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NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS
III. THE AXIOMS

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Expressum facit cessare tacitum

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. III. АКСИОМЫ.

Эта работа является третьей частью серии исследования под общим наименованием *Динамические аксиомы Ньютона и Эйлера*, первые две части которой опубликованы в 79-том томе этого *Ежегодника* 1985 г. (книга 2 — *Механика*). Целью этой серии является исследование роли динамических аксиом Ньютона и Эйлера в процессе логической консолидации математических основ динамики массовых точек и твердых тел, а также точного места, которое эти фундаментальные динамические постулаты занимают в системе аналитической механики. В этом смысле вопросная серия представляет принос к решению шестой проблеме Гильберта относительно аксиоматического построения аналитической механики. Специальное внимание обращено понятию инерциальности твердых систем отсчета как согласно Ньютона, так и Эйлера (предложения 11 – 13), особенно в случае динамики массовых точек и твердых тел с променливыми массами.

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. III. THE AXIOMS.

This paper is the third part of a series of studies under the general title *Newtonian and Eulerian dynamical axioms*, the first two parts of which have been published in the 79th volume of this *Annual* for 1985 (book 2 — *Mechanics*). The aim of the series is to examine the role of the Newtonian and Eulerian dynamical axioms in the process of the logical

consolidation of the mathematical foundations of mass-point and rigid body dynamics, as well as the exact place these basic dynamical postulates take up in the edifice of the science analytical mechanics. In such a sense the series in question represents a contribution to the solution of Hilbert's sixth problem concerning the axiomatical construction of analytical mechanics. Special attention is paid to the notion of inertiality of rigid systems of reference according to both Newton and Euler (Pr 11 – Pr 13), particularly in the case of dynamics of mass-points and rigid bodies with variable masses.

This paper is the third part of a series of studies under the general title *Newtonian and Eulerian dynamical axioms*, the first two parts of which have been published in the 79th volume of this *Annual* for 1985 (book 2 — *Mechanics*). The aim of the series is to examine the role the Newtonian and Eulerian dynamical axioms play in the process of the logical consolidation of the mathematical foundations of mass-point and rigid body dynamics, as well as the exact place these basic dynamical postulates take up in the edifice of the science analytical mechanics. This may be accomplished by a thorough analysis of all the aspects of Newtonian and Eulerian dynamical axioms. In such a sense the series in question represents a contribution to the solution of Hilbert's sixth problem concerning the axiomatical construction of analytical mechanics.

The Newtonian and Eulerian dynamical axioms have a crucial role in the historical development of analytical dynamics. As a matter of fact, the mass-point dynamics has been borne in 1687 with the publication of Newton's famous *Philosophiae Naturalis Principia Mathematica*; and the rigid body dynamics — in 1775, when Euler wrote his *Nova methodus motum corporum rigidorum determinandi*, a work unfortunately still obscure even among professional mechanicians. That is why in the first part of the series a historical review has been proposed on the meanders that analytical mechanics was destined to wander about before the laws or principles of momentum and of moment of momentum of mass-points and rigid bodies have been discovered.

In the second part a review has been proposed on the manner these fundamental dynamical laws are represented (or sooner misrepresented) in the traditional literary sources on analytical dynamics.

The present third part is dealing with the axioms themselves. It contains strict mathematical formulations of these axioms along with several preliminary definitions of mechanical entities, therein involved, and some immediate but important corollaries.

Sch 1. For the sake of brevity the symbols Sgn, sgn:, Ax, Df, Pr, Dm, and Sch replace the words *notation, denotes by definition, axiom, definition, proposition, proof*, and *scholium* respectively, and the letters, R and C are reserved for the fields of all real and all complex numbers respectively.

Sch 2. The bibliography of all three parts of the series has a unified numeration.

Sch 3. Numbers in *brevier* refer to the *Appendix* in the end of the article.

Sch 4. Quotations from the *Appendix* are made in the following manner: relation (17) and proposition 19 therein are cited simply as (17) and Pr 19 respectively in the *Appendix* itself, but as App(17) and AppPr 19 respectively elsewhere.

Sch 5. Similarly, relation (17) and proposition 19 from the *main text* of this paper are cited simply as (17) and Pr 19 respectively in the main text itself, but as M(17) and M Pr 19 respectively in the *Appendix*.

The whole of mass-point dynamics is based upon, and is developed from, the following two postulates.

Ax 1 N (*fisrt Newtonian dynamical axioms,* alias *law* or *principle of momentum of mass-point*). There exists such a rigid system of reference S that, all derivatives being taken with respect to S, for any mass-point P and for any system of forces $\underrightarrow{F}$ acting on P, the derivative with respect to the time of the momentum of P equals the basis of $\underrightarrow{F}$.

Df 1 N. Any system of reference, satisfying Ax 1 N, is called *inertial according to Newton.*

Ax 2 N (*second Newtonian dynamical axiom,* alias *law* or *principle of moment of momentum* (*kinetical moment*) *of a mass-point*). If S is an inertial according to Newton system of reference and all derivatives are taken with respect to S, then for any mass-point P and for any system of forces $\underrightarrow{F}$ acting on P, the derivative with respect to the time of the moment of momentum of P equals the moment of $\underrightarrow{F}$, both moments being taken with respect to the origin of S.

The whole of rigid body dynamics is based upon, and is developed from, the following two postulates.

Ax 1 E (*first Eulerian dynamical axiom,* alias *law* or principle of momentum of rigid body). There exists such a rigid system of reference S that, all derivatives being taken with respect to S, for any rigid body B and for any system of forces $\underrightarrow{F}$ acting on B, the derivative with respect to the time of the momentum of B equals the basis of $\underrightarrow{F}$.

Df 1 E. Any system of reference satisfying Ax 1 E is called *inertial according to Euler.*

Ax 2 E (*second Eulerian dynamical axiom,* alias *law* or *principle of moment of momentum* (*kinetical moment*) *of rigid body*). If S is an inertial according to Euler system of reference and all derivatives are taken with respect to S, then for any rigid body B and for any system of forces $\underrightarrow{F}$ acting on B, the derivative with respect to the time of the moment of momentum of B equals the moment of $\underrightarrow{F}$, both moments being taken with respect to the origin of S.

Sch 6. These formulations of the Newtonian and Eulerian dynamical axioms may be found nowhere in the current literature on analytical dynamics of mass-points and rigid bodies. Instead, amorphous redactions of imitations of Ax 1 N and possibly of Ax 1 E are proposed to the reader, the role of the inertial systems of reference according to Newton, as well as to Euler, being as a rule completely economized if not surpressed. As regards Ax 2 N and Ax 2 E, in the traditional literature on analytical dynamics these dynamical suppositions or hypotheses are taken down from their logical pedestal of dynamical axioms to the unenviable level of theorems, being labelled "the theorem of kinetical moment" of mass-points and of rigid bodies respectively. Moreover, even Ax 1 E is called, by some authors at least, "the theorem of momentum" of rigid bodies, with the claim that it is derivable from Ax 1 N. This is a most unpardonable logical error rooted in a deep ignorance of the real state of affairs in analytical mechanics, at least as far as its logical foundations are concerned. It is a topic we shall discuss at length subsequently.

Sch 7. For the time being we confine ourselves to the most categorical declaration that Ax 1 N, Ax 2 N, as well as Ax 1 E, Ax 2 E, are unprovable mathematical statements, as unprovable at least, as for instance Euclid's fifth postulate or Pascal's principle of mathematical induction are.

Sch 8. One of the aspects of these realities lies in the fact that the Newtonian and the Eulerian axioms involve mechanical terms which are unsusceptible to explicit mathematical definitions. Since this mathematical phenomenon is one of the most important, let us sumbit it to a closer analysis.

The meaning of both Newtonian and Eulerian dynamical axioms is out of reach unless and until the meaning of any term these verbal propositions involve is made clear. These terms are: *system of reference, rigid* system of reference, *derivative* of a vector function with respect to a system of reference, *mass-point* and *rigid body, momentum* and *moment of momentum* (*kinetical moment*) of a mass-point and a rigid body, *system of forces, basis* and *moment* of a system of forces with respect to a given point (pole), *origin* of a system of reference, *time,* and *acting* (a system of forces is "acting" on a mass-point and a rigid body). If all these terms were susceptible to explicit mathematical definitions, then the Newtonian and Eulerian dynamical axioms would turn out to be (true or false) mathematical theorems.

And if not?

The answer of this question, as regards more elementary mathematical theories than analytical mechanics (as, for instance, arithmetic and Euclidean geometry), was known to nobody until the end of the last century[1]. According to the proclaimed in 1899 Hilbert's *axiomatical principle*, a system of axioms for a mathematical theory must unconditionally include a certain number of void of explicit definitions terms of this theory. Now all the terms, numbered above and involved in Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E, are susceptible to strict explicit mathematical definitions with the only exception of the last two ones, namely *time* and *acting.* Those are primary notions of the mathematical theory called analytical mechanics, and they are defined implicitly namely by the aid of Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E (along with other mechanical axioms which will not be formulated manifestly here for the time being). At that, the term time is a *primary notion-object*[2] and the term acting is a *primary notion-relation*[3].

Sch 9. Ax 1 N and Ax 1 E are *existance statements.* Any of them asserts that there exists one at least system of reference with certain properties, and Df 1 N, Df 1 E give special appelations of these kinds of systems of reference.

There is a definite lack of information in Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E, however. Indeed, if a particular system of reference S is given, then neither Ax 1 N, Ax 2 N nor Ax 1 E, Ax 2 E give any possibility to decide whether S is inertial according to Newton or according to Euler respectively. This is a question that will be discussed in detail below.

Sch 10. There is also a lack of distinctness about the relation between inertialities according to Newton and according to Euler. In other words, on the basis of Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E only, one cannot answer the question whether there exists one at least inertial according to Newton system of reference which is, or is not, inertial according to Euler too. Alias, Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E are tolerant to any of these alternatives.

Sch 11. A mere glance at the Newtonian and Eulerian dynamical axioms displays at once that the notions of *mass-point* and of *rigid body* play a central role among all other notions these axioms involve. Their role is comparable to that the notion of integral plays in mathematical analysis. As well as it is impossible to build-up a logically irreproachable mathematical analysis without a strict mathematical definition of the term *intégral*, it is not lesser impossible to construct a logically unimpeachable analytical mechanics without strict mathematical definitions of the terms *mass-point* and *rigid body*[4,5].

Sch 12. A last general remark concerning the Newtonian and Eulerian dynamical axioms affects the striking similitude, the complete analogy between Ax 1 N, Ax 2 N, on the one hand, and Ax 1 E, Ax 2 E respectively, on the other hand. No great perspicacity is needed, indeed, to see that it is quite sufficient to substitude the term *rigid body* for the term *mass-point* in Ax 1 N, Ax 2 N in order to obtain automatically Ax 1 E, Ax 2 E respectively.

This formal resemblance between the Newtonian and the Eulerian dynamical axioms quite naturally brings forward the question: was there as much *marifet* needed, factually, as to ascribe Euler's name to the laws of momentum and of moment of momentum of rigid bodies? After all, once disposing with Ax 1 N and Ax 2 N, is it not a trivial whim to substitute in them the word *mass-point* by *rigid body* in order to obtain Ax 1 E and Ax 2 E respectively? Is there a great merit in such a procedure in order to perpetuate someone's name?[6]

Such an attitude toward the Eulerian dynamical axioms may be evinced only by someone who is entirely ignorant of the very essence of rigid body dynamics. *Cum grano salis,* to support such an outlook is as unwise as to uphold that a woman may be created out of a man by a mere substitution of the pronoun *she* for *he*.

Sch 13. First of all, neither Newton nor Euler have had the slightest idea of the above formulations of their dynamical laws[7]. Those are formulations that only modern mathematics could propose: it is hardly accidental that, as already underlined, they are nowhere to be seen even in the current mechanical literature. Euler did not dispose of Ax 1 N and Ax 2 N in order to substitute in them *rigid body* for *mass-point* and to obtain, in such a parrot way, Ax 1 E and Ax 2 E respectively.

Sch 14. The most that Euler took from Newton's *Principia* was Lex II. Besides, Euler did never have the slightest idea that Ax 1 E and Ax 2 E are beyond proof. A son of his epoch, he believed he had proved them. He never understood that his "demonstrationes" are, at the best, only plausible inferences. The nature of his reasonings is bordering on physical intuition. In his arguments, let us emphasize that once more, Newton's Lex II has played a most essential role.

