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APPROXIMATE ANALYTICAL INVESTIGATION OF THE ELASTIC-PLASTIC BEHAVIOUR OF FIBROUS COMPOSITES. I. THERMAL LOADING

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Клаус Херман, Иван Миховски. ПРИБЛИЖЕННОЕ АНАЛИТИЧЕСКОЕ ИССЛЕДОВАНИЕ УПРУГОПЛАСТИЧЕСКОГО ПОВЕДЕНИЯ ВОЛОКНИСТЫХ КОМПОЗИТОВ. I. ТЕРМИЧЕСКОЕ НАГРУЖЕНИЕ.

Предложена математико-механическая модель упругопластического поведения класса волокнистых композитов с пластической матрицей и параллельными упругими волокнами с низким объемным содержанием последних. Наряду с качественными заключениями относительно механизмов пластифицирования матрицы получен ряд количественных оценок поведения композитов в условиях термического и механического нагружения (части I и II соответственно).

Klaus Herrmann, Ivan Mihovsky. APPROXIMATE ANALYTICAL INVESTIGATION OF THE ELASTIC-PLASTIC BEHAVIOUR OF FIBROUS COMPOSITES. PART I. THERMAL LOADING.

A mechano-mathematical model of the elastic-plastic response of a class of fibrous composites is proposed. It concerns low fibre volume fraction composites with a ductile matrix and parallel elastic fibres. Along with the qualitative conclusions about the mechanisms of matrix plastification a series of quantitative results is derived as well, concerning the composites response under thermal and mechanical loading conditions (Parts I and II, respectively).

INTRODUCTION

Reinforcement of complaint materials by parallelly aligned continuous strong fibres provides an essential increase in their strength and stiffness and makes the fibrous composites thus obtained attractive for various load-bearing applications. On the other hand such applications involve, as a rule, high fracture resistance requirements. Fibrous composites with ductile matrices prove to satisfy these requirements sufficiently well.

Thus, matrix plasticity appears to be a desired property of the composites. It reduces their sensitivity to a variety of typical structural defects which are either introduced by the fabrication processes or created artificially. The plasticity of the matrix material improves the resistance of the composites to initiation of modes of local fracture, associated with the stress concentration effects due to such structural defects. At the same time, matrix plasticity is known to change essentially the overall thermomechanical response of the composites and, in particular, to reduce considerably their overall strength. In other words, matrix plasticity leads to an overall behaviour of the composite material and to the development of modes of failure, which are much less sensitive to the local structural defects. Therefore, this phenomenon should be considered to be due to the very nature of the plastic deformation process developing within the matrix phase. The mechanisms, involved in this process, change the entire pattern of fibre-matrix interactions and, correspondingly, the basic features of the phenomena of load transfer and distribution, respectively, developing within the composite structures. Thus, it is of definite interest to clear up the nature of these mechanisms and in addition the trends in their development, their dependence on the structural parameters and the loading status of the composites, and accordingly their influence on the overall thermomechanical response of the latter. An attempt in this regard is made in the present study which concerns also the associated questions of how these mechanisms affect the failure phenomena in the composites, and how and to what extent they reduce their sensitivity to the typical structural defects.

A general approach to the problem is developed and an approximate analytical version of this approach is realized. The approach concerns the class of unidirectionally fibre reinforced composites of relatively low fibre volume fraction and with continuous strong elastic fibres perfectly bonded to a matrix of a weaker ductile material. Furthermore, the class of thermal and mechanical loading conditions is considered under which axisymmetric stress-strain states develop within a composite unit cell consisting of a circular cylindrical fibre with a coaxial cylindrical matrix coating. Numerous aspects of the basic problem considered in the following have been already successfully studied, for example, in the works of Hill [1], Spencer [2], Mulhern et al. [3], Ebert et al. [4], Thomason [5], Dvorak & Rao [6], Strife & Prevo [7], Min [8], Morley [9]. It should be immediately underlined that these references exhaust by no means the large list of publications on the problem but, at the same time, the present study aims neither at describing the state of the arts nor at reviewing the existing literature. Reference is made to these articles since they, even in such a restricted amount, clearly indicate how different the approaches to the problem may be and, in addition, how this variety of approaches is derivable from practically the same adoptions about the composite structures as well as by means of the same basic concepts of the plasticity theory. The distinguishing features of these approaches concern, in fact, the ways in which they account for (or neglect) the specific effects of the continuous fibre reinforcement, namely the strengthening (including the stiffening), the stress concentration, and the shrinkage effect. From the point of view of this distinguishing criterion one may specify the approach below as an attempt for a more rigorous account for each of these effects as well as for the simultaneous account of all of them. The remaining adoptions and concepts involved in the analysis do not differ in their nature from these of the works just cited.

