ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 103

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 103

LOOPS RELATED TO REFLECTION GEOMETRIES

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Dedicated to Johannes Böhm on the occasion of his 90-th birthday.

Keywords: Reflection Geometry, Sperner Plane, Kinematic Space, Parallelism, Loops.
2000 Math. Subject Classification: 20N05, 51J15 51M10.

1. INTRODUCTION

In the book "Theory of K-Loops" by Hubert Kiechle we find the following construction theorem for left-loops and loops (cf. [5] (2.7)):

Theorem 1. *Let* $(G, \cdot)$ *be a group,* 1 *the neutral element,* U *a subgroup of* G *and let* $L \subseteq G$ *be a transversal of the pair* (G, U)*, i.e.*

(T1) $\forall x \in G\ \exists_1 \bar{x} \in L$ *with* $\bar{x} \cdot U = x \cdot U$.

(T2) $1 \in L$.

For $t, s \in L$ *let* $t \oplus s := \overline{t \cdot s}$*. Then* $(L, \oplus)$ *is a left loop (called derived left-loop), i.e.*

1) $\forall l \in L : 1 \oplus l = l \oplus 1 = l$.

2) $\forall s, t \in L, \exists_1 x \in L$ *with* $s \oplus x = t$.

$(L, \oplus)$ *is a loop (i.e. also the equation* $x \oplus s = t$ *has an unique solution) if and only if the set* L *satisfies the condition (T3) - stronger as (T1) - (then* L *is called L-transversal).*

(T3) $\forall x, y \in G : |L \cap x \cdot U \cdot y| = 1$.

The next theorem tells us that any left-loop can be obtained by the method of Theorem 1 (cf. [5] (2.6), (2.7)):

Theorem 2. *Let $(L,+)$ be a left-loop, for $a \in L$ let $a^+ : L \to L \ ; \ x \mapsto a+x$, $L^+ := \{a^+ \mid a \in L\}$, let $G :=< L^+ >$ be the group generated by the left-translations a^+ and let $U := \{\xi \in G \mid \xi(o) = o\}$. Then L^+ is a transversal of (G,U) and the derived left-loop is isomorphic to $(L,+)$.*

Under the notion "reflection geometry" or "Sperner plane" one finds all absolute planes. To any reflection geometry $(G,\mathfrak{D},\mathfrak{B})$ we associate firstly a so called *kinematic fibration* $\mathfrak{F}$ of the group $\mathfrak{D}^2$ and then a *kinematic space* $(\mathfrak{D}^2,\mathfrak{G},\cdot)$. To certain subgroups $F \in \mathfrak{F}$ we find in form of a plane $< \varepsilon >$ of the kinematic space $(\mathfrak{D}^2,\mathfrak{G},\cdot)$ a transversal of $(\ F)$. With this method we can associate to each reflection geometry in a natural way loops $(L,\oplus)$, which shall be studied in this paper. Also in the paper [6] by S. Pasotti, S. Pianta and E. Zizioli we find constructions of loops related to hyperbolic planes using transversals.

2. REFLECTION GROUPS , REFLECTION GEOMETRIES AND SPERNER PLANES

We recall some notions and facts taken from [1], [3] and [4]. Let $(G,\cdot)$ be a group, $J := \{\gamma \in G \mid \gamma^2 = id \neq \gamma\}$ and $\mathfrak{D} \subseteq J$ such that $< \mathfrak{D} >= G$, i.e. $\mathfrak{D}$ is a system of generators of G.

A subset $\mathfrak{b} \subseteq \mathfrak{D}$ is called *pencil* if there are $A,B \in \mathfrak{D}$, $A \neq B$ with $\mathfrak{b} = \overbrace{A,B} := \{X \in \mathfrak{D} \mid A \cdot B \cdot X \in J\}$. Let $\mathfrak{B}$ be the set of all pencils. A pencil $\mathfrak{b}$ is called *proper* or also *projective* if for all $\mathfrak{x} \in \mathfrak{B}$, $\mathfrak{b} \cap \mathfrak{x} \neq \emptyset$. Let $\mathfrak{B}_o$ be the set of all proper pencils.