All these circumstances being *bien entendu,* let us at last give a formal mathematical redaction of the Newtonian and Eulerian dynamical axioms, together with some immediate corollaries.

V denoting the *real standard vector space*[8], a *mass-point* P is defined as an ordered pair (r, m) of a vector function[9]

$$\boldsymbol{r}: R \longrightarrow V \tag{1}$$

(the *radius-vector* of P) and a scalar function

$$m: R \longrightarrow R \qquad (0 < m(t),\ t \in R) \tag{2}$$

(the *mass* of P). Under these notations it is written $P(\boldsymbol{r}, m)$.

Let $Oxyz$ be a *right-hand orientated orthonormal Cartesian system of reference*[10] and let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be the *unit vectors* of the axes Ox, Oy, Oz respectively. If $P(\boldsymbol{r}, m)$ is a mass-point, its radius-vector (1) being taken with respect to O, then the derivative

$$\boldsymbol{v} \text{ sgn}: \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} \tag{3}$$

of $\boldsymbol{r}$ with respect[11] to $Oxyz$ is called the *velocity* of P with respect to $Oxyz$.

The quantity

$$\boldsymbol{k} \text{ sgn}: m\boldsymbol{v} \tag{4}$$

is called the *momentum* of P with respect to $Oxyz$, and the quantity

$$\boldsymbol{l} \text{ sgn}: \boldsymbol{r} \times m\boldsymbol{v} \tag{5}$$

is called the *moment of momentum* (*kinetical moment*) of P with respect to $Oxyz$.

Let now P be under the action of the system

$$\underrightarrow{F} \text{ sgn}: \{\overrightarrow{F}_\nu\}_{\nu=1}^{n} \tag{6}$$

of forces

$$\overrightarrow{F}_\nu \text{ sgn}: (\boldsymbol{F}_\nu, \boldsymbol{M}_\nu) \qquad (\nu = 1, \ldots, n), \tag{7}$$

where by definition

$$\boldsymbol{F}_\nu = \boldsymbol{M}_\nu = \boldsymbol{O} \qquad (1 \leqq \nu \leqq n) \tag{8}$$

or otherwise

$$F_\nu \neq O, \; F_\nu M_\nu = O, \qquad (1 \leqq \nu \leqq n), \tag{9}$$

all moments M_ν ($\nu = 1, \dots, n$) being taken with respact to O.

For the sake of brevity let

$$F \text{ sgn} : \sum_{\nu=1}^{n} F_\nu, \qquad M \text{ sgn} : \sum_{\nu=1}^{n} M_\nu \tag{10}$$

be the basis and the moment (with respect to O) of (6) respectively.

If the system of reference $Oxyz$ is, by hypothesis, inertial according to Newton, then the mathematical expressions of Newton's dynamical axioms Ax 1 N and Ax 2 N are

$$\frac{d}{dt}(mv) = F \tag{11}$$

and

$$\frac{d}{dt}(r \times mv) = M \tag{12}$$

respectively, the derivatives in the left-hand sides of (11) and (12) being taken with respect to $Oxyz$.

Using the abbreviated notations (4), (5), instead of (11) and (12) one can write

$$\dot{k} - F = O \tag{13}$$

and

$$\dot{l} - M = O \tag{14}$$

respectively.

While the notion of *mass-point* is a most simple one, the notion of *rigid body* is, on the contrary, a most complicated one. We are devoid of the opportunity of entering here in any details in this connection and we are compelled to refer the reader to the articles [44, 87 – 89]. Still some explications are impreventable with a view to better comprehension of the exposition.

Let B be a rigid body and P be any of its points. If $r = \overline{OP}$, then (3) defines the *velocity* of P with respect to $Oxyz$. The definition of B requires the prescription of a function

$$\varkappa : V \longrightarrow [0, \infty) \tag{15}$$

(*density* of B at r). If $d\mu$ denotes an element of arc, area, or volume of B, according to the dimensions of B (1-*dimensional*, 2-*dimensional*, or 3-*dimensional* rigid body respectively[12]), then the differential

$$dm = \varkappa(\boldsymbol{r})d\mu \tag{16}$$

is called the *element of mass* (*mass-element*, *elementary mass*) of B at $\boldsymbol{r}$.

Extremely important for rigid body dynamics are the following kinetical quantities:

$$m \text{ sgn} : \int dm \tag{17}$$

(mass of B),

$$\boldsymbol{r}_G \text{ sgn} : \frac{1}{m}\int \boldsymbol{r}dm \tag{18}$$

(radius-vector with respect to O of the mass-center G of B),

$$\boldsymbol{K} \text{ sgn} : \int \boldsymbol{v}dm \tag{19}$$

(momentum of B with respect to $Oxyz$), and

$$\boldsymbol{L} \text{ sgn} : \int \boldsymbol{r} \times \boldsymbol{v}dm \tag{20}$$

(moment of momentum, alias kinetical moment, of B with respect to $Oxyz$).

It is seen that while the mass of a rigid body is invariant with respect to the chosen system of reference, all the other three quantities (18) – (20) are not. At that, all integrals in the right-hand sides of (17) – (20) are taken over the occupied by the rigid body[13] space.

If the rigid body B is under the action of the system (6) of forces (7) with (8), (9) and the notations (10) are accepted, and if the system of reference $Oxyz$ is, by hypothesis, inertial according to Euler, then the mathematical expressions of Euler's dynamical axioms Ax 1 E and Ax 2 E are

$$\frac{d}{dt}\int \boldsymbol{v}dm = \boldsymbol{F} \tag{21}$$

and

$$\frac{d}{dt}\int_{\cdot} \boldsymbol{r} \times \boldsymbol{v} dm = \dot{\boldsymbol{M}} \tag{22}$$

respectively, the derivatives in the left-hand sides of (21) and (22) being taken with respect to $Oxyz$.

Using the abbreviated notations (19), (20), instead of (21) and (22) one can write

$$\dot{\boldsymbol{K}} - \boldsymbol{F} = \boldsymbol{O} \tag{23}$$

and

$$\dot{\boldsymbol{L}} - \boldsymbol{M} = \boldsymbol{O} \tag{24}$$

respectively.

Sch 15. The complete analogy between (23), (24), on the one hand, and (13), (14) respectively, on the other hand, is obvious. This similitude is only formal though, as a mere glance at the definitions (4), (5), on the one hand, and (19), (20) respectively, on the other hand, at once displays. This juxtaposition throws a new light on the raised in Sch 12 — Sch 14 question concerning Euler's attributions in rigid body dynamics, which are now seen in its true colours.

The difference between the momenta and the moments of momenta of mass points, on the one hand, and of rigid bodies, on the other hand, are tremendous indeed. One should not be deceived by the fact that such radically different mathematical objects as (4) and (19), as well as (5) and (20), are called with the same names. If, as underlined, the analogy between (13) and (23), as well as between (14) and (24), is perfect, yet one is at a loss to see the nuclei, or the germs, or the embrios of $\boldsymbol{K}$ and $\boldsymbol{L}$ in $\boldsymbol{k}$ and $\boldsymbol{l}$ respectively, to say nothing about some similarity whatever. The blind Euler managed to see $\boldsymbol{K}$ and $\boldsymbol{L}$ through $\boldsymbol{k}$ and $\boldsymbol{l}$ respectively, as well as (21) and (22) through (11) and (12). How did he succeed in doing so?

The only answer we can give is: Frankly, we don't know.

We have our suspicions, though. They will be exposed somewhat later.

The following three propositions are almost obvious.

Pr 1 N. (11), (12)

$$\boldsymbol{F} = \boldsymbol{O} \tag{25}$$

imply

$$\boldsymbol{M} = \boldsymbol{O}. \tag{26}$$

Dm. (11) implies

$$\boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F}. \tag{27}$$

On the other hand, obviously

$$\boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}). \tag{28}$$

Now (27), (28) imply

$$\frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F} \tag{29}$$

and (12), (29), (25) imply (26).

Pr 2 N. (11), (12) imply

$$\boldsymbol{F}\boldsymbol{M} = 0. \tag{30}$$

Dm. (12), (29) imply

$$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M} \tag{31}$$

and (31) implies (30).

Pr 3 N. (11), (12) imply

$$\boldsymbol{r}\boldsymbol{M} = 0. \tag{32}$$

Dm. (31).

Pr 4 N. If P is a mass-point and $\underrightarrow{F}$ is a system of forces acting on P, then

$$\operatorname{rank} \underrightarrow{F} \neq 1 \tag{33}$$

and

$$\operatorname{rank} \underrightarrow{F} \neq 3. \tag{34}$$

Dm. If a system of forces $\underrightarrow{F}$ is given, then it gives rise to a mapping

$$\mu : V \longrightarrow V \tag{35}$$

defined by

$$\mu(\boldsymbol{r}) = \operatorname{mom}_{\boldsymbol{r}} \underrightarrow{F}, \tag{36}$$

where by definition

$$\text{(37)} \qquad \operatorname{mom}_{\boldsymbol{r}} \underset{\longrightarrow}{F} \qquad \operatorname{sgn}: \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}$$

is the *moment* of $\underset{\longrightarrow}{F}$ *with respect to* $\boldsymbol{r}$ (the $\boldsymbol{r}$*-moment of* $\underset{\longrightarrow}{F}$), $\boldsymbol{F}$ and $\boldsymbol{M}$ being, as until now, the basis and the moment (with respect to $\boldsymbol{o}$) of $\underset{\longrightarrow}{F}$ respectively. The mapping (35), defined by (36), is called the *momental field* of $\underset{\longrightarrow}{F}$. Now the *rank* of $\underset{\longrightarrow}{F}$ (symbolically — rank $\underset{\longrightarrow}{F}$) is defined as the maximal number of the linearly independent elements of the image $\mu(V)$ of V. According to the *rank-theorem* [90].

$$\text{(38)} \qquad \operatorname{rank} \underset{\longrightarrow}{F} = \begin{cases} 0 & \text{iff } \boldsymbol{F} = \boldsymbol{O},\ \boldsymbol{M} = \boldsymbol{O}, \\ 1 & \text{iff } \boldsymbol{F} = \boldsymbol{O},\ \boldsymbol{M} \neq \boldsymbol{O}, \\ 2 & \text{iff } \boldsymbol{F} \neq \boldsymbol{O},\ \boldsymbol{F}\boldsymbol{M} \neq \boldsymbol{O}, \\ 3 & \text{iff } \boldsymbol{F}\boldsymbol{M} = 0. \end{cases}$$

Now (38), Pr 1 N, Pr 2 N imply (33), (34).

Sch 16. Pr 4 N is very instructive. It manifests prohibitions in mass-point dynamics. According to it, a mass-point P and a system of forces $\underset{\longrightarrow}{F}$ acting on P being given, then necessarily rank $\underset{\longrightarrow}{F} = 0$ or rank $\underset{\longrightarrow}{F} = 2$.

Pr 5 N. P being a mass-point under the action of the system of forces $\underset{\longrightarrow}{F}$, the latter is equivalent to the zero-force or to a single non-zero force with a directrix passing through P.

Dm. Pr 4 N, [90] Pr 2, (31).

Sch 17. A direct, though not purely mathematical, corollary from Pr 4 N consists in the conclusion that *the rigid body dynamics cannot be derived from, or be reduced to, the mass-point one.* Indeed, as particular problems of rigid body dynamics display at once, the systems of acting on rigid bodies forces may be quite arbitrary; in particular, their ranks may equal 1 or 3. In other words, the systems of forces that are warrantable to competition as regards their actions on mass-points form an inessential part of the set of all systems of admissible to actions on rigid bodies forces. *Quod erat demonstrandum.*

Let us analyse the possibilities of the following alternative for Ax 2 N postulate.

Ax 2 N bis. If S is an inertial according to Newton system of reference, then for any mass-point $P(\boldsymbol{r}, m)$ and for any system of forces $\underset{\longrightarrow}{F}(\boldsymbol{F}, \boldsymbol{M})$ acting on P, the relation (31) holds, both $\boldsymbol{r}$ and $\boldsymbol{M}$ being taken with respect to the origin of S.

The following proposition is almost obvious.

Pr 6 N. The system of axioms Ax 1 N, Ax 2 N and Ax 1 N, Ax 2 N bis are equivalent.