In essence, the approach itself is a direct further development of the matrix plastification model previously proposed by the authors in [10, 11]. This basically qualitative model has proved to imply a series of useful conclusions concerning, for example, the development of the matrix plastification process (existence of a maximum plastic zone size), the mechanisms and the modes of failure of the composites (plastic instability of the matrix), and, in addition, the fibre-matrix cracks interactions phenomena (applicability of the Dugdale crack model, cf. [11, 12]). The development of the model in the present study leads to further conclusions concerning both the qualitative and the quantitative aspects of the considered problem. When coupled with appropriate numerical methods the general approach allows to achieve an improved accuracy of the results as well as an enlargement of the classes of the considered composite structures and loading conditions without principal changes in the structure of the governing equations. At the same time the object of the present investigation is not to deliver quantitative estimations of high accuracy but rather to bring a sufficient understanding of the very nature of the processes of matrix plastification and of their influence on the overall response of the composites. To clear up these questions is the principal aim and to this respect the general approach proves to be an effective tool even in its simplified approximate analytical version. The latter simulates adequately enough the specific features and trends of development of the matrix plastification process. The analysis predicts an overall response which is consistent with the commonly adopted understanding of the composites behaviour in the "rule of mixtures" sense.

Two model problems are considered in detail. These are the problems of matrix cooling (a simplified version of the cooling of the entire composite structure) and longitudinal extension. They simulate loading conditions which are typically involved in the processes of fabrication of the composites (thermal treatment) and in their load-bearing applications, respectively. The study is divided into two parts. This is due to the fact that the general approach reveals quite different specific patterns of the elastic-plastic response of the composites when applied to each of the two model problems considered. Each of these patterns proves to deserve due attention from the point of view of the corresponding analysis, its predictions, and the practical applications of the latter. The first part of the study deals with the thermally induced elastic-plastic behaviour of the considered class of fibrous composites.

STATEMENT OF THE PROBLEM

The class of composites and the composite unit cell, considered in the following, are as specified in the introduction. When referred to a cylindrical coordinate system $\{r, \theta, z\}$, where the z-axis coincides with the axis of the fibre, the cross-sections of the fibre and the matrix occupy the regions $\{0 \leqq r \leqq r_f, 0 \leqq \theta \leqq 2\pi\}$ and $\{r_f < r \leqq r_m, 0 \leqq \theta \leqq 2\pi\}$, respectively.

The fibre material is linearly elastic with Young's modulus E_f, Poisson's ratio ν_f, and linear thermal expansion coefficient α_f. The material of the matrix is elastic (E_m, ν_m, α_m) — perfectly plastic and obeys the von Mises yield condition. The thermoelastic properties of the fibre and the matrix as well as the tensile yield stress σ_y of the latter are considered as temperature independent.

The thermal loading is specified as matrix cooling, that means as a process of monotonous quasi- static decrease of the itself negative matrix temperature T_m, which is measured from the temperature of the initially unstressed state of the composite. The same scheme of loading has been considered in [11]. The genere-lization of the analysis of this model scheme with respect to the process of cooling of the entire cell, which is practically always involved in the fabrication of the composites, as well as to other more realistic modes of thermal loading is almost straightforward. No external loads are applied to the cell. Thus, the correspond-ing thermally induced stress-strain state of the cell is axisymmetric and, due to the assumed perfect fibre-matrix bond, allows to be treated by applying the plane cross-sections hypothesis. Correspondingly, the normal stresses in both the fibre and matrix phases are principal ones and depend upon the radial coordinate only.

THE MATRIX PLASTIFICATION MODEL

It was already mentioned that the analysis in the present investigation is based upon the matrix plastification model, developed in previous works of the authors [10, 11]. Thus, a brief general description of the model and of the associated basic concepts would be usefull both for the better understanding of the analysis and for its concise presentation. As it should be expected, the basic concepts of the model concern the principal features of the considered composites and, firstly, the main effect of the fibre reinforcement, namely the strengthening one. In fact, due to the associated decrease of the compliance of the composites connected with this effect, the longitudinal strains ε_z in the latters remain relatively small, i.e. comparable with the themselves small purely elastic strains in the stiff fibres. Then the elastic ε_z^e and the plastic ε_z^p-components of the itself small total ε_z-strain in the plastified matrix region are also small enough for a comparison, using relations like "much larger" or "negligibly small". Accordingly, the model states first of all that by considering the matrix plastification process one should permanently account for the current ε_z^e-strain instead of neglecting it with respect to the ε_z^p-strain, as it is the usual case in the common plasticity approaches. The way, in which the latter account is carried out, is associated with another principal feature of the considered composites, namely the limited elastic response of the matrix material. The natural development of a given process of progressive plastification in a point, i.e. in an elementary volume of such a material, involves, most generally speaking, trends of progressive decrease and increase in the elastic and the plastic strain increments, respectively. One may thus generally relate such a process with a certain specific instant of its development upon which the elastic strains may be viewed as keeping approximately constant values, since their further increments become small enough to prevent (upon superposing) further substantial changes in the values which they have achieved at this instant. In accordance with these mostly qualitative but realistic considerations the model assumes the following. For a given composite structure, given loading status, for a given elementary volume of the matrix phase, a specific value $\overset{*e}{\varepsilon}_z$ of the ε^e_z-strain exists such that upon a certain transitional regime of plastification, at the end of which the ε^e_z-strain in this volume achieves the value $\overset{*e}{\varepsilon}_z$, a second regime starts developing for which the relation $\varepsilon^e_z = \overset{*e}{\varepsilon}_z$ holds true. A further simplifying assumption of the model concerns the dependence of the $\overset{*e}{\varepsilon}_z$-value on the location of the elementary volume, i.e. on the spesific and actually unknown pattern of the transitional plastic stress redistribution which depends itself on this location. The model actually deals with the same $\overset{*e}{\varepsilon}_z$-value in the entire matrix region, where the second regime has started developing. The quantity $\overset{*e}{\varepsilon}_z$ may be thus considered as an average overall measure of the limited elastic response of a given composite under a given loading status. The determination of this quantity is, of course, a part of the analysis of the elastic-plastic response of the composites.

