2} \right) f_2\omega_2,$$

$$\mathbb{D}_h * f = \left(\frac{\partial}{\partial \bar{z}_1} + i \frac{\partial}{\partial \bar{z}_2} \right) f_1\omega_1 + \left(\frac{\partial}{\partial \bar{z}_1} - i \frac{\partial}{\partial \bar{z}_2} \right) f_2\omega_2.$$

Hence the conditions

$$\mathbb{D} * f = 0, \quad \mathbb{D}_h * f = 0$$

are equivalent to

$$\frac{\partial f_m}{\partial \bar{q}_n} = 0, \quad m = 1, 2; \ n = 1, 2.$$

Theorem 2. *Let f be an H-regular function on D. Then the following conditions are equivalent*:

1) *The function f is S-regular on D*;
2) *The function f has an expression*

$$f(q) = f_1(q_1)\omega_1 + f_2(q_2)\omega_2, \quad q = q_1\omega_1 + q_2\omega_2 \in D,$$

where f_m, $m = 1, 2$, are holomorphic functions on $\alpha(D)$, and on D

$$\frac{\partial f_1}{\partial q_2} = \frac{\partial f_2}{\partial q_1} = 0.$$

Proof. By Lemma 1

$$\mathbb{D}_s * f = \left(\frac{\partial}{\partial z_1} - i\frac{\partial}{\partial z_2}\right) f_1\omega_1 + \left(\frac{\partial}{\partial z_1} + i\frac{\partial}{\partial z_2}\right) f_2\omega_2$$

and 1) $\Longleftrightarrow$ 2) follows directly from Theorem 1.

Combining this theorem with the function theory on a polydisc (cf. for example [7]), we get the following

Proposition 1. *Let $q^0 = q_1^0\omega_1 + q_2^0\omega_2 \in \mathbb{H}_*$, and let $\alpha(D) := U_1(q_1^0) \times U_2(q_2^0)$ be an open polydisc with a centre $\alpha(q^0)$ and a polyradius (r_1, r_2). Then for every S-regular function f on D the following conditions are equivalent*:

1) *For $f := \varphi_1 + \varphi_2 j$ and $q = z_1 + z_2 j \in D$ follows*

$$\frac{\partial \varphi_1}{\partial z_1} = \frac{\partial \varphi_2}{\partial z_2}, \quad \frac{\partial \varphi_1}{\partial z_2} = -\frac{\partial \varphi_2}{\partial z_1};$$

2) *For $f := f_1\omega_1 + f_2\omega_2$ and $q := q_1\omega_1 + q_2\omega_2 \in D$ follows*

$$f_1(q) = \varphi(z_1 - z_2 i), \quad f_2(q) = \psi(z_1 + z_2 i),$$

where φ and ψ are holomorphic functions on $U_1(q_1^0)$ and $U_2(q_2^0)$, respectively.

3. QUASIMONOGENIC FUNCTIONS

The purpose of this section is a confirmation of known results on analyticity and monogenity in the commutative algebras. As well known, for a complex function $f := u + iv$ on a domain $A \subset \mathbb{C}^1$ with $u \in C^1(A)$, $v \in C^1(A)$ (as real functions of two real variables) to be holomorphic on A, it is necessary and sufficient that $df \wedge \omega = 0$ is fulfilled on A, where $\omega = dz$ is a differential of $z \in \mathbb{C}^1$. This condition was generalized by Roşculeţ [6] for an arbitrary commutative algebra. For a quaternionic function $f = f_1\omega_1 + f_2\omega_2$ on a domain $D \subset \mathbb{H}$ with $f \in C^1(D)$ we have an expression of the quaternionic differential $df = df_1\omega_1 + df_2\omega_2$, where $(q := q_1\omega_1 + q_2\omega_2 \in D)$

$$df_m = \sum_{n=1}^{2} \left(\frac{\partial f_m}{\partial q_n} dq_n + \frac{\partial f_m}{\partial \bar{q}_n} d\bar{q}_n\right)$$

are the complex differentials of the complex functions f_m, $m = 1, 2$.

Definition 4 ([6]). Let ω be a differential 1-form on $D \subset \mathbb{H}$ with values in the algebra $\mathbb{H}_*$. A function $f : D \to \mathbb{H}_*$ is called ω-quasimonogenic on D if $f \in C^1(D)$ and $df \wedge \omega = 0$ is fulfilled on D.

Proposition 2. *Let ω be a differential 1-form*

$$\omega = dq_1\omega_1 + dq_2\omega_2, \quad q := q_1\omega_1 + q_2\omega_2 \in \mathbb{H}_*, \tag{4}$$

and $f : D \to \mathbb{H}_*$ be an H-regular function on D. Then the following conditions are equivalent:

1) *The function f is ω-quasimonogenic on D;*
2) *The function f is S-regular on D.*

Proof. This follows immediately from Theorem 2 and by the formula

$$df \wedge \omega = (df_1 \wedge dq_1)\omega_1 + (df_2 \wedge dq_2)\omega_2.$$

Definition 5 ([6]). Let ω be a differential 1-form on $D \subset \mathbb{H}$ with values in the algebra $\mathbb{H}_*$. A function $f : D \to \mathbb{H}_*$ is called ω-monogenic on D if $f \in C^1(D)$ and on D

$$d(f * \omega) = 0.$$

Proposition 3. *Let ω be a differential 1-form from (4) and $f : D \to \mathbb{H}_*$ be an H-regular function on D. Then the following conditions are equivalent:*

1) *The function f is ω-quasimonogenic on D;*
2) *The function f is ω-monogenic on D;*
3) *The function f is S-regular on D.*

Proof. It follows from Proposition 3 and Definition 5, keeping in mind that $d(f * \omega) = df \wedge \omega$.