Dm. Ax 1 N and Ax 2 N imply Ax 2 N bis (Pr 2 N). Inversely, Ax 1 N and Ax 2 N bis, alias (11) and (31), imply (12), i.e. Ax 2 N by virtue of (28).

Sch 18. It has been mentioned in App 7 that Newton "thought wrongly that" Ax 2 N "is an immediate corollary from Lex II", i.e. from Ax 1 N. Now Pr 6 N displays that this idea of Newton is a half-truth: Ax 2 N is a corollary from Ax 1 N and Ax 2 N bis *coniunctim* rather than from Ax 1 N alone. It is clear that on the logical background of that epoch Newton's error is easy to be explained.

Sch 19. It must not be left unobserved that Ax 2 N bis, in its simplest form at least, is a much more natural and intuitively clear proposition than Ax 2 N. Indeed, Ax 2 N speaks nothing to the physical experience. On the contrary, if the mass-point $P(\boldsymbol{r}, m)$ is acted on by a single non-zero force

$$\overrightarrow{F} = (\boldsymbol{F}, \boldsymbol{M}), \tag{39}$$

then Ax 2 N bis simply states that its directrix must unconditionally pass through P. This supposition is as natural as to seem obvious. No wonder the authors of text-books on analytical dynamics never bothered to formulate it explicitly.

Sch 20. The relation (31) is certainly satisfied in the case

$$\boldsymbol{r} \times \boldsymbol{F}_\nu = \boldsymbol{M}_\nu \qquad (\nu = 1, \dots, n), \tag{40}$$

as it is immediately seen by virtue of (10). Since any of the relations (40) represents (provided the radius-vector $\boldsymbol{r}$ of P is fluent) the equation of the directrix of the force $\overrightarrow{F}_\nu$ $(\nu = 1, \dots, n)$ respectively, these relations give expression to the requirement that any of these directrices must pass through the mass-point P. This is what all physicists bear in mind when asserting that there "would be but a single law of motion"[14]. Although *physically* quite natural, *mathematically* this requirement follows from no other hypothesis of mass-point dynamics and is, consequently, a new dynamical supposition.

Being unprovable, it may be raised to the rank of a new dynamical axiom, namely:

Ax 2 N bis bis. If a system of forces is acting on a mass-point P, then P lies on the directrix of any of these forces.

Sch 21. Naturally, it is possible to build a mass-point dynamics founded on Ax 1 N and Ax 2 N bis bis instead of Ax 1 N and Ax 2 N. The range of action of this hypothetical dynamics is, however, considerably narrower than the Newtonian one. It is true that Ax 1 N and Ax 2 N bis bis imply Ax 2 N, but the inverse is not true: Ax 1 N and Ax 2 N may be satisfied while Ax 2 N bis bis may not.

Sch 22. In such a way we are faced with the alternative: on the one hand, to develop the Newtonian mass-point dynamics on the basis of Ax 1 N, Ax 2 N; on the other hand, to develop an entirely identical dynamics on the basis of Ax 1 N, Ax 2 N bis. Both ways are completely equal in rights, as regards the Newtonian mass-point dynamics solely. If, however, the last is regarded together with the Eulerian rigid body dynamics, then the first way is preferable from an aesthetic point of view at least. Indeed, as it has been mentioned above, there is a complete parallelism between Ax 1 N, Ax 2 N, on the one hand, and Ax 1 E, Ax 2E, on the other hand. This parallelism, however, vanishes into thin air, if one chooses Ax 2 N bis instead of Ax 2 N, since Ax 2 E bis analogous to Ax 2 N bis simply does not exist: in the Eulerian rigid body dynamics there is no true proposition (axiom or theorem) similar to Ax 2 N bis.

In other words, in rigid body dynamical the analogue of Ax 2 N bis, if any, is simply and purely false: whereas (31) implies (30), in rigid body dynamics, as underlined in Sch 17, there is no obligatory relation connecting the basis and the moment of a system of forces acting on a rigid body — both these quantities may be absolutely arbitrary in a particular dynamical problem concerning rigid bodies.

Sch 23. Let us now display how "proofs" of the Eulerian dynamical axioms Ax 1 E and Ax 2 E are fabricated by most authors of text-books, treatises and monographs on analytical dynamics. To this end let $P_\nu(\boldsymbol{r}_\nu, m_\nu)$ $(\nu = 1, \dots, n)$ be mass-points on which the systems of forces $F_\nu(\boldsymbol{F}_\nu, \boldsymbol{M}_\nu)$ $(\nu = 1, \dots, n)$ respectively are acting, the moments $\overrightarrow{\boldsymbol{M}}_\nu$ $(\nu = 1, \dots, n)$ being taken with respect to O. Then Ax 1 N and Ax 2 N imply

(41) $$\frac{d}{dt}(m_\nu \boldsymbol{v}_\nu) = \boldsymbol{F}_\nu \qquad (\nu = 1, \dots, n)$$

and

(42) $$\frac{d}{dt}(\boldsymbol{r}_\nu \times m_\nu \boldsymbol{v}_\nu) = \boldsymbol{M}_\nu \qquad (\nu = 1, \dots, n)$$

respectively, provided $\boldsymbol{v}_\nu = \dot{\boldsymbol{r}}_\nu$ $(\nu = 1, \dots, n)$, the derivatives being taken with respect to the inertial according to Newton system of reference $Oxyz$. Adding (41) and (42) together and adopting the notations (10) one obtains

(43) $$\frac{d}{dt}\sum_{\nu=1}^{n} m_\nu \boldsymbol{v}_\nu = \boldsymbol{F}$$

and

(44) $$\frac{d}{dt}\sum_{\nu=1}^{n} \boldsymbol{r}_\nu \times m_\nu \boldsymbol{v}_\nu = \boldsymbol{M}$$

respectively. The quantities

(45) $$\boldsymbol{k}_n \ \text{sgn} : \sum_{\nu=1}^{n} m_\nu \boldsymbol{v}_\nu$$

and

$$(46)\qquad \boldsymbol{l}_n \ \mathrm{sgn} : \sum_{\nu=1}^{n} \boldsymbol{r}_\nu \times m_\nu \boldsymbol{v}_\nu$$

are by definition the momentum and the moment of momentum (kinetical moment) respectively of the system S_n of mass-points P_ν ($\nu = 1, \dots , n$). Now (43) — (46) imply

$$(47)\qquad \dot{\boldsymbol{k}}_n - \boldsymbol{F} = \boldsymbol{O}$$

and

$$(48)\qquad \dot{\boldsymbol{l}}_n - \boldsymbol{M} = \boldsymbol{O}$$

respectively.[15]

The formal analogy between the laws (47), (48), on the one hand, and (23), (24) respectively, on the other hand, is obvious. Not so obvious is the similitude between the relations (43), (44), on the one hand, and (21), (22) respectively, on the other hand. Some words in this connection are therefore not pointless.

Let us imagine, to this end, that a partition of a rigid body B is accomplished by the aid of three series of mutually perpendicular planes into a system of parallelepipeds, $n = m^3$ in number, in such a manner that all dimensions of any of them tend to zero with increasing m. Let p_ν denote the ν-th of these parallelepipeds, m_ν — its mass, and P_ν — any point inside of p_ν ($\nu = 1, \dots , n$).

If now one condescends to follow the logical process of those authors of text-books on analytical dynamics which are pretending to "prove" the Eulerian dynamical axioms Ax 1 E and Ax 2 E, then one could fancy that the rigid body B is "substituted" by a system of mass-points $P_\nu(\boldsymbol{r}_\nu, m_\nu)$, provided $\boldsymbol{r}_\nu = OP_\nu$ ($\nu = 1, \dots , n$). According to these authors, this peculiar imitation of B is as much more adequate as greater n is, with the tendency to transmute into a complete identity with the infinitely increasing n. According to this current of thoughts the discrete $\boldsymbol{r}_\nu$, $\boldsymbol{v}_\nu$, and m_ν ($\nu = 1, \dots , n$) in the right-hand sides of (45) and (46) are transformed into the general, deprived of individuality, $\boldsymbol{r}$, $\boldsymbol{v}$, and dm respectively in the right-hand sides of (19) and (20) respectively. According to the same ideas, the dynamical equations (43) and (44) are transmuted into the dynamical equations (21) and (22) respectively: the Eulerian dynamical axioms are proved!

There are, however, two at least points in connection with this *Hokus Pokus* that badly need explications.

First of all, let us remind the way the relations (43) and (44) have been obtained. We arrived at them by adding together the equations (41) and (42) respectively, in other words the latter are inescapable for our gains. If one does not dispose of (41) and (42) in any particular case, then one simply and purely has no right, both logical and ethical, to appeal to (43) at all. Now in the above reasonings one is entirely denuded of the possibility to write down (41) and (42), since there is no information about the forces in the right hand sides of these equations. In other words, one knows *nothing*, but *nothing indeed* (*nichts, rien de rien, nada, niente, ничто*) about both the bases $\boldsymbol{F}_\nu$ and the moments $\boldsymbol{M}_\nu$ of those hypothetical forces (7) which are acting (in the heads of eminent authors at least) on the mass-points P_ν of p_ν $(\nu = 1, \dots, n)$. All that is known is the system of forces acting on the rigid body B itself. As far as the problem is concerned how are these latter forses decomposed (mentally at least, if not actually, alias physically) or distributed in order to act on those mass-points which, in their *Mannigfaltigkeit*, are intended to replace B in the above reasonings — its only answer is *ignoramus et ignorabimus.*

In such a manner, the whole mental procedure, described above, remains hanging in the air. It is rotten through and through. *In my end is my begining*, the proverb says. As regards the efforts to prove the Eulerian dynamical axioms, their end is in their beginning. Reasonings of the sort just exposed should not be written in black and white. The addle eggs must be cast out of the nest.

As regards the passage from m_ν $(\nu = 1, \dots, n)$ to dm, mentioned above, the things stand topsyturvy.

Physically the notion of *density* of a rigid body B at any of its points P is defined as follows. Let

$$\Delta_1,\ \Delta_2, \dots, \Delta_n, \dots \tag{49}$$

be an infinite sequence of parts of B, any of which involves P and is deposited in the preceding one, and let their dimensions in every direction tend to zero with increasing n. Let $\boldsymbol{v}_\nu$ and m_ν denote the volume and the mass of Δ_ν $(\nu = 1, 2, \dots, n)$ respectively. Then the fraction

$$\varkappa_\nu = \frac{m_\nu}{v_\nu} \qquad (\nu = 1, 2, \dots) \tag{50}$$

is called the *mean density* of Δ_ν. Its limit

$$\varkappa = \lim_{\nu \to \infty} \varkappa_\nu \tag{51}$$

is called the *density* of B *at* P. (The physicists take for gospel truth the hypothesis that $\varkappa$ is a function of P only, *independently* of the particular sequence (49) by means of which it is defined.)

Mathematically, however, the whole procedure is unrecognizably reversed. The *density* of a rigid body B at any of its points P is a beforehand given function (15) of the radius-vector $\boldsymbol{r}$ of P that takes part in the very definition of B. (Naturally, certain additional conditions about this function must be hypothesized, in the first place its integrability in a certain sense, in view of the definition (16) of the elementary mass dm and its participation in all the integrals (17) — (20), etc.) This function being known, the mass of the part Δ_ν of B is defined by the integral

$$m_\nu := \int_{\Delta_\nu} dm = \int_{\Delta_\nu} \varkappa(\boldsymbol{r}) d\mu \tag{52}$$

according to (17), taken over Δ_ν ($\nu = 1, 2, \ldots$), the meaning of $d\mu$ being explained above.

This analysis displays once more the complete bankruptcy of the mechanical texts pretending to give "proofs" of the Eulerian dynamical laws of momentum and of kinetical moment.