When specified with respect to the considered composite unit cell these basic concepts of the model imply the following qualitative description of the development of the matrix plastification process for both the model problems mentioned. Due to the stress concentration effect of the fibre, plastic deformations appear in the matrix at first at the fibre-matrix interface and a transitional regime of matrix plastification starts developing. The plastic zone associated with this regime has, due to the symmetry, the form of an annulus $r_f \leqq r \leqq r_c$ and spreads into the matrix phase. At the instant when $\varepsilon^e_z|_{r=r_f} = \overset{*e}{\varepsilon}_z$, i.e. when the ε^e_z-strain achieves its limiting value (and this instant is first achieved at the fibre-matrix interface), the second regime starts developing with a plastic zone $r_f \leqq r \leqq R_c$, $R_c \leqq r_c$, within which the relation $\varepsilon^e_z = \overset{*e}{\varepsilon}_z$ holds true. The second plastic zone spreads into the matrix phase as well having the first one, which occupies now the annulus $R_c \leqq r \leqq r_c$ at its front $r = R_c$. Thereby the transitional plastic zone $R_c \leqq r \leqq r_c$ is further considered as a thin layer, i.e. $R_c \approx r_c$. The latter plays the role of an elastic-plastic boundary, to which a softened version of fulfillment of the standard elastic-plastic transitional conditions is applicable (cf. [10, 11]).

Finally, the following remark is due with respect to the thermal problem considered below. The elastic part ε^e_z of the total axial strain ε_z in this case involves itself a part $\varepsilon^{e,sts}_z$, due to the thermal stresses, and a part $\varepsilon^{e,temp}_z$, due to the thermal contraction or expansion, respectively. When referred to the thermal problem the considerations, made above with respect to the ε^e_z-strain, should be now viewed as concerning not the entire ε^e_z-strain but its $\varepsilon^{e,sts}_z$-part only. Moreover, the strain

$\varepsilon_z^{e,temp}$ is stress independent.

ELASTIC BEHAVIOUR AND ELASTIC-PLASTIC TRANSITION

The assumptions, specifying the class of fibrous composites under consideration, allow to treat the products and the powers of the ratios E_m/E_f and r_f/r_m as small quantities. Appropriate simplifications are carried out accordingly in the following sections and the results derived are presented in forms, containing the principal terms only.

The linear-elastic solution of the considered problem is obtainable as a simple generalization of the plane strain ($\varepsilon_z \equiv 0$) solution of Herrmann [13]. The process of matrix cooling implies the following elastic distribution of the stresses σ_i^{me} and σ_i^{fe}, $i = r, \theta, z$, in the matrix and in the fibre respectively:

$$
(1) \qquad \begin{aligned}
\left.\begin{matrix}\sigma_r^{me}\\ \sigma_\theta^{me}\end{matrix}\right\} &= \frac{E_m}{1+\nu_m}\frac{C}{r_m^2}\left(1 \mp \frac{r_m^2}{r^2}\right),\\
\sigma_z^{me} &= E_m(\varepsilon_z - \alpha_m T_m) + \nu_m(\sigma_r^{me} + \sigma_\theta^{me}),\\
\sigma_r^{fe} &= \sigma_\theta^{fe} = \sigma_r^{me}|_{r=r_f},\\
\sigma_z^{fe} &= E_f\varepsilon_z + 2\nu_f\sigma_r^{me}|_{r=r_f},
\end{aligned}
$$

where

$$
(2) \qquad C = -r_f^2\alpha_m T_m(1+\nu_m).
$$

In fact, eqn (2) represents the exact value of the principal term of C for a composite with $\nu_m = \nu_f$. Generally, this term involves the multiplying factor $[1-(\nu_m-\nu_f)/(1+\nu_m)(1+E_c)]$ as well, where

$$
(3) \qquad E_c = E_f r_f^2/E_m r_m^2.
$$

The latter factor is neglected in the following analysis, since, as one may actually prove, it does not affect substantially the basic features of composite's behaviour. Along with the self-equilibrium condition of the axial stresses σ_z^{ie}, $i = f, m$, the stress distribution from eqns (1) implies the relations

$$
(4) \qquad \varepsilon_z = \alpha_m T_m/(1+E_c),
$$

$$
(5) \qquad \varepsilon_z^{sts} = -\alpha_m T_m E_c/(1+E_c),
$$

where $\varepsilon_z^{sts} = \varepsilon_z - \varepsilon_z^{temp}$ is the part of the ε_z-strain, due to the stresses, and $\varepsilon_z^{temp} = \alpha_m T_m$. Eqns (4) and (5) are obviously approximations of the thermoelastic response of the composite unit cell in the common "rule of mixtures" sense. Furthermore, in accordance with the von Mises' yield condition, the foregoing relations define the *temperature of initial matrix plastification* T_m^{pl} at the fibre-matrix interface as

