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ACUTE TRIANGLES IN THE CONTEXT OF THE ILLUMINATION PROBLEM

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We consider strong *at*-subsets of the Euclidean space $\mathbf{R}^n$ and estimate from below the growth of the maximal cardinality of such subsets (our method essentially differs from that of [6]). We then apply some properties of strong *at*-sets to the illumination problem.

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1. INTRODUCTION AND RESULTS

Let X be a subset of of the n-dimensional Euclidean space $\mathbf{R}^n$, where $n \geq 2$.

We shall say that X is an *at*-set (in $\mathbf{R}^n$) if any three-element subset of X forms either an acute-angled triangle or a right-angled triangle.

We shall say that X is a strong *at*-set (in $\mathbf{R}^n$) if any three-element subset of X forms an acute-angled triangle.

It directly follows from the above definitions that each subset of an *at*-set (respectively, of a strong *at*-set) is also an *at*-set (respectively, a strong *at*-set).

It is natural to envisage the question concerning the maximal value among the cardinalities of *at*-subsets of $\mathbf{R}^n$.

Denote by $q(n)$ the maximum of the cardinalities of all *at*-sets in $\mathbf{R}^n$. Answering two questions posed by P. Erdös and V. L. Klee, it was demonstrated in the paper by Danzer and Grünbaum [5] that the inequality $q(n) \le 2^n$ holds true. Moreover, the equality $\operatorname{card}(X) = 2^n$ for an *at*-set $X \subset \mathbf{R}^n$ is valid if and only if X coincides with the set of all vertices of some right rectangular n-dimensional parallelepiped in $\mathbf{R}^n$. Thus, one can directly see that $q(n)$ has an exponential growth with respect to n. For more details, see the above-mentioned paper [5] or [2] or Chapter 15 of the remarkable book [1].

Denote by $k(n)$ the maximum of the cardinalities of all strong *at*-sets in $\mathbf{R}^n$. It is easy to show that $k(2) = 3$ and it is also known that $k(3) = 5$. It immediately follows from the result of Danzer and Grünbaum [5] that one of the upper bounds for $k(n)$ is $2^n - 1$, i.e., one has the trivial inequality

$$k(n) \le 2^n - 1.$$

In the general case the precise value of $k(n)$ is still unknown. However, it was proved that $k(n)$ also has an exponential growth with respect to n; in this connection, see [6] or Chapter 15 of the same book [1].

It should be noticed that in [1] and [6] an exponential growth of $k(n)$ is proved with the aid of a probabilistic argument which seems to be somewhat artificial in this case. Indeed, a deterministic proof of the same fact can be presented by using another approach. We would like to give below a sketch of a different proof of the same fact. The suggested proof is simple, purely combinatorial, and so does not rely on any facts from probability theory.

In what follows, the symbol V_n will stand for the set of all vertices of the unit cube $C_n = [0,1]^n$ of the space $\mathbf{R}^n$, so we have $\operatorname{card}(V_n) = 2^n$. First of all, we are going to present a precise formula for the total number r_n of right-angled triangles whose vertices belong to V_n. Clearly, this number coincides with the total number of all right-angled triangles whose vertices belong to the set of vertices of any n-dimensional right rectangular parallelepiped P in $\mathbf{R}^n$.

Let t_n stand for the number of all right-angled triangles in C_n, the right angle of which is a fixed vertex v from V_n and the other two vertices also belong to V_n. Consider some facet C_{n-1} of C_n incident to v. Obviously, we have t_{n-1} right angles with the same vertex v, all of which lie in C_{n-1}. Further, each of the above-mentioned angles is a projection of exactly two right angles which do not lie in C_{n-1}. Besides, there are precisely $2^{n-1} - 1$ right angles, all of which have a fixed common side, namely, the edge of C_n passing through v and orthogonal to C_{n-1}.

Thus, we come to the following recurrence formula:

$$t_n = 3t_{n-1} + 2^{n-1} - 1.$$

This formula allows us to readily deduce (e.g., by induction) that

$$t_n = (3^n + 1)/2 - 2^n.$$

Therefore, ranging v over the whole of V_n, we finally get

$$r_n = 2^n((3^n + 1)/2 - 2^n).$$

As an immediate consequence of the above formula, we obtain that the total number of all those acute-angled triangles whose vertices belong to V_n is equal to

$$\frac{2^n!}{3!(2^n-3)!} - r_n = \frac{2^n!}{3!(2^n-3)!} - 2^n((3^n+1)/2 - 2^n).$$

Now, let us try to apply the formula for r_n in evaluating from below the function $k(n) = k$.

Let $X_1, X_2, \ldots, X_p$ be an injective enumeration of all $(k+1)$-element subsets of V_n, so

$$p = \frac{2^n!}{(k+1)!(2^n-(k+1))!},$$

and let, for each natural index $i \in [1,p]$, the symbol a_i denote the number of the right-angled triangles in X_i. Since no X_i is a strong *at*-set, we obviously may write

$$1 \le a_i \qquad (1 \le i \le p).$$

At the same time, it is clear that

$$a_1 + a_2 + \cdots + a_p = \frac{(2^n-3)!}{(k-2)!(2^n-3-(k-2))!} \cdot r_n.$$

The above equality is easily deduced if we consider the set of all pairs (Z, X_i), where X_i ranges over the family of all $(k+1)$-element subsets of V_n and Z is a three-element subset of X_i which forms a right-angled triangle. Calculating in two possible ways the cardinality of the set of all these pairs, we come to the required equality.