We claim (*Three reflection Axiom*):

(S) If $\mathfrak{b}$ is a pencil and $A,B,C \in \mathfrak{b}$, then $A \cdot B \cdot C \in \mathfrak{D}$ and then if $\mathfrak{B}_o \neq \emptyset$, we call the pair $(G,\mathfrak{D})$ *reflection group.*

Proposition 1. If $(G,\mathfrak{D})$ is a reflection group then

1. $\mathfrak{D}^4 = \mathfrak{D}^2$, i. e. $\mathfrak{D}^2$ is a subgroup of G.
2. For $\mathfrak{b} \in \mathfrak{B}$ the set $\mathfrak{b}^2 := \{X \cdot Y \mid X, Y \in \mathfrak{b}\}$ is a commutative subgroup of $\mathfrak{D}^2$.
3. For $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathfrak{B}$ we have: $\mathfrak{b}_1^2 \cap \mathfrak{b}_2^2 = \{1\} \iff \mathfrak{b}_1 \neq \mathfrak{b}_2$.
4. For $\xi \in G$ and $\mathfrak{b} \in \mathfrak{B}$ we have $\xi \cdot \mathfrak{b} \cdot \xi^{-1} \in \mathfrak{B}$.

To a reflection group $(G,\mathfrak{D})$ we associate the following geometric structure $(G,\mathfrak{D},\mathfrak{B})$ called *reflection geometry*: The elements of $\mathfrak{D}$ are called *lines*, of $\mathfrak{B}$ *points* and of $\mathfrak{B}_o$ *projective points.* The incidence between a line $L \in \mathfrak{D}$ and a point

$\mathfrak{b} \in \mathfrak{B}$ is given by $L \in \mathfrak{b}$. A reflection geometry $(G, \mathfrak{D}, \mathfrak{B})$ is called *Sperner plane* if every line $G \in \mathfrak{D}$ is incident with in at least three distinct projective points.

If in a reflection geometry there exist more than one pencil then there exist $A, B, C \in \mathfrak{D}$ with $A \cdot B \cdot C \neq C \cdot B \cdot A$.

Let $A, B \in \mathfrak{D}$ with $A \neq B$ and $\mathfrak{b} := \overbrace{A, B}$. We call A and B *orthogonal* and denote that by $A \perp B$ if $A \cdot B \in J$ and then $\mathfrak{b}$ is called an *orthogonal pencil*.

A Sperner plane is called *regular* if for every $A \in \mathfrak{D}$, the set $A^\perp := \{X \in \mathfrak{D} \mid X \perp A\}$ is a pencil and then $A^\perp$ is called the *pole* of the line A.

Let $\mathfrak{P} := \mathfrak{D}^2 \cap J$. If $p \in \mathfrak{P}$ then $\mathfrak{p} := \{X \in \mathfrak{D} \mid p \cdot X \in J\}$ is an orthogonal pencil. Hence the set of all orthogonal pencils can be identified with the set $\mathfrak{P}$ of all involutions contained in the group $\mathfrak{D}^2$.

Proposition 2. Let $(G, \mathfrak{D}, \mathfrak{B})$ be a Sperner plane and let $\alpha \in G$ then

1. If $(G, \mathfrak{D}, \mathfrak{B})$ is regular then $\mathfrak{B}_o \subseteq \mathfrak{P}$.
2. The map
$$\widetilde{\alpha} \ : \ G \ \to \ G \ : \ \xi \ \mapsto \ \alpha \cdot \xi \cdot \alpha^{-1}$$
maps lines onto lines, points onto points, preserves incidence and orthogonality hence $\widetilde{\alpha}$ is a motion of the Sperner plane.

If $\alpha \in \mathfrak{D}$ resp. $\alpha \in \mathfrak{P}$ then $\widetilde{\alpha}$ is called *line-reflection* resp. *point-reflection*. If to $a, b \in \mathfrak{P}$ there is a $m \in \mathfrak{P}$ with $\widetilde{m}(a) = b$ then m is called *midpoint* of a and b. A regular Sperner plane with $\mathfrak{B}_o = \mathfrak{P}$ is called *midpoint plane* if for any two distinct points there exists exactly one midpoint.