$$(6) \qquad T_m^{pl} = -\sigma_y/\sqrt{3}\alpha_m E_m.$$

The corresponding ε_z^{pl}- and $\varepsilon_z^{sts,pl}$-values are

$$(7) \qquad \varepsilon_z^{pl} = -\sigma_y/\sqrt{3}E_m(1+E_c),$$

$$(8) \qquad \varepsilon_z^{sts,pl} = \sigma_y E_c/\sqrt{3}E_m(1+E_c).$$

ANALYSIS OF THE ELASTIC-PLASTIC BEHAVIOUR

According to the matrix plastification model described above the $\varepsilon_z^{e,sts}$-strain at the fibre-matrix interface achieves upon a certain transitional regime the value $\overset{*e}{\varepsilon}_z$. Its initial value is the value $\varepsilon_z^{sts,pl}$ defined by eqn (8). At this instant the second plastic zone $r_f \leqq r \leqq R_c$ starts spreading into the matrix phase. The relation $\varepsilon_z^{e,sts} = \overset{*e}{\varepsilon}_z$ holds true within this zone. In accordance with the generalized thermoelastic Hooke's law the stresses σ_i^{mp}, $i = r, \theta, z$, in the plastic zone satisfy the relation

$$(9) \qquad \sigma_z^{mp} = E_m\overset{*e}{\varepsilon}_z + \nu_m(\sigma_r^{mp} + \sigma_\theta^{mp}).$$

Eqn (9) allows a reduction of the von Mises' yield criterion to the form

$$(10) \qquad \left(\frac{\sigma_\theta^{mp} - \sigma_r^{mp}}{2}\right)^2 + \left(\frac{\sigma_\theta^{mp} + \sigma_r^{mp}}{2} - \frac{E_m\overset{*e}{\varepsilon}_z}{1-2\nu_m}\right)^2 \frac{(1-2\nu_m)^2}{3} - \frac{\sigma_y^2}{3} = 0.$$

The latter equation is identically satisfied by stresses of the form

$$(11) \qquad \left.\begin{matrix}\sigma_r^{mp}\\ \sigma_\theta^{mp}\end{matrix}\right\} = \frac{E_m\overset{*e}{\varepsilon}_z}{1-2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\Phi}\cos(\omega \pm \Phi)$$

where

$$(12) \qquad \sin\omega = \frac{\sigma_\theta - \sigma_r}{2} \Big/ \frac{\sigma_y}{\sqrt{3}},$$

(13) $$\tan\Phi = (1-2\nu_m)/\sqrt{3}.$$

Due to the elastic restriction specified by eqn (9) the yield condition defines an ellipse in the $(\sigma_\theta,\ \sigma_r)$-plane, eqn (10) or eqns (11), respectively. The points of the yield ellipse have coordinates $(\sigma_\theta^{mp},\ \sigma_r^{mp})$ and are representative points in the stress-space for the stress-states in the points of the plastified matrix phase. Thus, a specific process of plastic stress redistribution in a point of the matrix phase defines via the angle ω, eqn (12), a specific law of motion along the yield ellipse of a corresponding representative point. Thereby the angle ω is easily seen to be a function of the loading parameter, i.e. T_m, as well as of the radial coordinate r and is further depending on both the geometrical and the mechanical characteristics of the composite constituents. The r-dependence of the angle ω is obtainable upon integrating the equilibrium equation

(14) $$\frac{d\sigma_r^{mp}}{dr} + \frac{\sigma_r^{mp} - \sigma_\theta^{mp}}{r} = 0$$

in the interval $r_f \leqq r \leqq R_c$ with the boundary condition

(15) $$\omega|_{r=R_c} \equiv \omega_{R_c} = \arccos\left[-E_m\overset{*}{\varepsilon}{}_z^e/\sigma_y(1+\nu_m)\right].$$

The latter condition reflects the assumption (cf. Herrmann & Mihovsky [10, 11]) that $\varepsilon_z^{e,sts}$ is the only non-negligible elastic strain in the second plastic zone (where, as adopted, $\varepsilon_z^{e,sts} = \overset{*}{\varepsilon}{}_z^e$) and that the matrix material is plastically incompressible.

The result of the integration reads

(16) $$\frac{R_c^2}{r^2} = \frac{\sin\omega}{\sin\omega_{R_c}}\exp[(\omega-\omega_{R_c})\mathrm{cotan}\ \Phi],$$

where the plastic zone radius R_c is to be further determined as a function of the loading parameter T_m.

With respect to the values of ω at the fibre-matrix interface eqn (16) implies

(17) $$\frac{R_c^2}{r_f^2} = \frac{\sin\omega_{r_f}}{\sin\omega_{R_c}}\exp[(\omega_{r_f}-\omega_{R_c})\mathrm{cotan}\ \Phi],$$

where the notation is introduced

(18) $$\omega_{r_f} = \omega(r)|_{r=r_f}.$$

It is clear from the very nature of the considered thermal loading process that progressive matrix cooling should result in progressive shrinkage, i.e. in progressive decrease of the itself negative radial stress acting over the fibre-matrix interface. At the same time the shrinkage effect is limited itself in the sense that, as eqns (11) prove, a maximum shrinkage, i.e. a minimum value of the latter stress is achievable at the instant when $\omega_{r_f} = \pi - \Phi$. This specific instant for the composite unit cell is shown in [10, 11] to correspond to a critical state of the cell when failure modes start developing in the latter due to the plastic instability of the matrix at the fibre-matrix interface. Further, these considerations imply the conclusion that with progressive thermal loading the angle ω_{r_f} increases (cf. the structure of the $\sigma_r^{mp}|_{r=r_f}$-stress, eqns (11)), running actually within the interval