Now, since we have the trivial inequality

$$\frac{(2^n-3)!}{(k-2)!(2^n-3-(k-2))!} \le \frac{(2^n)!}{(k-2)!(2^n-(k-2))!},$$

we infer that

$$a_1 + a_2 + \ldots + a_p \le \frac{(2^n)!}{(k-2)!(2^n-(k-2))!} \cdot r_n.$$

Consequently,

$$\frac{2^n!}{(k+1)!(2^n-(k+1))!} \le \frac{2^n!}{(k-2)!(2^n-(k-2))!} \cdot 2^n((3^n+1)/2 - 2^n).$$

The last inequality directly implies

$$(2^n-(k+1))^3 \le (k+1)^3 \cdot 2^n((3^n+1)/2 - 2^n)$$

or, equivalently,

$$\frac{2^n}{1 + (2^n((3^n+1)/2 - 2^n))^{1/3}} \le k+1.$$

Further, taking into account the two trivial relations

$$1 + (2^n((3^n + 1)/2 - 2^n))^{1/3} \leq 2 \cdot (2^n((3^n + 1)/2 - 2^n))^{1/3},$$

$$(3^n + 1)/2 - 2^n < 3^n,$$

we can conclude that

$$\frac{1}{2} \cdot (\frac{2}{6^{1/3}})^n \leq k + 1.$$

Since $2 > 6^{1/3}$, we see that $k + 1$ (and, consequently, $k = k(n)$) has an exponential growth with respect to n.

Remark 1. The argument presented above and the argument given in [6] are not effective in the sense that they do not allow one to indicate or geometrically describe any strong *at*-subset X of V_n whose cardinality is of an exponential growth with respect to n. In this connection, it would be interesting to have some concrete examples of such subsets X of V_n and to give their geometric characterization.

Remark 2. The notions of *at*-sets and of strong *at*-sets can be introduced for any Hilbert space H over the field $\mathbf{R}$ of all real numbers. In this more general situation the question concerning maximal cardinality of such sets also makes sense and deserves to be investigated. In particular, for an infinite-dimensional H the question is interesting from the purely set-theoretical view-point.

Strong *at*-sets in $\mathbf{R}^n$ are also of interest in connection with the well-known problem of illumination of the boundary of a compact convex body in $\mathbf{R}^n$. There is a rich literature devoted to this important problem of combinatorial geometry. See, for example, [2], [3], and [4].

Actually, the famous hypothesis of Hadwiger says that the minimum number of rays in $\mathbf{R}^n$ which suffice to illuminate the boundary of every compact convex body in $\mathbf{R}^n$ is equal to 2^n and, moreover, any n-dimensional parallelepiped P in $\mathbf{R}^n$ needs at least 2^n rays. Notice that the set of all singular boundary points of P is infinite (moreover, it is of cardinality continuum).

In this context, we would like to recall the following old result of Hadwiger.

Theorem 1. *If the boundary of a convex body $T \subset \mathbf{R}^n$ is smooth, then $n + 1$ rays in $\mathbf{R}^n$ suffice to illuminate the boundary of T.*

Actually, Theorem 1 states that if $n + 1$ rays $l_1, l_2, \ldots, l_n, l_{n+1}$ are given in $\mathbf{R}^n$, which have common end-point 0 and do not lie in a closed half-space of $\mathbf{R}^n$, then $l_1, l_2, \ldots, l_n, l_{n+1}$ are enough to illuminate the boundary of any convex smooth body in $\mathbf{R}^n$ (the compactness of the body is not required here).

Recall also that Hadwiger's above-mentioned result was strengthened by Boltyanskii (see, e.g., [2]). Namely, Boltyanskii established the following statement.

Theorem 2. *If the boundary of a convex body $T \subset \mathbf{R}^n$ has at most n singular points, then $n + 1$ rays in $\mathbf{R}^n$ suffice to illuminate the boundary of T.*

Boltyanskii's theorem does not admit further generalizations to the case where the boundary of a compact convex body $T \subset \mathbf{R}^n$ may have more than n singular points (see [4] and [7]). In addition to this, the $n+1$ rays of Theorem 2 substantially depend on the convex body T.

It is natural to ask whether there is a compact convex body in $\mathbf{R}^n$ with a finite number of singular boundary points, which needs a large number of rays for illuminating its boundary (i.e., the number of illuminating rays must be of an exponential growth with respect to the dimension n of $\mathbf{R}^n$).

Let X be a strong *at*-subset of $\mathbf{R}^n$ with cardinality equal to $k(n)$. Recall that $k(n)$ is of an exponential growth with respect to n. By starting with this X, one can obtain the following statement.

Theorem 3. *There exists a compact convex body $B \subset \mathbf{R}^n$ such that:*
(1) X coincides with the set of all singular boundary points of B;
(2) at least $k(n)$ rays are necessary to illuminate the boundary of B.

Let us present a sketch of the proof of Theorem 3.

Denote by M the convex hull of the set X. Clearly, M is an n-dimensional convex polyhedron in $\mathbf{R}^n$ and the set of all vertices of M coincides with X. For every point $x \in X$, denote by $M(x)$ the polyhedral angle of M with vertex x, and let $C(x)$ be a convex cone with the same vertex x, such that $M(x) \subset C(x)$. We may assume that the conical hypersurface of $C(x)$ is smooth (of course, except for its vertex x). If each $C(x)$ slightly differs from $M(x)$, then the boundary of the compact convex body

$$B' = \cap\{C(x) : x \in X\}$$

has isolated singular points x, where $x \in X$, and continuum many other singular points y, where $y \in Y$. We may suppose that the distance between the sets X and Y is strictly positive. Now, all singular boundary points of B' belonging to Y can be deleted by using a standard trick, without touching the points of X. So, proceeding in this way, we will be able to replace B' by the compact convex body B satisfying both conditions (1) and (2) of Theorem 3.

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