Sch 24. And yet, this approach has been broadly used in mechanics in times gone by. So, for instance, the *mass* of a system S_n of masspoints $P_\nu(\boldsymbol{r}_\nu, m_\nu)$ ($\nu = 1, \ldots, n$) is defined by

$$m \text{ sgn} : \sum_{\nu=1}^{n} m_\nu; \tag{53}$$

the radius-vector $\boldsymbol{r}_G$ of its *mass-center* G by

$$\boldsymbol{r}_G \text{ sgn} : \frac{1}{m} \sum_{\nu=1}^{n} m_\nu \boldsymbol{r}_\nu; \tag{54}$$

and its *kinetic energy* by

$$T \text{ sgn} : \frac{1}{2} \sum_{\nu=1}^{n} m_\nu v_\nu^2, \tag{55}$$

provided $\boldsymbol{v}_\nu = \dot{\boldsymbol{r}}_\nu$ ($\nu = 1, \ldots, n$). Completely similar to (53) — (55), the *mass* and the *mass-center* of a rigid body B are, as already mentioned, defined by (17) and (18) respectively, and its *kinetic energy* by

$$T \text{ sgn} : \frac{1}{2} \int v^2 dm. \tag{56}$$

Now, not a word could be said justly against the definitions (17), (18), and (56), no matter that, treating the continual case, they are suggested by the definitions (53) – (55) respectively, treating the discrete one. The role of (53) — (55) for the formulation of the *definitions* (17), (18), and (56) respectively, however, is purely suggestive, inductive, heuristic, by no means logical. As regards the theorems, none of them can be proved for continua on the basis of facts known in the discrete case only, i.e. by analogy.

Sch 25. The question, raised at the end of Sch 15, namely — how did Euler arrive at the idea of his dynamical laws, may now be concretisized: did he use the logical process described in Sch 23, — or in other words, did he attain to Ax 1 E and Ax 2 E through (43) and (44) respectively? Although it is rather tempting to answer this last question in the affirmative, the justice requires some cautiousness. It is true that the equations (43), (44) are Euler's discoveries. It is also true that the heuristic approaches, described in Sch 24, have been widely used in Euler's days for definitional goals. At that, the associative abilities of Euler's mind were proverbial; as regards the formal analogies, he was a universally recognized master (let us remind the established by him relation between the exponential and the trigonometrical functions, or his summations of divergent series — to cite a few examples out of a legion). At last, it is true that in the epoch immediately foregoing the French revolution the mental picture of a rigid body as composed of a large number of mass-points has been most popular. And yet, the mathematical creation is a phenomenon that belongs to psychology rather than to mathematics itself. Let us remind that someone had said: he, who states categorically something that lies outside pure mathematics, is at least imprudent.

It has been underlined in Sch 9 that, S being a particular rigid system of reference, neither Ax 1 N, Ax 2 N, nor Ax 1 E, Ax 2 E, give any possibility to decide whether S is inertial or not according to Newton or to Euler respectively. We are now in a position to subject this problem to a detailed analysis.

The following propositions play an auxiliary role in solving this problem.

Pr 7. If α and β are rigid systems of reference, the function

$$\boldsymbol{p} : R \longrightarrow V \tag{57}$$

is differentiable, and $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then

$$\frac{d_\alpha \boldsymbol{p}}{dt} = \frac{d_\beta \boldsymbol{p}}{dt} + \overline{\omega}_{\alpha\beta} \times \boldsymbol{p} \qquad (t \in R). \tag{58}$$

Dm. Let by definition

$$\beta \text{ sgn} : \{\overrightarrow{b}_\nu\}_{\nu=1}^{3}, \tag{59}$$

where

$$\overrightarrow{b}_\nu \text{ sgn} : (\boldsymbol{b}_\nu, \boldsymbol{B}_\nu) \qquad (\nu = 1, 2, 3), \tag{60}$$

$$\boldsymbol{b}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3), \tag{61}$$

$$\boldsymbol{B}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3), \tag{62}$$

$$b_1(t) \times b_2(t) \cdot b_3(t) \neq 0 \qquad (t \in R), \tag{63}$$

$$b_\mu(t) B_\nu(t) + b_\nu(t) B_\mu(t) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ t \in R), \tag{64}$$

$$\frac{d}{dt}(b_\mu(t) b_\nu(t)) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ t \in R) \tag{65}$$

(Sch 10). Then $\overline{\omega}_{\alpha\beta}$ is defined as the only solution of the system of vector equations

$$\frac{d_\alpha b_\nu}{dt} = \overline{\omega}_{\alpha\beta} \times b_\nu \qquad (\nu = 1,\ 2,\ 3;\ t \in R), \tag{66}$$

namely

$$\overline{\omega}_{\alpha\beta} = \frac{1}{2} \sum_{\nu=1}^{3} b_\nu^{-1} \times \frac{d_\alpha b_\nu}{dt} \qquad (t \in R) \tag{67}$$

[91], the reciprocal vectors b_ν^{-1} of b_ν ($\nu = 1,\ 2,\ 3$) being defined as in Sch 10. It is proved that $\overline{\omega}_{\alpha\beta}$ satisfies also the system

$$\frac{d_\alpha b_\nu^{-1}}{dt} = \overline{\omega}_{\alpha\beta} \times b_\nu^{-1} \qquad (\nu = 1,\ 2,\ 3;\ t \in R). \tag{68}$$

The definition App (17) implies

$$\frac{d_\beta p}{dt} = \sum_{\nu=1}^{3} \left(\frac{d}{dt}(p b_\nu^{-1}) \right) b_\nu \qquad (t \in R). \tag{69}$$

It is proved that if the functions

$$p_\nu :\ R \longrightarrow V \qquad (\nu = 1,\ 2) \tag{70}$$

are differentiable, then for any rigid system of reference α

$$\frac{d(p_1 p_2)}{dt} = \frac{d_\alpha p_1}{dt} p_2 + p_1 \frac{d_\alpha p_2}{dt} \qquad (t \in R). \tag{71}$$

The relation (71) implies

$$\frac{d}{dt}(p b_\nu^{-1}) = \frac{d_\alpha p}{dt} b_\nu^{-1} + p \frac{d_\alpha b_\nu^{-1}}{dt} \qquad (t \in R) \tag{72}$$

($\nu = 1, 2, 3$), and (72), (68) imply

$$\frac{d}{dt}(\boldsymbol{p}\boldsymbol{b}_\nu^{-1}) = \frac{d_\alpha \boldsymbol{p}}{dt}\boldsymbol{b}_\nu^{-1} + \boldsymbol{p} \times \overline{\omega}_{\alpha\beta} \cdot \overline{\boldsymbol{b}_\nu}^{-1} \qquad (t \in R) \tag{73}$$

($\nu = 1, 2, 3$). Now (69), (73) imply

$$\frac{d_\beta \boldsymbol{p}}{dt} = \sum_{\nu=1}^{3} \left(\frac{d_\alpha \boldsymbol{p}}{dt}\boldsymbol{b}_\nu^{-1}\right) \boldsymbol{b}_\nu + \sum_{\nu=1}^{3} (\boldsymbol{p} \times \overline{\omega}_{\alpha\beta} \cdot \boldsymbol{b}_\nu^{-1})\boldsymbol{b}_\nu \tag{74}$$

($t \in R$), and (74), App (16) with $\frac{d_\alpha \boldsymbol{p}}{dt}$ and $\boldsymbol{p} \times \overline{\omega}_{\alpha\beta}$ respectively instead of $\boldsymbol{p}$ imply

$$\frac{d_\beta \boldsymbol{p}}{dt} = \frac{d_\alpha \boldsymbol{p}}{dt} + \boldsymbol{p} \times \overline{\omega}_{\alpha\beta} \qquad (t \in R), \tag{75}$$

whence (58).

Pr 8. If α and β are rigid systems of reference and $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then

$$\frac{d_\alpha \boldsymbol{p}}{dt} = \frac{d_\beta \boldsymbol{p}}{dt} \qquad (t \in R) \tag{76}$$

for any differentiable function (57) if, and only if,

$$\overline{\omega}_{\alpha\beta} = \boldsymbol{o} \qquad (t \in R). \tag{77}$$

Dm. Pr 7.

Sch 26. The relation (58) is usually called the *connection between the local derivatives of a vector function with respect to two systems of reference.*

Sch 27. Let us note in passing that

$$\overline{\omega}_{\beta\alpha} = -\overline{\omega}_{\alpha\beta}. \tag{78}$$

Indeed, Pr 7 implies

$$\frac{d_\beta \boldsymbol{p}}{dt} = \frac{d_\alpha \boldsymbol{p}}{dt} + \overline{\omega}_{\beta\alpha} \times \boldsymbol{p} \qquad (t \in R). \tag{79}$$

Now (58) and (79) imply

$$(\overline{\omega}_{\alpha\beta} + \overline{\omega}_{\beta\alpha}) \times \boldsymbol{p} = \boldsymbol{o}. \tag{80}$$

Since (80) is satisfied for any differentiable function (57), it implies (78).

Pr 9. If α and β are rigid systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, P is any moving point and

$$\boldsymbol{r} = \boldsymbol{AP}, \qquad \boldsymbol{r}_B = \boldsymbol{AB}, \qquad \overline{\rho} = \boldsymbol{BP}, \tag{81}$$

then

$$\frac{d_\alpha \boldsymbol{r}}{dt} = \frac{d_\alpha \boldsymbol{r}_B}{dt} + \overline{\omega}_{\alpha\beta} \times \overline{\rho} + \frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R). \tag{82}$$

Dm. (81) and the obvious identity $\boldsymbol{AP} = \boldsymbol{AB} + \boldsymbol{BP}$ imply

$$\boldsymbol{r} = \boldsymbol{r}_B + \overline{\rho}, \tag{83}$$

whence (82).

Sch 28. If α is a rigid system of reference, n is a natural number, and the function (57) is $n+1$ times differentiable, then by definition

$$\frac{d_\alpha^{n+1} \boldsymbol{p}}{dt^{n+1}} \text{ sgn : } \frac{d_\alpha}{dt} \frac{d_\alpha^n \boldsymbol{p}}{dt}. \tag{84}$$

The left-hand side of (84) is called the $(n+1)$*th derivative* of $\boldsymbol{p}$ *with respect to* α or the *local* (*with respect to* α) $(n+1)$*th derivative* of $\boldsymbol{p}$.

Sch 29. If $\boldsymbol{r}$ denotes the radius-vector of a moving point with respect to the origin of the rigid system of reference α, then the first and the second local derivatives of $\boldsymbol{r}$ with respect to α are usually called respectively the *local velocity* and the *local acceleration* of P *with respect to* α. If there is no danger of collision of notations, they are traditionally denoted by $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively.

Pr 10. If α and β are rigid systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α,

$$\overline{\varepsilon}_{\alpha\beta} \text{ sgn : } \frac{d_\alpha \overline{\omega}_{\alpha\beta}}{dt} \tag{85}$$

is by definition the *instantaneous angular acceleration* of β with respect to α, P is a mass-point with two times differentiable mass

$$m : R \longrightarrow R \tag{86}$$

and (81) hold, then

$$(87) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times m\overline{\rho})$$

$$+\overline{\omega}_{\alpha\beta} \times \left(m\frac{d_\beta \overline{\rho}}{dt} + \frac{d_\beta (m\overline{\rho})}{dt}\right) + \frac{d_\beta}{dt}\left(m\frac{d_\beta \overline{\rho}}{dt}\right) \qquad (t \in R).$$

Dm. (82) implies

$$(88) \qquad m\frac{d_\alpha \boldsymbol{r}}{dt} = m\frac{d_\alpha \boldsymbol{r}_B}{dt} + \overline{\omega}_{\alpha\beta} \times m\overline{\rho} + m\frac{d_\beta \overline{\rho}}{dt}$$

$(t \in R)$, whence

$$(89) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho}$$

$$+\overline{\omega}_{\alpha\beta} \times \frac{d_\alpha (m\overline{\rho})}{dt} + \frac{d_\alpha}{dt}\left(m\frac{d_\beta \overline{\rho}}{dt}\right) \qquad (t \in R)$$

in view of (85). On the other hand, Pr 7 implies

$$(90) \qquad \frac{d_\alpha (m\overline{\rho})}{dt} = \frac{d_\beta (m\overline{\rho})}{dt} + \overline{\omega}_{\alpha\beta} \times m\overline{\rho} \qquad (t \in R),$$

$$(91) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\beta \overline{\rho}}{dt}\right) = \frac{d_\beta}{dt}\left(m\frac{d_\beta \overline{\rho}}{dt}\right) + \overline{\omega}_{\alpha\beta} \times m\frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R).$$

Now (89) — (91) imply (87).

Sch 30. Now we are capable of proving some important propositions shedding some light on the problem of inertiality according to Newton and Euler of rigid systems of reference. With a view to a better comprehension of these propositions they are somewhat dismembered.