(19) $$\omega_{R_c} \leqq \omega_{r_f} \leqq \pi - \Phi.$$

The latter conclusion is meaningful if, of course, the angles ω_{R_c} and Φ satisfy the relation $\omega_{R_c} < \pi - \Phi$. Since the quantity $\overset{*e}{\varepsilon_z}$ should be expected to belong actually to the interval $[\varepsilon_z^{sts,pl}, \sigma_y/E_m]$, then eqns (13) and (15) prove immediately that the latter relation is valid if $\nu_m > 0.1$, which is the practical case for the commonly used matrix materials. Moreover, in accordance with this conclusion eqn (17) proves the existence of a maximum plastic zone size R_c^* and defines the latter as

(20) $$R_c^{*2} = r_f^2 \frac{\sin\Phi}{\sin\omega_{R_c}} \exp[(\pi - \Phi - \omega_{R_c})\text{cotan}\,\Phi].$$

For reasons of simplicity the analysis below is restricted to cases for which $R_c^* < r_m$. From its quantitative side this analysis aims at the prediction of the thermally induced elastic-plastic response of the unit cell, i.e. the $\varepsilon_z(T_m)$-dependence. This aim is achieved in the following in a step-wise way, which involves at first the determination of the $\omega_{r_f}(\varepsilon_z)$- and the $R_c(\varepsilon_z)$-dependences.

The procedure of obtaining the $\omega_{r_f}(\varepsilon_z)$-dependence involves the following basic steps. First, the condition of continuity of the radial displacements u_r^i, $i = f, m$ at the fibre-matrix interface is constructed by the aid of the known axisymmetric relations $u_r^i|_{r=r_f} = r_f \varepsilon_\theta^i|_{r=r_f}$, where ε_θ^i, $i = f, m$, are the circumferential strains in the fibre and the matrix, respectively. Further, the strain rates ξ_θ^i, $i = f, m$, are obtained as derivatives of the strains ε_θ^i with respect to the loading parameter T_m. Thereby the elastic part ξ_θ^{me} of the ξ_θ^m-strain rate at the interface $r = r_f$ is neglected (cf. the text following eqn (15)). The strain rate ξ_θ^f is defined via the generalized Hooke's law and eqns (1), now with $\sigma_r^{mp}|_{r=r_f}$ instead of $\sigma_r^{me}|_{r=r_f}$ for the stresses at the fibre-matrix interface. The plastic part ξ_θ^{mp} of the ξ_θ^m-strain rate is defined in accordance with the associated flow rule concept along with the yield function, used as a plastic potential (cf. [11]). Moreover, the thermal part of the ξ_θ^m-strain rate can be neglected without affecting the basic trends of the ω_{r_f}-behaviour. The u_r-continuity condition is thus reduced to the form

(21) $$\Lambda d\varepsilon_z = f(\omega_{r_f}) d\omega_{r_f},$$

where the notations are introduced

$$\Lambda = \frac{E_f\sqrt{3}}{2\sigma_y(1+\nu_f)(1-2\nu_f)}, \tag{22}$$

$$f(\omega_{r_f}) = \frac{\sin(\omega_{r_f}+\Phi)\cos\omega_{r_f}}{\sin(\omega_{r_f}+\Phi) - 2\nu_f \sin\Phi \cos\omega_{r_f}}. \tag{23}$$

Eqn (21) has to be solved in the interval $[\omega_{R_c}, \pi - \Phi]$ with the approximate boundary condition

$$\varepsilon_z|_{\omega_{r_f}=\omega_{R_c}} = \tilde{\varepsilon}_z^{pl} = -\overset{*}{\varepsilon}{}_z^{e}/E_c. \tag{24}$$

This boundary condition results from the assumption that the behaviour of the unit cell in the interval between the initial matrix plastification and the occurrence of the second plastic zone, i.e. in the transitional regime, is not substantially affected by the only presence of the corresponding transitional plastic zone and thus may be considered as following the linear-elastic dependence, given by eqn (4) or eqn (5) respectively. Such an assumption practically identifies the $\overset{*}{\varepsilon}{}_z^{e}$ and $\varepsilon_z^{sts,pl}$ strains and further defines by means of eqns (6) and (8) (the latter with $\varepsilon_z^{sts,pl} = \overset{*}{\varepsilon}{}_z^{e}$ now) the instant of occurrence of the second plastic zone (when $\omega_{r_f} = \omega_{R_c}$, cf. eqn (24)), as corresponding to the values $\tilde{T}_m^{pl}$ and $\tilde{\varepsilon}_z^{pl}$ of T_m and ε_z^{pl} respectively, which are

$$\tilde{T}_m^{pl} = -\overset{*}{\varepsilon}{}_z^{e}(1+E_c)/\alpha_m E_c, \tag{25}$$

$$\tilde{\varepsilon}_z^{pl} = -\overset{*}{\varepsilon}{}_z^{e}/E_c. \tag{26}$$

An approximate series expansion procedure for solving the boundary value problem, specified by eqns (21) and (24), is applied. It consists of the following steps. Eqn (21) is first solved for values of ω_{r_f}, close to $\pi - \Phi$, upon an expansion of the function $f(\omega_{r_f})$ into the powers of the small differences $(\pi - \Phi - \omega_{r_f})$. The solution thus obtained is then extrapolated over the entire interval $[\omega_{R_c}, \pi - \Phi]$ in order to fit the boundary condition, eqn (24). Accordingly, the following form of the desired approximate dependence is obtained

