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A CHARACTERIZATION OF THE COMPLEX SPACE FORMS*

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Грозьо Станилов, Веселин Видев. ХАРАКТЕРИСТИКА КОМПЛЕКСНЫХ ПРОСТРАНСТВЕННЫХ ФОРМ

В почти эрмитовой геометрии вместе с классическим оператором Якоби мы вводим в рассмотрении также линейный симметрический оператор $\lambda_{X,JX}$, где X касательный вектор в точке $p \in M$. Доказываем следующая теорема: Келеровое многообразие размерности $2n \geqq 4$ есть комплексная пространственная форма тогда и только тогда, когда для любого X в любой точки $p \in M$ оператор $\lambda_{X,JX}$ имеет собственные векторы в плоскости $X \wedge JX$.

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In the almost Hermitian geometry together with the classical Jacobi operator λ_X we define also the linear symmetric operator $\lambda_{X,JX}$, where X is a tangent vector at a point $p \in M$. Then we prove the following theorem: A Kaehlerian manifold of dimension $2n \geqq 4$ is a complex space form iff for every X at any point p the operator $\lambda_{X,JX}$ has eigen vectors in the plane $X \wedge JX$.

Let (M, g, J) be an almost Hermitian manifold with curvature tensor R. If X is a tangent (unit) vector at a point p of M, the well-known Jacobi operator λ_X is defined as linear mapping $\lambda_X : M_p \to M_p$ by the equality

$$\lambda_X(u) = R(u, X, X), \quad u \in M_p.$$

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In the same way we define the linear operator

$$\lambda_{X,JX}(u) = \frac{1}{2}(R(u, X, JX) + R(u, JX, X)),$$

which is also a symmetric operator.

Now we state the following problem: to describe the manifolds for which at every point $p \in M$ and for any vector $X \in M_p$ the linear operator $\lambda_{X,JX}$ has eigen vectors u belonging to the holomorphic plane $E^2(p; X \wedge JX)$. If we take

$$u = \alpha X + \beta JX, \quad \alpha^2 + \beta^2 = 1,$$

from the condition

$$\lambda_{X,JX}(u) = cu,$$

i. e.

$$\frac{1}{2}(R(u, X, JX) + R(u, JX, X)) = cu,$$

by multiplication with X and JX we get the result: the vectors

$$u_1 = \frac{1}{\sqrt{2}}(X + JX), \quad u_2 = \frac{1}{\sqrt{2}}(X - JX)$$

are eigen vectors with corresponding eigen values

$$c_1 = -\frac{1}{2}H(X), \quad c_2 = \frac{1}{2}H(X).$$

Here $H(X)$ is the holomorphic sectional curvature of X.

Now we express that the vectors $\dfrac{X \pm JX}{\sqrt{2}}$ are eigen vectors of the operator $\lambda_{X,JX}$:

$$\frac{1}{2}\left(R\left(\frac{X \pm JX}{\sqrt{2}}, X, JX\right) + R\left(\frac{X \pm JX}{\sqrt{2}}, JX, X\right)\right) = \mp\frac{H(X)}{2}\frac{X \pm JX}{\sqrt{2}}.$$

Using the properties of R we see that these equalities are equivalent to the relations

(1) $$R(X, JX, JX) = H(X)X, \quad R(JX, X, X) = H(X)JX.$$

Thus we can formulate the following

Theorem 1. *If the operator $\lambda_{X,JX}$ has eigen vectors in the plane $E^2(p; X \wedge JX)$ then the Jacobi operator λ_X has as eigen vector JX and the Jacobi operator λ_{JX} has as eigen vector X.*

Now we try to prove the converse of this assertion. Namely, using (1) we get immediately:

$$\lambda_{X,JX}\left(\frac{X \pm JX}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}(\pm R(JX, X, JX) + R(X, JX, X))$$
$$= \frac{1}{2\sqrt{2}}(\mp H(X)X - H(X)JX) = \mp\frac{1}{2}H(X)\frac{X \pm JX}{\sqrt{2}}.$$

Thus we have the following

Theorem 2. *If the operator* λ_X *has as eigen vector* JX *and the operator* λ_{JX} *has as eigen vector* X *then the vectors* $\dfrac{X \pm JX}{\sqrt{2}}$ *are eigen vectors of the operator* $\lambda_{X,JX}$.