Pr 11 N. If α and β are inertial according to Newton systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α and $\boldsymbol{r}_B = \boldsymbol{AB}$, then

$$(92) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) = \boldsymbol{o} \qquad (t \in R)$$

for any function (86) and

$$(93) \qquad \overline{\omega}_{\alpha\beta} = \boldsymbol{o} \qquad (t \in R).$$

Dm. α and β being inertial according to Newton by hypothesis, Df 1 N and Ax 1 N imply

$$\frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \boldsymbol{F} \qquad (t \in R), \tag{94}$$

$$\frac{d_\beta}{dt}\left(m\frac{d_\beta \boldsymbol{r}}{dt}\right) = \boldsymbol{F} \qquad (t \in R), \tag{95}$$

provided (81), m denoting the mass of any mass-point P and $\boldsymbol{F}$ — the basis of any system of forces acting on P. Then (94), (95), and (87) imply

$$\frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times m\overline{\rho}) + \overline{\omega}_{\alpha\beta} \times \left(m\frac{d_\beta\overline{\rho}}{dt} + \frac{d_\beta(m\overline{\rho})}{dt}\right) \qquad (t \in R). \tag{96}$$

Since Ax 1 N holds for any mass-point and for any system of forces acting on it, the corollary (96) from (94), (95) holds for any m and $\overline{\rho}$. Let us first choose

$$\overline{\rho} = \boldsymbol{o} \qquad (t \in R). \tag{97}$$

Then (96) implies (92), and (96), (92) imply

$$\overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times m\overline{\rho}) + \overline{\omega}_{\alpha\beta} \times \left(m\frac{d_\beta\overline{\rho}}{dt} + \frac{d_\beta(m\overline{\rho})}{dt}\right) = \boldsymbol{o} \tag{98}$$

$(t \in R)$. Thereupon let us choose

$$m = 1 \qquad (t \in R), \tag{99}$$

$$\frac{d_\beta\overline{\rho}}{dt} = \boldsymbol{o} \qquad (t \in R). \tag{100}$$

Then (98) — (100) imply

$$\overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho}) = \boldsymbol{o} \qquad (t \in R). \tag{101}$$

Scalar multiplication of (101) with $\overline{\rho}$ implies

(102) $$(\overline{\omega}_{\alpha\beta} \times \overline{\rho})^2 = 0 \qquad (t \in R),$$

whence

(103) $$\overline{\omega}_{\alpha\beta} \times \overline{\rho} = \boldsymbol{o} \qquad (t \in R).$$

In particular , if one puts in (103) successively $\overline{\rho} = \boldsymbol{b}_1$ and $\overline{\rho} = \boldsymbol{b}_2$, then one obtains the system of vector equations

(104) $$\overline{\omega}_{\alpha\beta} \times \boldsymbol{b}_\nu = \boldsymbol{o} \qquad (\nu = 1,\ 2;\ t \in R).$$

Since by hypothesis $\boldsymbol{b}_1 \times \boldsymbol{b}_2 \neq \boldsymbol{o}$ according to (63), the system (104) has exactly one solution, namely (93).

Pr 12 N. If α is an inertial according to Newton system of reference and β is a rigid system of reference, with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then (92) for any function (86) and (93) imply that β is inertial according to Newton too.

Dm. α being inertial according to Newton by hypothesis, Df 1 N and Ax 1 N imply (94) provided (81), m denoting the mass of any mass-point P and $\boldsymbol{F}$ — the basis of any system of forces acting on P. Then (87), (92) — (94) imply (95), i.e. β is inertial according to Newton (Df 1 N, Ax 1 N).

Pr 13 N. If α and β are rigid systems of reference with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then necessary and sufficient conditions in order that α and β are simultaneously inertial according to Newton are (92) for any function (86) and (93).

Dm. Pr 11 N, Pr 12 N.

Sch 31. Before proceeding further, let us make some remarks in connection with Pr 13 N.

In the formulation of the Newtonian dynamical axioms no hypotheses have been made concerning the mathematical nature of the masses of the mass-points. Following the Newtonian tradition, however, for a long period of time the classical mechanics has worked under the acceptance (not explicity formulated, it is true) that the masses are absolute constants, especially, that they are invariant with respect to the time. In any case, such has been the state of affairs in mass-point and rigid body dynamics until the end of the last century.

In 1904, however, the Russian nechanician Meshtcherski proposed the differential equation

(105) $$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} + \overline{\Phi}_1 + \overline{\Phi}_2 \qquad (t \in R)$$

for the motion of mass-points with variable masses. At that, by definition

$$\overline{\Phi}_\nu \operatorname{sgn}: \frac{dm_\nu}{dt} v_\nu \qquad (\nu = 1,\ 2) \tag{106}$$

are additional forces generated by the alterations of the masses, where $\frac{dm_1}{dt}$ is the rate of change of the outgo of m and $\frac{dm_2}{dt}$ — that of its income; v_1 is the relative velocity of the "particles separating from the mass-point" (according to Meshtcherski's mechanical ideas) and v_2 the relative velocity of the "particles added to the mass-point"; $\overline{\Phi}_1$ is called the "reactive traction" and $\overline{\Phi}_2$ — the "arresting force". It is supposed that the equation (105) is related to an inertial according to Newton system of reference. After Meshtcherski's work a new branch of analytical dynamics germinated: variable mass dynamics (although mainly mass-point problems have been discussed).

Sch 32. As it will be shown soon, the necessary and sufficient conditions, formulated in Pr 13 N, for the simultaneous inertiality according to Newton of two rigid systems of reference do not coincide in the cases of constant masses, on the one hand, and of variable masses, on the other hand. In view of the importance of this circumstance for the Newtonian mass-point dynamics in general, we shall subject it to a close analysis.

To this end we shall formulate two additional dynamical hypotheses which are mutually exclusive, i.e. inconsistent *coniunctim*. Afterwards we shall re-redact Pr 13 N separately for any of these cases.

Hpth NC. If m is the mass of any mass-point, then

$$\frac{dm}{dt} = 0 \qquad (t \in R). \tag{107}$$

Hpth NV. There exists one at least mass-point, the mass of which satisfies

$$\frac{dm}{dt} \neq 0 \qquad (t \in R). \tag{108}$$

Pr 14 NC. Under the conditions and notations of Pr 13 N, the supposition Hpth NC implies that the relation (92) is equivalent with

$$\frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} = \boldsymbol{o} \qquad (t \in R). \tag{109}$$

Dm. Clear.

Pr 14 NV. Under the conditions and notations of Pr 13 N, the supposition Hpth NV implies that the relation (92) is equivalent with

$$\frac{d_\alpha \boldsymbol{r}_B}{dt} = \boldsymbol{o} \qquad (t \in R). \tag{110}$$

Dm. Since (92) must hold for any function (86), let (99) hold. Then (92) implies (109). Now (86) is equivalent with

$$\frac{dm}{dt}\frac{d_\alpha \overline{\boldsymbol{r}}_B}{dt} + m\frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} = o \qquad (t \in R) \tag{111}$$

and (111), (109), (108) imply (110).

Sch 33. If (109), (93) hold, then it is said that the motion of β with respect to α is a *rectilinear uniform translation*. On the other hand, if (110), (93) hold, then obviously β is *at rest* with respect to α. Indeed, (110) implies that the origin B of β does not move with respect to α, whereas (93) and (66) imply that the axis vectors $\boldsymbol{b}_\nu$ ($\nu = 1,\ 2,\ 3$) of β do not move with respect to α.

Using this terminology, we may re-redact Pr 13 N, splitting it into two propositions, the one corresponding to Hpth NC and the other — to Hpth NV.

Pr 15 NC. If there exists no mass-point with variable mass, then a necessary and sufficient condition in order that two rigid systems of reference are simultaneously inertial according to Newton is that they move with a rectilinear uniform translation with respect to each other.

Dm. Pr 13 N, Pr 14 NC, Sch 33.

Pr 15 NV. If there exists one at least mass-point with a variable mass, then a necessary and sufficient condition in order that two rigid systems of reference are similtaneously inertial according to Newton is that they are at rest with respect to each other.

Dm. Pr 13 N, Pr 14 NV, Sch 33.

Sch 34. Let us note that the necessary and sufficient conditions of Pr 14 NV are obviously considerably more restrictive than those of Pr 14 NC. In such a manner, as regards these two propositions, we are faced with a problem that, poetically at least, may be compared with the Gordian knot.

Pr 14 NC represents a fundamental credo of the classical Newtonian mass-point dynamics. Moreover, appropriate versions of this theorem belong to the basic acceptance of the Eulerian rigid body dynamics too, as well as of the theory of elasticity and of fluid mechanics, in other words, of the classical rational mechanics as a whole. In the mathematical reference book [91], for instance, chosen at random by the way, one reads:

"Всякая система отсчета, к-рая движется относительно И. с. о. прямолинейно и равномерно, является И. с. о."[16] (p. 562).

Now Pr 14 NV seems to destroy this dynamical credo. Indeed, according to it, if α is an inertial according to Newton system of reference, then the rigid system β is inertial according to Newton if, and only if, it does not move with respect to α (Pr 15 NV).

Sch 35. Physical arguments are *personae non grata* in mathematics. And yet, they may serve as a compass or, if one likes it better, as Ariadne's thread, even for pure mathematicians. For doubtlessly Newton and Euler have been striving at the shaping of a rational mechanics, applicable in the real world they were living in. In this connection let us underline that Pr 14 NC has successfully sustained the trials of practical examinations for two clear centuries.

Sch 36. Things standing as they ar, Pr 14 NV will persist in being *anguis in herba*, a logical trap for analytical dynamics until its contradictions with Pr 14 NC are abolished. It is obvious that in the eyes of a Newtonian purist the very idea that two rigid systems of reference are similtaneously inertial according to Newton only in case of mutual rest would seem a little short of heresy. One must not forget, however, that *nothing in mathematics is heresy enough to be worthy of the name*, the greatest virtue of a genuine mathematician consisting in his only ideology to have no ideology.

In its long adventuresome life mathematics has overlived quite a lot of mental shocks in order to be impressed of any. The fiest one has been when mathematics was *in cunabùla*: $\sqrt{2}$ turned out to be no broken number! Then the collapse of the hopes for trisecting angles, doubling cubes, and squaring circles. And the fifth postulate — a far cry from what it has been imagined! To say nothing about tangentless curves, Mengenlehre-antinomies, the choice axiom, or the crash of Hilbert's axiomatical expectations... There is hardly something on God's earth to disturb mathematician's peace of mind nowadays.

Exits out of the logical pitfall that Pr 14 NC has driven mass-point dynamics in may be sought in several directions.

Sch 37. The first line of conduct may be capitulatory one: the avowal that the acceptance of (110) and (93) in the capacity of necessary and sufficient conditions of inertiality is OK. This is the easiest and at the same time the silliest solution.

In the second place, one could hypothesize the impossibility of (108) in the frames of Ax 1 N and Ax 2 N. This is equivalent to the acceptance of Hpth NC along with Ax 1 N and Ax 2 N, alias with the postulate that no Newtonian mass-point dynamics with variable masses exists.

Third, and last as we can see, one could come at the idea that a slight reformulation of Ax 1 N may render a helpful assistance.

Since there is an extremism in the air in the cases of the first two possibilities, we shall fix our attention on the last of these alternatives.

Sch 38. *Entre paranthèses*, the second of the above three opportunities is not as radical as it may seem at first glance. Indeed, the following questions may quite naturally arise. Is up to now classical mechanics to such an extent and so closely intimate with variable mass-point problems — sensible at that, not concocted, though the meaning of the last requirement is somewhat vague — that it could by no means divorce them? Is the classical example of Meshtcherski's dynamical equation (105) as blameless indeed, as it may seem at first sight? Is it not an underhand constant mass-point problem disguised as variable, as a matter of faet? For $\frac{dm}{dt}$ does not take part in this equation, as it should, and nothing in it suggests that m is variable with the time. Let us quote an excerpt from Ax 1 N: "for any mass-point P and for any system of forces $\underrightarrow{F}$, acting on it". In other words, the genuine mathematical equivalent of Ax 1 N in the variable mass-point case should be

$$(112)\qquad \frac{dm}{dt}\boldsymbol{v} + m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F},$$

the function (86) being prescribed for any particular mass-point problem and $\boldsymbol{F}$ — involving all the forces acting on P. It is true that $\frac{dm_1}{dt}$ and $\frac{dm_2}{dt}$ are at hand in (105), but m_1 and m_2 have nothing to do with m. In the same time it is also true that additional forces $\overline{\Phi}_1$ and $\overline{\Phi}_2$ are supplemented to $\boldsymbol{F}$ in the right-hand side of (105), unwarranted by (112). And so on, and so forth, etcetera... All these questions badly need a thorough mathematical analysis. Instead of it, in the mechanical literature one finds only texts written *currente calamo.*

Sch 39. In the constant mass-point case (107) the equation (112) reduces to

$$(113)\qquad m\boldsymbol{w} = \boldsymbol{F},$$

provided $\boldsymbol{w} = \dot{\boldsymbol{v}}$. In other words, if (107) holds, then (11) and (113) coincide.