$$\omega_{r_f}(\Delta\varepsilon_z) = \pi - \Phi - \left[(\pi - \Phi - \omega_{R_c})^2 + \frac{2b\Lambda}{\cos\Phi}\Delta\varepsilon_z\right]^{1/2}, \tag{27}$$

where

(28) $$b = 2\sqrt{3}\nu_f(1-2\nu_m)/[3+(1-2\nu_m)^2],$$

(29) $$\Delta\varepsilon_z = \varepsilon_z - \overline{\varepsilon}_z^{pl}.$$

The quantity $\Delta\varepsilon_z$ is thus the part of the total axial strain ε_z which develops upon the occurrence of the second plastic zone. The critical value $\Delta\overset{*}{\varepsilon}_z$ of $\Delta\varepsilon_z$ at which the unit cell undergoes a transition to failure, follows from eqn (27) with $\omega_{r_f} = \pi - \Phi$ to be

(30) $$\Delta\overset{*}{\varepsilon}_z \doteq -(\pi - \Phi - \omega_{R_c})^2 \frac{\cos\Phi}{2b\Lambda}.$$

With the aid of a similar expansion technique one obtains upon introducing ω_{r_f} from eqn (27) into eqn (17) the $R_c(\varepsilon_z)$-dependence in the form

(31) $$R_c^2(\Delta\varepsilon_z) = R_c^{*2}\left[1 - \left(1 - \frac{r_f^2}{R_c^{*2}}\right)\left(1 - \frac{\Delta\varepsilon_z}{\Delta\varepsilon_z^*}\right)\right].$$

It should be pointed out that eqns (27) and (31) approximate the actual $\omega_{r_f}(\varepsilon_z)$- and $R_c(\varepsilon_z)$-dependences rather roughly but, at the same time, they keep and clearly indicate the basic features of the latter, due to their simple analytical forms.

Further, the determination of the $\varepsilon_z(T_m)$-dependence is a matter of simple computations, based upon the condition of self-equilibrium of the axial stresses

(32) $$r_f^2\sigma_z^f + (r_m^2 - R_c^2)\sigma_z^{me} + 2\int_{r_f}^{R_c} \sigma_z^{mp} r\,dr = 0.$$

Thereby the stress σ_z^f is to be defined from eqns (1) with $\sigma_r^{mp}|_{r=r_f}$ instead of $\sigma_r^{me}|_{r=r_f}$ and with $\sigma_r^{mp}|_{r=r_f}$ given by eqns (11) with $\omega = \omega_{r_f}$ along with eqn (27) for $\omega_{r_f}(\Delta\varepsilon_z)$. The axial stress σ_z^{me} in the elastically deformed matrix region $R_c \leqq r \leqq r_m$ is obtainable from eqns (1) upon definition of a new C-value from the σ_r-continuity condition at the elastic-plastic boundary $r = R_c$. The latter condition reflects the softened version of the fulfillment of the elastic-plastic transitional conditions mentioned above (cf. [10, 11]).

The axial stress σ_z^{mp} in the plastic zone and the radius R_c of the latter are defined by eqns (9) and (31) respectively.

Upon corresponding computations and appropriate simplifications eqn (32) implies the relation

(33) $$\Delta\varepsilon_z = \alpha_m \Delta T_m \frac{1 - R_c^2/r_m^2}{1 + E_c - R_c^2/r_m^2},$$

and by introducing for R_c from eqn (31) it is obtained

(34) $$\Delta\varepsilon_z = \alpha_m \Delta T_m \left[1 + E_c + \alpha_m \Delta T_m \frac{E_c}{1 + E_c} \frac{R_c^{*2}}{r_m^2} \left(1 - \frac{r_f^2}{R_c^{*2}}\right) \frac{1}{\Delta\varepsilon_z^*}\right]^{-1},$$

where the notation is used

(35) $$\Delta T_m = T_m - \widetilde{T}_m^{pl}.$$

The explicite form of the $\varepsilon_z^{sts}(T_m)$-dependence is obtainable straightforwardly from eqn (35) and the relation $\Delta\varepsilon_z^{sts} = \Delta\varepsilon_z - \alpha_m \Delta T_m$.

The critical temperature of failure of the unit cell $T_m^* = \widetilde{T}_m^{pl} + \Delta T_m^*$ follows from eqns (25) and (34) (with $\Delta\varepsilon_z = \Delta\varepsilon_z^*$ for ΔT_m^*) respectively. Both quantities $\Delta\varepsilon_z^*$ and ΔT_m^* and therefore T_m^* are dependent on the specific value of $\overset{*e}{\varepsilon}_z$ for the unit cell and thus for the composite structure also. Consequently, eqn (34) represents the desired approximate analytical form of the thermally induced elastic-plastic response of the composite unit cell in the considered model problem of matrix cooling.