Example. Let us consider the complex space form that is the Kaehlerian manifold of constant holomorphic sectional curvature μ. In this case the curvature tensor is of the form

$$R(X,Y,Z,U) = \frac{\mu}{4}\big(g(Y,Z)g(X,U) - g(X,Z)g(Y,U) + g(JY,Z)g(JX,U)$$
$$- g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)\big).$$

Then the Jacobi operator is represented by

$$\lambda_X(u) = \frac{\mu}{4}\big(u - g(u,X)X + 3g(u,JX)JX\big).$$

We can see immediately λ_X has as eigen vector JX and λ_{JX} has as eigen vector X.

We shall prove the converse of this assertion. Namely, the main result of this paper is the following characterization of the complex space forms:

Theorem 3. *Let* (M,g,J) *be a Kaehlerian manifold of dimension* $2n \geq 4$. *Then it is a complex space form iff for any unit vector* $X \in M_p$ *at every point* $p \in M$ *the operator* $\lambda_{X,JX}$ *has eigen vectors in the plane* $E^2(p; X \wedge JX)$.

Proof. Let X, Y, JX, JY is an orthonormal quadruple. Using (1) we get the relation $R(X,JX,JX,Y) = 0$. We apply this relation for the vectors $X+Y$, $X-Y$, $JX+JY$, $JX-JY$ and after some calculations we have

$$R(X, JX+JY, JX+JY, X) = R(Y, JX+JY, JX+JY, Y).$$

Since the manifold is a Kaehlerian one, the last equality is equivalent to the relation $H(X) = H(Y)$.

We take now the vector $u = \cos\alpha X + \sin\alpha Y$ in the plane $E^2(p; X \wedge Y)$. Following (1) we can write the relation $R(Ju,u,u) = H(u)Ju$ and after some long calculations we get the equality

$$\cos^3\alpha R(JX,X,X) + \cos^2\alpha\sin\alpha R(JX,X,Y) + \cos^2\alpha\sin\alpha R(JX,Y,X)$$
$$+ \cos\alpha\sin^2\alpha R(JX,Y,Y) + \cos^2\alpha\sin\alpha R(JY,X,X) + \sin^3\alpha R(JY,Y,Y)$$
$$+ \cos\alpha\sin^2\alpha R(JY,X,Y) + \cos\alpha\sin^2\alpha R(JY,Y,X) = H(u)(\cos\alpha JX + \sin\alpha JY).$$

If we multiply by JX we get the relation

$$\cos^2\alpha H(X) + 2\sin^2\alpha R(JX,Y,Y,JX) + \sin^2\alpha R(JY,Y,X,JX) = H(u)$$

and after multiplying by JY we have

$$\cos^2\alpha R(JX,X,Y,JY) + 2\cos^2\alpha R(JX,Y,Y,JX) + \sin^2\alpha H(Y) = H(u).$$

From the last two equalities we can get

$$\cos^4\alpha H(X) - \sin^4\alpha H(Y) = H(u)(\cos^2\alpha - \sin^2\alpha)$$

and since $H(X) = H(Y)$ then $H(u) = H(X) = H(Y)$.

To finish the proof we must show that the relation

$$H(v) = H(X) = H(Y)$$

holds good for any vector

$$v = aX + bY + cJX + dJY \quad (a^2 + b^2 + c^2 + d^2 = 1)$$

in the 4-dimensional space spaned by vectors X, Y, JX, JY.

If we put

$$aX + cJX = \cos\alpha X', \quad \cos\alpha = \sqrt{a^2 + c^2},$$
$$bY + dJY = \sin\alpha Y', \quad \sin\alpha = \sqrt{b^2 + d^2},$$

we reach to $v = \cos\alpha X' + \sin\alpha Y'$ and since $X \wedge JX = X' \wedge JX'$, $Y \wedge JY = Y' \wedge JY'$, then

$$H(v) = H(X') = H(Y') = H(X) = H(Y).$$

If the dimension $2n > 4$, we take vector Z orthogonal to X, Y, JX, JY and apply the same considerations for the orthonormal quadruple X, Z, JX, JZ.

Evidently, the main result can be expressed also in the following way: A Kaehlerian manifold of dimension $2n \geqq 4$ is a complex space form iff for every $X \in M_p$ at every point $p \in M$ the Jacobi operator λ_X has as eigen vector JX.

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