The question now arises, whether Pr 14 NC could be saved in the variable mass-point case (108) too if (113) is substituted for the first Newtonian axiom (11)? In other words, let us try the possibilities of the following variant of Ax 1 N:

Ax 1 N bis. There exixts such a system of reference S that, all derivatives being taken with respect to S, for any mass-point P and for any system of forces $\underrightarrow{F}$ acting on P, the product of the mass and the acceleration of P equals the basis of $\underrightarrow{F}$.

It is easy to prove now that Ax 1 N bis implies a theorem analogous to Pr 15 NC, making, however, no use of the hypothesis Hpth NC. Beforehand the following definition must be recognized.

Df 1 N bis. Any system of reference satisfying Ax 1 N bis is called *inertial according to Newton.*

Pr 11 N bis. Ax 1 N bis being accepted, if α and β are inertial according to Newton systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous velocity of β with respect to α and $\boldsymbol{r}_B = \boldsymbol{AB}$, then (109) and (93) hold.

Dm. The demonstration imitates that of Pr 11 N. The relation (82) implies

$$(114)\qquad \frac{d_\alpha^2\boldsymbol{r}}{dt} = \frac{d_\alpha^2\boldsymbol{r}_B}{dt^2} + \overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho})$$

$$+ 2\overline{\omega}_{\alpha\beta} \times \frac{d_\beta\overline{\rho}}{dt} + \frac{d_\beta^2\overline{\rho}}{dt^2} \qquad (t \in R),$$

whence

$$(115) \qquad m\frac{d_\alpha^2 \boldsymbol{r}}{dt^2} = m\frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} + m\overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + m\overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho})$$

$$+ 2m\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} + m\frac{d_\beta^2 \overline{\rho}}{dt^2} \qquad (t \in R).$$

The systems of reference α and β being by hypothesis inertial according to Newton to the effect of Df 1 N bis, the latter together with Ax 1 N bis imply

$$(116) \qquad m\frac{d_\alpha^2 \boldsymbol{r}}{dt^2} = \boldsymbol{F} \qquad (t \in R),$$

$$(117) \qquad m\frac{d_\beta^2 \overline{\rho}}{dt^2} = \boldsymbol{F} \qquad (t \in R)$$

provided (81), m denoting the mass of any mass-point P and F — the basis of any system of forces acting on P. Then (115) — (117) imply

$$(118) \qquad \frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} + \overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho})$$

$$+ 2\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} + \frac{d_\beta^2 \overline{\rho}}{dt^2} \qquad (t \in R)$$

after canceling m.

Disposing of the equation (118) applying to any $\overline{\rho}$, in the particular case (97) it implies (109), and (109) and (118) imply

$$(119) \qquad \overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho}) + 2\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} \times \frac{d_\beta^2 \overline{\rho}}{dt^2} = \boldsymbol{o}$$

($t \in R$). Thereupon (119) and the choice (100) imply (101). Afterwards (93) is proved in the same way as in the proof of Pr 11 N.

Pr 12 N bis. Ax 1 N bis being accepted, if α is an inertial according to Newton system of reference and β is a rigid system of reference with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then (109) and (93) imply that β is inertial according to Newton too.

Dm. α being inertial according to Newton by hypothesis, Df 1 N bis and Ax 1 N bis imply (116) provided (81), m denoting the mass of any mass-point P and $\boldsymbol{F}$ — the basis of any system of forces acting on P. Then (115), (116), (109), (93) imply (117), i.e. β is inertial according to Newton (Df 1 N bis, Ax 1 N bis).

Pr 13 N bis. Ax 1 N bis being accepted, if α and β are rigid systems of reference with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then necessary and sufficient conditions in order that α and β are simultaneously inertial according to Newton are (109) and (93).

Dm. Pr 11 N bis, Pr 12 N bis.

Pr 15 N bis. Ax 1 N bis being accepted, a necessary and sufficient condition in order that two rigid systems of reference are simultaneously inertial according to Newton is that they move with a rectilinear uniform translation with respect to each other.

Dm. Pr 13 N bis, Sch 33.

Sch 40. Coming back to the general case let us note that by virtue of Df 1 N the only criterion for the inertiality according to Newton of a rigid system of reference is the answer to the question whether is satisfies Ax 1 N or not: if yes, then it is inertial; if not, it isn't. That is why no use of Ax 2 N has been made in the proof of the formulated in Pr 13 N criterion.

And yet, a pending question remains in connection with Ax 2 N and it is: is the axiom stable with respect to the established by Pr 13 N inertiality criterion? Since the meaning of this formulation is somewhat vague, let us make it more precise.

Suppose that α and β are rigid systems of reference for which the conditions (92), (93) hold and let α be inertial according to Newton. Then α *eo ipso* satisfies Ax 2 N. On the other hand β is inertial according to Newton in view of Pr 12 N, hence it also must *eo ipso* satisfy Ax 2 N. Now the question arises: it *must*, but *does it* indeed?

In other words, if $P(\boldsymbol{r}, m)$ is any mass-point and $\underrightarrow{F}(\boldsymbol{F}, \boldsymbol{M})$ is any system of forces acting on it, then it is certain that

$$(120) \qquad \frac{d_\alpha}{dt}\left(\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \boldsymbol{M} \qquad (t \in R)$$

(under the notations already repeatedly used) by virtue of Ax 2 N, α being inertial according to Newton by hypothesis and $\boldsymbol{M}$ being taken with respect to the origin A of α. On the other hand, Pr 12 N warrants that β is inertial according to Newton too and, consequently, the equation

$$(121) \qquad \frac{d_\beta}{dt}\left(\overline{\rho} \times m\frac{d_\beta \overline{\rho}}{dt}\right) = \boldsymbol{M}_B \qquad (t \in R)$$

(where

$$(122) \qquad \boldsymbol{M}_B = \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}_B$$

is the moment of $\underrightarrow{F}$ with respect to the origin B of β) must also be satisfied by virtue of Ax 2 N. The meaning of the question, brought up above, is: *is it satisfed indeed*, in other words, *can* (121) *be proved*?

It is obvious that this question must unconditionally be answered *in the affirmative.*

Technically this problem is equivalent with the question: do (120), (92), (93), (122) imply (121)?

There are two ways leading to the answer.

The first of them is the direct one. Let (120), (92) hold for any function (86), as well as (93) and (122). Because of (93) the identity (88) becomes

$$m\frac{d_\alpha \boldsymbol{r}}{dt} = m\frac{d_\alpha \boldsymbol{r}_B}{dt} + m\frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R) \tag{123}$$

and (123), (83) imply

$$\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt} = (\boldsymbol{r}_B + \overline{\rho}) \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} + (\boldsymbol{r}_B + \overline{\rho}) \times m\frac{d_\beta \overline{\rho}}{dt} \tag{124}$$

($t \in R$), i.e.

$$\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt} = \boldsymbol{r}_B \times m\frac{d_\alpha \boldsymbol{r}}{dt} + \overline{\rho} \times m\frac{d_\beta \overline{\rho}}{dt}$$
$$+ \overline{\rho} \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} + \boldsymbol{r}_B \times m\frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R). \tag{125}$$

On the other hand, (93) and Pr 8 imply (76) for any differentiable function (57).

The equation (125) therefore implies

$$\frac{d_\alpha}{dt}\left(\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \frac{d_\alpha}{dt}\left(\boldsymbol{r}_B \times m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right)$$
$$+ \frac{d_\beta}{dt}\left(\overline{\rho} \times m\frac{d_\beta \overline{\rho}}{dt}\right) + \frac{d_\beta \overline{\rho}}{dt} \times m\frac{d_\alpha \boldsymbol{r}_B}{dt}$$
$$+ \overline{\rho} \times \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \frac{d_\alpha \boldsymbol{r}_B}{dt} \times m\frac{d_\beta \overline{\rho}}{dt} + \boldsymbol{r}_B \times \frac{d_\beta}{dt}\left(m\frac{d_\beta \overline{\rho}}{dt}\right) \tag{126}$$

($t \in R$). But obviously

$$\frac{d_\alpha}{dt}\left(\boldsymbol{r}_B \times m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) = \boldsymbol{r}_B \times \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right), \tag{127}$$

$$\frac{d_\beta \overline{\rho}}{dt} \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} + \frac{d_\alpha \boldsymbol{r}_B}{dt} \times m\frac{d_\beta \overline{\rho}}{dt} = o, \tag{128}$$

(129) $$\frac{d_\beta}{dt}\left(m\frac{d_\beta\overline{\rho}}{dt}\right) = \boldsymbol{F}$$

($t \in R$), the latter equation by virtue of Pr 13 N, i.e. (95). Now (126) — (129), (92), (120) imply

(130) $$\boldsymbol{M} = \frac{d_\beta}{dt}\left(\overline{\rho}\times m\frac{d_\beta\overline{\rho}}{dt}\right) + \boldsymbol{r}_B \times \boldsymbol{F} \qquad (t \in R)$$

and (130), (122) imply (121).

The second way to the proof of (121) is extraordinary tricky. According to Pr 6, (120) may be written in the equivalent form

(131) $$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}$$

and (131), (83), (122) imply

(132) $$\overline{\rho} \times \boldsymbol{F} = \boldsymbol{M}_B.$$

On the other hand, β is inertial according to Newton by virtue of Pr 13 N, hence (95) holds. Now (132), (95) imply

(133) $$\overline{\rho} \times \frac{d_\beta}{dt}\left(m\frac{d_\beta\overline{\rho}}{dt}\right) = \boldsymbol{M}_B \qquad (t \in R),$$

whence (121).

Sch 41. *Acta est fabula.* For the time being this is almost all one could say apropos of ***inertiality according to Newton.*** Now it is high time to proceed to the discussing of analogous problems concerning ***inertiality according to Euler.***

Comparing (11), (12), on the one hand, with (21), (22) respectively, on the other hand, one should observe that these latter problems promise to be considerably harder. This is true. In the same time it is also true that up to now we have accumulated a certain experience in such matters.

Let the Cartesian system of reference $Oxyz$ be inertial according to Euler and let $\Omega\xi\eta\zeta$ be a Cartesian system of reference invariably connected with the rigid body B, its origin coinciding with the mass-center G of B. As it is well known from rigid body kinematics, then the following identity takes place

(134) $$\boldsymbol{v} = \boldsymbol{v}_G + \overline{\omega} \times \overline{\rho},$$

provided $\boldsymbol{r} = \boldsymbol{OP}$ for any point P of B, $\boldsymbol{r}_G = \boldsymbol{OG}$, $\boldsymbol{v} = \dot{\boldsymbol{r}}$, $\boldsymbol{v}_G = \dot{\boldsymbol{r}}_G$,

(135) $$\boldsymbol{r} = \boldsymbol{r}_G + \overline{\rho},$$

$\overline{\omega}$ denoting the instantaneous angular velocity of $\Omega\xi\eta\zeta$ with respect to $Oxyz$, all derivatives being taken with respect to $Oxyz$.

The identity (135) implies

$$\int \overline{\rho} dm = o \tag{136}$$

in view of the definition (18). Then (134), (136), (17) imply

$$\int \boldsymbol{v} dm = m\boldsymbol{v}_G \tag{137}$$

and (137), (21) imply

$$\frac{d}{dt}(m\boldsymbol{v}_G) = \boldsymbol{F} \qquad (t \in R), \tag{138}$$

$\boldsymbol{F}$ denoting the basis of any system of forces acting on B.