BASIC FEATURES OF THE COMPOSITE BEHAVIOUR

The basic features of the elastic-plastic response of the composite predicted by the foregoing analysis will be briefly considered in this section. It should be mentioned, first of all, that with the aid of the obvious relation $\Delta\varepsilon_z^{sts} = \Delta\varepsilon_z - \alpha_m \Delta T_m$ one may immediately transform eqn (33) into the relation $E_m(r_m^2 - R_c^2)\Delta\varepsilon_z^{sts} + E_f r_f^2 \Delta\varepsilon_z = 0$. Thereby the latter relation is nothing else but an explicit representation of the predicted composite response in the "rule of mixtures" sense. In accordance with this representation the plastified matrix region influences the redistribution of the axial forces via its radius R_c but does not explicity contribute to this redistribution. Its own contribution appears to be just negligible within the frame of the present approximate analysis. Furthermore, the following statement should be made with respect to the structure of the $\Delta\varepsilon_z(\Delta T_m)$-dependence obtained above. The strain $\Delta\varepsilon_z$ defined by eqn (34) is easily seen to decrease monotonically as a concave negative function when the itself negative temperature difference ΔT_m decreases. The curve $\Delta\varepsilon_z(\Delta T_m)$ proves to deviate smoothly from the linear elastic $\varepsilon_z(T_m)$-dependence defined by eqn (4). With the formal limit transition $\Delta T_m \longrightarrow -\infty$ the strain $\Delta\varepsilon_z$ approaches asymptotically a limit value $\Delta\hat{\varepsilon}_z$ which may be easily shown to satisfy the relation $\Delta\hat{\varepsilon}_z < \Delta\varepsilon_z^*$ (with $\Delta\varepsilon_z^* < 0$, cf. eqn (30)). The latter means that the composite cell achieves its critical state of failure at finite values of ΔT^*_m and T^*_m respectively.

A purely qualitative schematic illustration of the total elastic-plastic response, derived above, is presented in Fig. 1 where the straight line I describes the behaviour of a homogeneous cylinder of the matrix material under the considered cooling process. No thermal stresses develop in such a cylinder and its axial strain is due to the thermal contraction only. The line II corresponds to purely elastic fibre and matrix materials, eqn (4). Each of the series of the concave curves III,$_i$ corresponds to eqn (34) with an initially specified $\overset{*}{\varepsilon}{}^{e}_{z,i}$-value. Each of these lines coincides with the line II over the corresponding interval $[0, \tilde{T}^{pl}_{m,i}]$ or $[0, \tilde{\varepsilon}^{pl}_{z,i}]$, respectively (cf. eqns

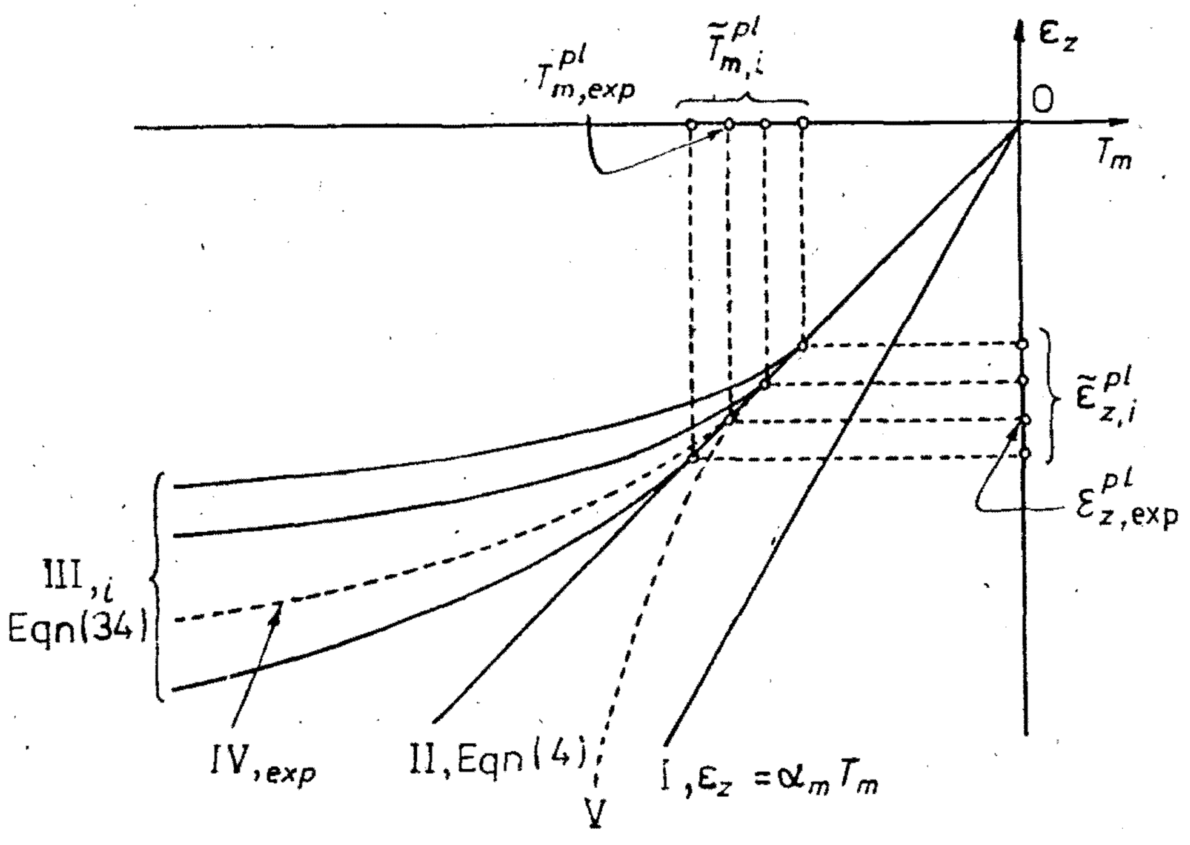

Fig. 1. Schematic qualitative illustration of the elastic-plastic response of fibrous composites due to matrix cooling