Proposition 3. Let $(G, \mathfrak{D}, \mathfrak{B})$ be a midpoint plane, let $o \in \mathfrak{P}$ be fixed and for $x \in \mathfrak{P}$ let x' be the midpoint of o and x and let $x^+ := \widetilde{x'} \circ \widetilde{o}$. If for $a, b \in \mathfrak{P}$

$$a \oplus b := a^+(b)$$

then $(\mathfrak{P}, \oplus)$ is a loop, even a K-loop.

3. THE KINEMATIC SPACE OF A SPERNER PLANE

Now we associate to a reflection group $(G, \mathfrak{D})$ a spatial structure. By 3. and 4. of Proposition 1 the set $\mathfrak{F} := \{\mathfrak{b}^2 \mid \mathfrak{b} \in \mathfrak{B}\}$ forms a *kinematic fibration of the group* $\mathfrak{D}^2$, i.e.

1) $\bigcup \mathfrak{F} = \mathfrak{D}^2$.
2) $\forall U, V \in \mathfrak{F}, \quad U \cap V = \{1\}$ or $U = V$.
3) If $\xi \in \mathfrak{D}^2$ and $U \in \mathfrak{F}$ then $\xi \cdot U \cdot \xi^{-1} \in \mathfrak{F}$.

Therefore if we call the elements of $\mathfrak{D}^2$ *points* and of $\mathfrak{G} := \{\gamma \cdot F \mid \gamma \in \mathfrak{D}^2, F \in \mathfrak{F}\}$ *lines* then $(\mathfrak{D}^2, \mathfrak{G})$ is an incidence space and the triple $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$ is called *kinematic space* (cf. [2]).

A subset $\Delta \subseteq \mathfrak{D}^2$ is called *subspace* if for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$ the line $\overline{\alpha, \beta}$ joining the points α and β is contained in Δ.

If $\alpha \in \mathfrak{D}^2 \setminus \{1\}$ let $[\alpha]$ denote the unique fiber of $\mathfrak{F}$ with $\alpha \in [\alpha]$.

By Proposition 8. of [4] we have

Proposition 4. $\alpha, \beta, \gamma \in \mathfrak{D}^2$ are collinear if and only if

$$\beta \cdot \alpha^{-1} \cdot \gamma = \gamma \cdot \alpha^{-1} \cdot \beta.$$

Let $\mathfrak{F}_o := \{\mathfrak{b}^2 \mid \mathfrak{b} \in \mathfrak{B}_o\}$ the subset of fibers coming from proper pencils. The elements of the subset $\mathfrak{G}_o := \{\xi \cdot U \mid \xi \in \mathfrak{D}^2,\ U \in \mathfrak{F}_o\}$ are called *projective lines.*

For $\varepsilon \in \mathfrak{D}^3$ let $< \varepsilon > := \{\xi \in \mathfrak{D}^2 \mid \varepsilon \cdot \xi \in \mathfrak{D}\} = \varepsilon^{-1} \cdot \mathfrak{D}$.

By Proposition 10 of [4] we have

Proposition 5. Let $\epsilon \in \mathfrak{D}^3$, $F \in \mathfrak{F}_o$ a projective fiber, $\xi \in \mathfrak{D}^2$, $G := \xi \cdot F$ hence $G \in \mathfrak{G}_o$ then

1. $< \varepsilon > \cap F \neq \emptyset$.
2. $< \varepsilon > \ \cap G \neq \emptyset$.
3. If $F \subseteq < \varepsilon >$ and $G \neq F$ then $| < \varepsilon > \ \cap G| = 1$.
4. $< \varepsilon >$ is a plane.
5. If G is contained in the plane $< \varepsilon >$ then G meets any line $A \in \mathfrak{G}$ which is contained in the plane $< \varepsilon >$.