In the case of an arbitrary algebra it is possible to introduce various (not equivalent) definitions of the notion "analytic function". Possibly, best definitions were given by Scheffers in 1893 and Ward in 1940. See also for details the other generalizations in [3, 6, 15].

In the considered algebra $\mathbb{H}_*$ we have

Theorem 3. *Let ω be a differential 1-form from (4) and $f : D \to \mathbb{H}_*$ be an H-regular function on D. Then the following conditions are equivalent:*

1) *The function f is S-regular on D;*
2) *The function f is analytic on D in the sense of Scheffers;*
3) *The function f is analytic on D in the sense of Ward.*

Proof. The conditions 2) and 3) are equivalent from the commutativity of the algebra $\mathbb{H}_*$. (See [15] for details.) On the other hand, from the proof of Theorem 3.1, [4], it follows that 1) is equivalent to 3) and this completes the proof.

We conclude this section with the next Cauchy integral formula for polydisc.

Theorem 4. *Let f be an S-regular function on a domain $D \subset \mathbb{H}$. Then for every polydisc $\alpha(U) := U_1(q_1) \times U_2(q_2) \subset \alpha(D)$ with a centre $q := q_1\omega_1 + q_2\omega_2 \in D$*

it holds

$$(5) \qquad f(q) = \frac{1}{(2\pi i)^2} \int\limits_{\partial U_1 \times \partial U_2} \frac{f(t_1\omega_1 + t_2\omega_2)\, dt_1 dt_2}{(t_1 - q_1)(t_2 - q_2)}.$$

Proof. By Proposition 1 and the proof of the Cauchy integral formula for holomorphic functions [7] we obtain (5) by evident way.

4. APPLICATIONS

The applications of the algebra $\mathbb{H}_*$ (known to the author) are in the following directions:

1. Coding system [12].
2. Inertial navigation [11].
3. Kinematics of the rigid body [13].

Some basic results on the ring of the integer numbers $a\omega_1 + b\omega_2 \in \mathbb{H}_*$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, were obtained by E. Ivanoff [9] and L. Chakaloff [14]. For quaternionic applications see [1, 2, 8]. This list of references does not exhaust the subject.

We shall end with one application of the S-regular funcions in the special relativity. The exponential function in the algebra $\mathbb{H}_*$ was defined by Nôno and Inenaga [5] by the following way:

$$\exp(q) := \sum_{n=0}^{\infty} \frac{P_n(q)}{n!}, \quad q := z_1 + z_2 j,$$

where

$$P_0(q) = 1, \quad P_1(q) = q, \quad P_{n+1}(q) = P_n(q) * q.$$

This series converges absolutely in $\mathbb{H}$ and uniformly on every compact subset of $\mathbb{H}$. Thus, the function "exp" is S-regular in $\mathbb{H}$ [5] and

$$\exp(z_1 + z_2 j) = e^{z_1}(\cos z_2 + \sin z_2 j).$$

We shall obtain the upper formula in a different way. By (3) and the absolutely convergence of the series it follows

$$\exp(z_1 + z_2 j) = \sum_{n=0}^{\infty} \frac{q_1^n \omega_1 + q_2^n \omega_2}{n!}$$
$$= e^{z_1 - z_2 i}\omega_1 + e^{z_1 + z_2 i}\omega_2 = e^{z_1}\cos z_2 + e^{z_1}\sin z_2 j.$$

In particular,

$$(6) \qquad \exp(\theta k) = \operatorname{ch}\theta + \operatorname{sh}\theta j, \quad \theta \in \mathbb{R},$$

where $k = i * j$.

Let us consider a mapping $L : \mathbb{R}^4 \to \mathbb{R}^4$, defined by

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},$$

where $v = \text{const} > 0$, $c = \text{const} > 0$. This is the Lorentz' transformation from the special relativity. Putting $ct = t_1$, $ct' = t'_1$, we have

$$\sqrt{1 - \left(\frac{v}{c}\right)^2}(x' + t'k) = (x + t_1 k) * \left(1 - \frac{v}{c}k\right).$$

We write

$$q = q_1 + q_2 j, \quad q_1 = x + t_1 k, \quad q_2 = z + yk,$$

and analogously,

$$q' = q'_1 + q'_2 j, \quad q'_1 = x' + t'_1 k, \quad q'_2 = z' + y'k.$$

Hence, by (6), for the Lorentz' transformation follows the expression

$$q = q_1 * \exp(\theta k) + q_2 j, \quad \theta := \operatorname{arth}\frac{v}{c}.$$

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