(25) and (26)) and smoothly deviates from this line in the way, shown in the graph, at the corresponding points $(\tilde{T}^{pl}_{m,i}, \tilde{\varepsilon}^{pl}_{z,i})$. The line IV is the assumed experimentally obtained $\varepsilon_z(T_m)$-curve for the considered composite. As it is usually accepted in the engineering practice, the linear part of this curve is constructed in accordance with the linear-elastic "rule of mixture" approach. Thus it coincides over this part with the straight line II. Let the line IV deviate from the line II at the point $(T^{pl}_{m,exp}, \varepsilon^{pl}_{z,exp})$, where "*exp*" stays for the experimentally measured values of T^{pl}_m and ε^{pl}_z. Then, upon identifying these values with the $\tilde{T}^{pl}_m$ and $\tilde{\varepsilon}^{pl}_z$ values in eqns (25) and (26), respectively, one defines a corresponding, say $\overset{*}{\varepsilon}{}^{e}_{z,exp}$-value of the quantity $\overset{*}{\varepsilon}{}^{e}_{z}$. It

is this latter value of $\overset{*e}{\varepsilon}_z$ to deal with when applying the foregoing general approach to a given composite structure.

A more sophisticated approach to the identification of the actual value of the $\overset{*e}{\varepsilon}_z$-strain involves a comparison between the actual $\Delta\varepsilon_z(\Delta T_m)$-curve and the series of theoretical curves $\text{III}_{,i}$. Upon introducing an appropriate best fitting criterion and by means of a corresponding processing of these curves one may define the theoretical curve which fits the experimental one in the best way with respect to the chosen criterion. The value of $\overset{*e}{\varepsilon}_z$, to which this theoretical curve corresponds, will then be the actual one for the considered composite. It should be mentioned that the strain $\Delta\varepsilon_z^{sts}$ increases as a concave positive function when the negative temperature difference ΔT_m decreases. One may easily derive the basic features of the $\Delta\varepsilon_z^{sts}(\Delta T_m)$-dependence by the aid of the foregoing equations. Furthermore, in some cases the linearization of the composite response in the elastic-plastic range may be of interest. A simple linearized version of eqn (34) is presented, for example, by the relation $\Delta\varepsilon_z/\Delta\varepsilon_z^* = \Delta T_m/\Delta T_m^*$. Such a linearization replaces the family of concave curves $\text{III}_{,i}$ in Fig.1 by a corresponding family of straight lines with the same points of deviation from the line II. The approaches to the identification of the actual $\overset{*e}{\varepsilon}_z$-value, described above, apply to the linearized case as well.

CONCLUDING REMARKS

The results, obtained in the previous sections, represent in the whole an approximate analytical solution of the considered problem of thermal loading of a composite structure. The general approach, developed in the study, involves a specific parameter $\overset{*e}{\varepsilon}_z$ for the composite structure as well as the loading status and reveals the ways to its identification under the implicit assuption that the real thermomechanical response of the composite corresponds to a concave strain-temperature curve (cf. curve IV in Fig. 1). Whether this is the actual case or, in other words, whether the predictions of the approach (the curve $\text{III}_{,i}$ in Fig. 1) are at least in qualitative agreement with the real composite response is a principal question. A positive answer to this question would not only support the validity of the approach in the whole but would obviously reveal further possibilities for achieving a better quantitative fitting between the predicted and the actual response. Thereby the following statement could be made with regard to this problem. To the authors' knowledge there exist at present no experimental data which could be used in a reliable way for a comparison with the prediction for the model problem considered. At the same time the behaviour of the composite under thermal loading with a concave $\varepsilon_z(T_m)$-curve is explainable in quite a natural way. The progressive matrix plastification results in a softening of the matrix in the sense that the stresses in the itself expanding plastic zone remain limited. This implies a corresponding relative increase in the strengthening effect of the fibre and thus of the overall stiffness of the composite structure. The concave curves $\text{III}_{,i}$ and IV in Fig. 1 reflect, in fact, exactly the latter effect. It is difficult to explain in a similar way an imaginary behaviour of the composite to which a convex curve, such as curve V in Fig. 1, would correspond. It should be mentioned in addition that the theoretically predicted response of the composite allows for a direct realistic interpretation in the "rule of mixture" sense. This fact may be considered to confirm to a further extent the potential of the developed approach for reliable predictions of the elastic-plastic response of the composites. Finally it should be noted that, as Part II of the present study proves, when applied to the problem of longitudinal extension of a fibrous composite the same approach predicts a stress-strain curve which is in entire qualitative agreement with the typical experimental observations. Certain additional aspects of the thermally induced response of the composites will be considered and simultaneously compared with the corresponding aspects of the behaviour of such composites under longitudinal extension in the closing section of Part II of the present study. These aspects concern basically the general features of the matrix plastification processes and their influence on the fracture phenomena in fibrous composites.

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