In other words, the equation (138) may be chosen in the capacity of a mathematical formulation of Euler's law of momentum, alias of Ax 1 E. It expresses the famous theorem of Euler according to which *the mass-center of any rigid body is moving like a mass-point with mass equal to the mass of the body and acted on by all forces acting on the body.*

Sch 42. The reader should not let himself be misled by the resemblance between (138) and (11). It is only formal. In other words, directly contrary to the wide-spread belief of all physicists, mechanicians, and mathematicians, the mass-center of a rigid body is no mass-point.

Indeed, if it was, then Ax 2 N would imply

$$\frac{d}{dt}(\boldsymbol{r}_G \times m\boldsymbol{v}_G) = \boldsymbol{M} \qquad (t \in R), \tag{139}$$

whence (30) by virtue of Pr 2 N, which is an absurdity.

By the way, another absurdity is obtainable in the following manner. First, (134) and (135) imply

$$\int \boldsymbol{r} \times \boldsymbol{v} dm = \int (\boldsymbol{r}_G + \overline{\rho}) \times (\boldsymbol{v}_G + \overline{\omega} \times \overline{\rho}) dm \tag{140}$$

$$= \int \boldsymbol{r}_G \times \boldsymbol{v}_G dm + \int \boldsymbol{r}_G \times (\overline{\omega} \times \overline{\rho}) dm + \int \overline{\rho} \times \boldsymbol{v}_G dm + \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho}) dm$$

($t \in R$). On the other hand, (17) and (136) imply

$$\int \boldsymbol{r}_G \times \boldsymbol{v}_G dm = \boldsymbol{r}_G \times \boldsymbol{v}_G \int dm = \boldsymbol{r}_G \times m\boldsymbol{v}_G, \tag{141}$$

$$\int \boldsymbol{r}_G \times (\overline{\omega} \times \overline{\rho}) dm = \boldsymbol{r}_G \times (\overline{\omega} \times \int \overline{\rho} dm) = \boldsymbol{o}, \tag{142}$$

$$\int \overline{\rho} \times \boldsymbol{v}_G dm = \int \overline{\rho} dm \times \boldsymbol{v}_G = \boldsymbol{o} \tag{143}$$

($t \in R$) and (140) — (143) imply

$$\int \boldsymbol{r} \times \boldsymbol{v} dm = \boldsymbol{r}_G \times m\boldsymbol{v}_G + \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho}) dm \tag{144}$$

($t \in R$). Now (144) and (22) imply

$$\frac{d}{dt}(\boldsymbol{r}_G \times m\boldsymbol{v}_G) + \frac{d}{dt} \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho}) dm = \boldsymbol{M} \tag{145}$$

($t \in R$) and (145), (135) imply the absurdity

$$\frac{d}{dt} \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho}) dm = \boldsymbol{o} \qquad (t \in R). \tag{146}$$

Sch 43. The analogy between (138) and Ax 1 N being entirely formal, it is all the same enough for our goal, namely to use it in order to economize all the reasoning and reconings spent in connection with Pr 11 N — Pr 15 N bis, to say nothing about the enigmatical Hpth NC and Hpth NV.

The reader has certainly become aware of the trade dodge, long ago notorious as *Steiner's tea-kettle principle*: reduce unknown to known. In our case it is practicable in the following manner: Pr 11 N — Pr 15 N bis being demonstrable on the basis of Ax 1 N only and (138) imitating Ax 1 N up to the least, to prove anew their rigid body analogies would certainly mean useless efforts and needless time-wasting. The formal analogy between (138) and (11) secures the validity of these theorems in the rigid body case too (with the obvious *mutatis mutandis*, of course) without any specific proofs whatever.

We shall save the reader the bitter cup of reiteration of all those formulations. They are obvious and reducible to substitutions of the terms *rigid body* and *inertial according to Euler* for the terms *mass-point* and *inertial according to Newton* respectively in these propositions. Naturally, all the fuss in connection with variable masses jumps out again and is settled in the same way. No, the rigid body case deserves no special attention after the pains we have taken in connection with the mass-point dynamics. Instead, we shall turn our interest to another topic.

Sch 44. The economy we have realized by avoiding explicit formulations of the rigid body analogues of Pr 11 N — Pr 15 N bis imposes the following convention. If some of them has to be quoted, we shall cite it under the same number as in the corresponding mass-point case, substituting E for N. In such a manner Pr 14 NV, say, becomes Pr 14 EV, etc.

Sch 45. Summing up we may now state that, on the basis of the criteria of Pr 11 N — Pr 15 N bis (Pr 11 E — Pr 15 E bis), it is enough and to spare to know that a particular system of reference α is inertial according to Newton (Euler) in order to decide, for any rigid system of reference β, whether it is inertial according to Newton (Euler) or not. The point now consists in this peculiar system α.

Physically the choice of an inertial (no matter whether according to Newton or to Euler) system of reference is a matter of experiment. There are quite a lot of physical phenomena (declination toward east of a body falling freely in the northern hemisphere, the effect of Ber[17], etc.) indicating that no invariably connected with the Earth system of reference may be qualified as inertial. On the other hand, there are not a few physically quite trustworthy grounds to state that any rigid system of reference, the origin of which coincides with the mass-center of the Sun while its axes are directed toward immovable (far distant) stars, may be accepted in the capacity of an inertial one.

Mathematically, however, α being any particular rigid system of reference, there is no reason either to incriminate it as non-inertial or to make a fetish of it as inertial. In other words, any such system may be qualified as inertial, as well as non-inertial. Alias, all rigid systems of reference are allowed to competition as regards the title "inertial". Mathematically this qualification is a matter of definition, of definition only, and of nothing but definition.

Nevertheless there is a but there. In mathematical affairs there is an authoritative rule, the principle of economy. In other words, the most desirable case is the most simple one. But what in our case does actually most simple mean? The answer is given in the following two propositions.

Pr 16. If

$$e_\nu \in V \qquad (\nu = 1,\ 2), \tag{147}$$

$$e_\mu e_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1,\ 2), \tag{148}$$

$$e_3 \operatorname{sgn} :\ e_1 \times e_2, \tag{149}$$

then

$$e_\mu e_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1,\ 2,\ 3), \tag{150}$$

$$e_1 \times e_2 \cdot e_3 > 0. \tag{151}$$

Dm. Clear.

Pr 17. If (147) — (149),

$$\overrightarrow{e}_\nu \ \text{sgn} :\ (e_\nu,\ o) \qquad (\nu = 1,\ 2,\ 3), \tag{152}$$

$$\varepsilon \ \text{sgn} :\ \{\overrightarrow{e}_\nu\}_{\nu=1}, \tag{153}$$

then ε is a rigid right-hand orientated orthonormal Cartesian system of reference and o is its origin.

Dm. App (1) — (6), App (12) — (14), App (9), Pr 16.

We now manifest the following dynamical axiom.

Ax 3 N. The defined by (153) system of reference ε, provided (147) —(149), (152), is inertial according to Newton.

Sch 46. On the basis of Pr 11 N — Pr 15 N bis and Ax 3N one is capable of determining, for any rigid system of reference, whether it is inertial according to Newton or not. Especially:

Pr 18 N. If

$$a_\nu \in V \qquad (\nu = 1,\ 2,\ 3), \tag{154}$$

$$a_1 \times a_2 \cdot a_3 \neq 0, \tag{155}$$

$$A_\nu \in V \qquad (\nu = 1,\ 2,\ 3), \tag{156}$$

$$a_\mu A_\nu + a_\nu A_\mu = 0 \qquad (\mu, \nu = 1,\ 2,\ 3), \tag{157}$$

App (1) — (2), then α is inertial according to Newton.

Dm. Ax 3 N, Pr 14 NC, Pr 14 NV.

Sch 47. In such a way, the problem about inertiality according to Newton is settled. As regards Euler, we are faced with the following alternative:

1. There exists no system of reference inertial according to both Newton and Euler.

2. There exists one at least system of reference inertial according to both Newton and Euler.

Tertium non datur.

For the time being we are ignorant which of these two possibilities is true. As a matter of fact, any of them could be true, or could be untrue. Indeed, if α is any particular rigid system of reference, then we can decide, on the basis of Ax 3 N, whether it is inertial according to Newton or not. As regards its inertiality according to Euler, for the time being at least, we know as much as nothing. Before answering the question which of the above possibilities is true, let us make a little thinking.

If the first situation is realized and S_N and S_E are systems of reference inertial according to Newton and Euler respectively, then no proposition of the Newtonian mass-point dynamics does hold in S_E and no proposition of the Eulerian rigid body dynamics does hold in S_N (under the supposition that these propositions are not invariant with respect to the systems of reference). In other words, in the first case there exist two entirely distinct dynamical theories which are completely alienate from one another. In particular, no dynamical problem, simultaneously treating a mass-point and a rigid body *coniunctum*, can be solved by the direct application of both Ax 1 N, Ax 2 N and Ax 1 E; Ax 2 E (although there are indirect methods to this end). It is obvious that such a perspective does not seem a very attractive one.

Besides, in this first case a rather complicated problem arises in connection with the *time*-notion. As already explained in Sch 8, this notion is incapable of an explicit mathematical definition, being a primary notion-object of analytical mechanics, definable implicitly by means of Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E namely.

As a matter of fact, there are two rather than one time-notions, the one defined by the aid of Ax 1 N, Ax 2 N, and the other — by means of Ax 1 E, Ax 2 E. Correspondingly one should speak of *Newtonian time* and of *Eulerian time* and nobody knows apriory do they have in general something in common at all. This circumstance once again makes the first of the above possibilities entirely unacceptable.

In the second case any rigid system of reference is either inertial or non-inertial both according to Newton and Euler. Indeed, let the system of reference S be inertial both according to Newton and Euler and let the system Σ be inertial according to Newton. Then by virtue of Pr 11 N — Pr 15 N bis, the motion of Σ with respect to S is necessarily a rectilinear uniform translation or possibly a rest respectively. This condition, however, is sufficient for the inertiality of Σ according to Euler by virtue of the corresponding criterion among Pr 11 E — Pr 12 E bis. And *vice versa*, if Σ is inertial according to Euler, then by virtue of Pr 11 E — Pr 15 E bis its motion with respect to S is necessarily a rectilinear uniform translation or possibly a rest respectively. This condition is, however, sufficient for the inertiality of Σ according to Newton by virtue of the corresponding criterion among Pr 11 N — Pr 15 N bis. Hence, the Newtonian and Eulerian dynamical axioms hold for exactly the same sets of rigid systems of reference. In other words, in the second of the above cases there will exist a general dynamics, the Newton — Eulerian mass-point and rigid body dynamics. That is why this possibility is beyond comparison more tempting than the first one.

These considerations justify the acceptance of the following axiom.

Ax 3 E. The defined by (153) system of reference ε, provided (147) — (149), (152), is inertial according to Euler.

The following proposition is an immediate corollary from Ax 3 N, Ax 3 E and from the argumentation adduced above.

Pr 19 NE. Any system of reference which is inertial according to Newton is inertial according to Euler and *vice versa.*

Pr 19 NE and Pr 18 N imply:

Pr 18 E. If (154) — (157) and App (1) — (2), then the system of reference α is inertial according to Euler.

Pr 19 NE justifies the advisability of the following definition:

Df 2 NE. A system of reference is called *inertial* if it is inertial according to Newton.

Pr 20 E. A system of reference is inertial if, and only if, it is inertial according to Euler.

Dm. Pr 19 NE, Df 2 NE.

Sch 48. A question of intransient interest for analytical dynamics is the formulation and use of both Newtonian and Eulerian dynamical axioms and of their corollaries for non-inertial rigid and non-rigid systems of reference. This is, however, a topic we shall not discuss here.

Sch 49. A problem similar to that formulated and solved in Sch 40, comes into being in rigid body dynamics too. Making a long story short, it may be formulated in the following manner. If B is any rigid body and $\underset{\rightarrow}{F}(\boldsymbol{F}, \boldsymbol{M})$ is any system of forces acting on it, then it is certain that

$$(158) \qquad \frac{d_\alpha}{dt}\int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \boldsymbol{M} \qquad (t \in R)$$

(under the notations already repeatedly used) by virtue of Ax 2 E, α being inertial according to Euler by hypothesis and $\boldsymbol{M}$ being taken with respect to the origin A of α. On the other hand, Pr 12 E warrants that if (92), (93) hold, then β is inertial according to Euler and consequently the equation

$$(159) \qquad \frac{d_\beta}{dt}\int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt} dm = \boldsymbol{M}_G \qquad (t \in R),$$

provided (122), must hold. The problem mentioned above now is: does it hold indeed.