4. PARALLELISMS IN A KINEMATIC SPACE

Let $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$ be the kinematic space belonging to the kinematic fibration $\mathfrak{F} := \{\mathfrak{b}^2 \mid \mathfrak{b} \in \mathfrak{B}\}$ of the group $\mathfrak{D}^2$. Then for $G \in \mathfrak{G}$, $G^{-1} \cdot G \in \mathfrak{F}$ and therefore we can define a left and a right parallelism. For $A, B \in \mathfrak{G}$ let

$$A \parallel_l B \Longleftrightarrow A^{-1} \cdot A = B^{-1} \cdot B, \qquad A \parallel_r B \Longleftrightarrow A \cdot A^{-1} = B \cdot B^{-1}.$$

If $A \in \mathfrak{G}$ and $\beta \in \mathfrak{D}^2$ then there is exactly one line $B := (\beta \parallel_l A)$ with $B \parallel_l A$ and $\beta \in B$, namely $B := (\beta \parallel_l A) = \beta \cdot A^{-1} \cdot A$.

By the last remarks of [4] we have:

Theorem 3. *Let $\varepsilon \in \mathfrak{D}^3$ and let $G \in \mathfrak{G}_o$ a projective line. Then:*

1. *If $H \in \mathfrak{G}$ and $H \parallel_l G$ or $H \parallel_r G$ then $H \in \mathfrak{G}_o$.*

2. *There is exactly one projective line denoted by $(\varepsilon \parallel_l G)$ resp. $(\varepsilon \parallel_r G)$ contained in $< \varepsilon >$ such that $(\varepsilon \parallel_l G) \parallel_l G$ resp. $(\varepsilon \parallel_r G) \parallel_r G$.*

3. *If $L \in \mathfrak{G}$ with $L \parallel_l G$ and $L \neq (\varepsilon \parallel_l G)$ or with $L \parallel_r G$ and $L \neq (\varepsilon \parallel_r G)$ then $|L \cap < \varepsilon > | = 1$.*

Theorem 4. *Let $(G, \mathfrak{D}, \mathfrak{B})$ be a Sperner plane, let $\mathfrak{b} \in \mathfrak{B}_o$ be a proper pencil, let $F := \mathfrak{b}^2$, let $A \in \mathfrak{b}$ and let $T := (< A > \backslash F) \cup \{1\}$. Then*

1. *F is a commutative subgroup of $\mathfrak{D}^2$ and a projective line of the corresponding kinematic space $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$,*

2. *$< A >$ is a plane of $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$ with $F \subseteq < A >$.*

3. *$\forall \alpha, \beta \in \mathfrak{D}^2$, $\alpha \cdot F \cdot \beta$ is a projective line and $\alpha \cdot F \cdot \beta \cap < A > \neq \emptyset$.*

4. *For each $\alpha \in \mathfrak{D}^2 \setminus F$ the line $(\alpha \parallel_l F) = \alpha \cdot F$ meets the plane $< A >$ in exactly one point $\bar{\alpha}$ and we have $\bar{\alpha} = \alpha \Leftrightarrow \alpha \in A$. For $\alpha \in F$ let $\bar{\alpha} := 1$.*

5. *T is a transversal of $(\mathfrak{D}^2, F)$ hence $(T, \oplus)$ with $\alpha \oplus \beta := \overline{\alpha \cdot \beta}$ for $\alpha, \beta \in T$ is a left loop.*

6. *T is a L-transversal of $(\mathfrak{D}^2, F)$ hence $(T, \oplus)$ is a loop and 1 is the neutral element of $(T, \oplus)$.*

5. PROPERTIES OF THE LOOP OF A SPERNER PLANE

Let $(T, \oplus)$ be the loop corresponding to a Sperner plane according to Theorem 4. Then if $\alpha, \beta \in T$ and if $\alpha \cdot \beta \in T$ we obtain $\alpha \oplus \beta = \alpha \cdot \beta$. Now let $\alpha \in T \setminus \{1\}$ and $[\alpha] = \overline{1, \alpha}$. Then $[\alpha]$ is a subgroup of $\mathfrak{D}^2$ and $[\alpha] \subseteq T$. Hence on $[\alpha]$ coincide the operations $\cdot$ and $\oplus$ and we have:

Theorem 5. *$(T, \oplus)$ is a fibered loop with the fibration $\mathfrak{F}' := \mathfrak{F} \setminus \{F\}$ and each fiber $X \in \mathfrak{F}'$ is a commutative subgroup of the loop $(T, \oplus)$.*

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Received on October 31, 2015

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