In order to solve it let us note that because of (93) the identity (82) implies

$$(160) \qquad \frac{d_\alpha \boldsymbol{r}}{dt} dm = \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \frac{d_\beta \overline{\rho}}{dt} dm \qquad (t \in R)$$

and (160), (83) imply

$$\int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \int (\boldsymbol{r}_B + \overline{\rho}) \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm \tag{161}$$

$$+ \int (\boldsymbol{r}_B + \overline{\rho}) \times \frac{d_\beta \overline{\rho}}{dt} dm \qquad (t \in R),$$

i.e.

$$\int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \int \boldsymbol{r}_B \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt} dm \tag{162}$$

$$+ \int \overline{\rho} \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \int \boldsymbol{r}_B \times \frac{d_\beta \overline{\rho}}{dt} dm \qquad (t \in R).$$

On the other hand, (93) and Pr 8 imply (76) for any differentiable function (57).

The equation (162) therefore implies

$$\frac{d_\alpha}{dt} \int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \frac{d_\alpha}{dt} \int \boldsymbol{r}_B \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm \tag{163}$$

$$+ \frac{d_\beta}{dt} \int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt} dm + \int \frac{d_\beta \overline{\rho}}{dt} \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm$$

$$+ \int \overline{\rho} \times \frac{d_\alpha}{dt} \left(\frac{d_\alpha \boldsymbol{r}_B}{dt} dm \right) + \int \frac{d_\alpha \boldsymbol{r}_B}{dt} \times \frac{d_\beta \overline{\rho}}{dt} dm + \int \boldsymbol{r}_B \times \frac{d_\beta}{dt} \left(\frac{d_\beta \overline{\rho}}{dt} dm \right)$$

$(t \in R)$. Obviously

$$\frac{d_\alpha}{dt} \int \boldsymbol{r}_B \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm = \boldsymbol{r}_B \times \frac{d_\alpha}{dt} \left(m \frac{d_\alpha \boldsymbol{r}_B}{dt} \right), \tag{164}$$

$$\int \frac{d_\beta \overline{\rho}}{dt} \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \int \frac{d_\alpha \boldsymbol{r}_B}{dt} \times \frac{d_\beta \overline{\rho}}{dt} dm = \boldsymbol{o}, \tag{165}$$

$$(166) \qquad \int \frac{d_\beta}{dt}\left(\frac{d_\beta \overline{\rho}}{dt}dm\right) = \frac{d_\beta}{dt}\int \frac{d_\beta \overline{\rho}}{dt}dm = \boldsymbol{F}$$

($t \in R$), the latter equation by virtue of Pr 13 E. Now (163) — (166), (92), (158) imply

$$(167) \qquad \boldsymbol{M} = \frac{d_\beta}{dt}\int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt}dm + \boldsymbol{r}_B \times \boldsymbol{F} + \int \overline{\rho} \times \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) dm$$

($t \in R$) and (167), (122) imply

$$(168) \qquad \frac{d_\beta}{dt}\int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt}dm = \boldsymbol{M}_G + \int \overline{\rho} \times \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}dm\right)$$

($t \in R$). It is immediately seen that (168) would imply (159) if, and only if, the relation

$$(169) \qquad \int \overline{\rho} \times \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}dm\right) = \boldsymbol{o} \qquad (t \in R).$$

Now (169) would surely hold if

$$(170) \qquad \frac{d}{dt}(dm) = 0 \qquad (t \in R),$$

i.e. in the constant mass case. Indeed, then (17) implies (107) and (107), (92), (170) imply

$$(171) \qquad \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}dm\right) = \frac{d_\alpha^2 \boldsymbol{r}_B}{dt}dm \qquad (t \in R)$$

and (171), (109) imply (169).

In such a manner, the affirmative answer of this problem depends on the mass-constancy problem. In view of the logical difficulties of the latter we leave things here as they are.

Finally let us note again that the considerations in this article, as already mentioned, are intended to revive the mathematician's interest in Hilbert's sixth problem concerning the axiomatical construction of rational mechanics in general, and of analytical mechanics in particular, as well as to constribute, humble as it is, to its headway. In our mind, such efforts are not useless on the background of not a few *argumenta ad ignorantiam*, one has the chance to see printed in black and white in the literary sources on this domain. The maxim *hoc volo; sic jubeo, sit pro ratione voluntas*, often carried out in everyday life, sounds ridiculously in the mathematical routine. *Volens nolens,* the mechanicians must become reconciled with the fact that rational mechanics "in its relation to experience, intuition, abstraction, and everyday life does not differ in essence from other branches of mathematics"[18], that the axiomatical consolidation of its logical foundations is hence forthcoming.

APPENDIX

1. As a matter of fact, this problem has been looked on as a *circulus vitiosus* or, more picturesquely, as a dog striving to bite its tail. Indeed, a geometrical notion Z, for instance, is defined by means of one or several geometrical notions Y, X, etc. Going back, one arrives in the long run at several geometrical notions A, B, C, etc., which are so fundamental, so elementar, and so simple, that there are no other geometrical notions by means of which these A, B, C, etc. could be defined explicitly. In such a manner, at first sight at least, the circuit seems to close and the geometers' honourable intentions for an irreproachable consolidation of the logical foundations of their science seem to be a complete failure.

A way out of this dead-lock has been discovered by Hilbert. In his non-pareil work [13] marking a new mathematical era simultaneously with the change of two centuries, he proclaimed a new mathematical principle ordained to break up the ancient mental stereotypes as only the theory of relativity did. According to Hilbert's *axiomatical principle*, in the process of logical consolidation of the foundations of any mathematical theory T certain ***primary notions-objects*** $A_1, \ldots, A_a$ and certain ***primary notions-relations*** $B_1, \ldots, B_b$ of T must be discovered, or selected, or proclaimed, which are ***unsusceptible to explicit definitions*** by the aid of any other notions of T. These ***primary notions*** of T must be ***defined implicitly*** by the aid of a ***system of axioms*** of T, i.e. a set of statements Ax 1, ..., Ax N, involving $A_1, \ldots, A_a$ and $B_1, \ldots, B_b$ and stating elementary properties of $A_1, \ldots, A_a$, suggested by the intuition, or by the naive ideas primarily incarnated in $A_1, \ldots, A_a$, or on the basis of God knows what reasons. The question about the authenticity, or reliability, or trustworthiness, etc. of Ax 1, ..., Ax N does not come into being at all: according to the axiomatical principle of Hilbert this question is pointless, i.e. unsubstantiated, devoid of sense, empty of matter. The Ax 1, ..., Ax N of T are true by definition, or by hypothesis, or by decree, etc., inasmuch as two adamant conditions are satisfied. First, the system of statements Ax 1, ..., Ax N must be unconditionally ***consistent***, i.e. free from inner contradictions. Second, its logical corollaries must form a system ***identical*** to T rather than to some of its far away cousins: any theorem of T must be demonstrable on the basis of Ax 1, ..., Ax N. (A system of axioms for the Euclidean geometry is proposed in the *Appendix* of the article [85, p. 160 – 161], while a system of axioms for arithmetic of natural numbers is given ibidem, p. 161 – 162.)

2. As, for instance, the notion ***point, line***, and ***plane*** in geometry.

3. As for instance, the notion *incident* (*zusammengehört, liegt,* see [85, p. 160]) in geometry.

4. The rigid body dynamics is the main subject in analytical mechanics. As it is well known, traditionally the latter is divided into three parts: kinematics, statics and dynamics. In their turn, any of them is divided into two parts: kinematics of points and rigid bodies, statics of mass-points and rigid bodies, dynamics of mass-points and rigid bodies. The first ones of all these, namely kinematics of points, statics of mass-points, and dynamics of mass-points respectively, belong to the most trivial parts of analytical mechanics.

On the other hand, statics of rigid bodies is a trivial part of analytical mechanics of rigid bodies too, having to deal, first, with the most restricted case when the rank of the system of forces acting on the rigid body (both active and passive forces, alias forces determined by the conditions of the particular statical problem under consideration and reactions of the geometrical constraints imposed on the rigid body respectively) is equal to zero, and, second, with algebraic mathematical conditions if equilibrium, rather than with systems of differential equations of motion as in the dynamical case.

As regards kinematics of rigid bodies, its role in the system of analytical mechanics may be assessed as an auxiliary one. Indeed, its predetermination is to supply the analytical mechanics with the necessary geometry. As a matter of fact, rigid body kinematics could be qualified as the geometry of motion. Its main aim is to define and describe such fundamental for analytical mechanics mathematical entities, as for instance the notions of *affine* and *rigid Cartesian systems of reference*, *motion* of such systems, *local derivatives* of vector functions with respect to these systems, *affine* and *rigid kinematical bodies* along with their basic attributes, as for instance *partial* and *total instantaneous angular velocities*, as well as the proofs of *Eulerian theorems* concerning the *relations between linear* and *angular velocities* and of *Euler's kinematical equations* involving the Eulerian angles and their time-derivatives, and so on, and so forth, etcetera, to say nothing about the definition of the most important for the whole of rigid body dynamics notion of *kinetical rigid body*, with its basic attributes: *mass*, *mass-center*, *momentum*, and *kinetical moment*.

In such a manner, in the long run, analytical dynamics of rigid bodies remains the specific part of analytical mechanics in general — its genuine core, as a matter of fact.

5. Before proceeding farther, let us say some more words concerning the rigid body notion in its dynamical as well as kinematical aspect.

As already emphasized in the previous parts of this article, the rigid body concept is traditionally looked upon by the authors of writings on analytical mechanics as an apriori notion of this science. This attitude is due to the fact they have not yet overcome the mentality of the puberty period in the history of mechanics, pretending rational mechanics to be physics in its substance. It is not. The erroneous belief that it is resulted in the deplorable state of affairs as far as the axiomatical consolidation of its logical fundaments are concerned and has postponed the execution of Hilbert's program towards its axiomatical construction [85, p. 158 – 159, 166] *ad calendas graecas*.

Rational mechanics in general, analytical mechanics in particular, are mental, not experimental, as well as geometry is mental, not instrumental, "and in its relations to experience, intuition, abstraction, and everyday life it does not differ in essence from" [1, p. 336] the theory of numbers, say. "In this audience, I am sure, mathematics itself needs no defence. It is unnecessary to persuade you that mathematics is trying to be physics or trying to be engeneering. It should also be unnecessary to point out that mathematics, however abstract and however precise, is a science of *experience*, for experience is not confined to the gross senses: Also the human mind can experience, and we need not be so naive as to see in an oscilloscope an instrument more precise than the brain of a man" [*ibid.*].

Analytical mechanics is pure mathematics *par excellence,* and this is borne by the fact that it is now, *in our days,* as much a deductive science — neither more nor less — as arithmetic and geometry for instance are. It is true that *long ago,* in its embryonic inductive state, analytical mechanics belonged to physics. (One should not forget that Newton christened his first-borne child namely *Philosophiae Naturalis Principia Mathematica,* and that *Philosophia Naturalis* meant exactly *physics* in his days.) In the same time it is also true that the degree of this appurtenance, of these affiliations, has not been higher than those of arithmetic and geometry. For, once upon a time, there has been a period when arithmetic and geometry were parts of physics too: in their experimental and instrumental age respectively, when commutativity of multiplication has been established by check-ups, and volumes of solids have been determinated by the aid of sand and water. Let us not forget the historical truth that even Leibniz knew by a physical experiment rather than by proof that 2 times 2 makes 4.

6. Such objections are not made up, or fabricated, or concocted. They correspond to, they reflect scientific reality. They have been not once nor twice brought forward before us even by highly educated professional mathematicians who, however, as far as analytical mechanics is concerned, behave (a not infrequent phenomenon) as haughtily as only dilletanti could (improvising mechanics, as a matter of fact, entirely forgetting that they have settled their accounts with rational mechanics as late as they have left their student's desks).

7. Newton's dynamical ideas culminated in his famous postulate:

Lex II. *Mutationem motus proportionalem esse vi motrici impresse et fieri secundum lineam rectam qua vis illa imprimitur* (alias, the alteration of motion is ever proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed).

Now, a mere glance at Lex II and its modern version Ax 1 N at once displays an essential flaw in Newton's formulation: the total absence of the notion of system of reference in it, to say nothing of derivatives with respect to such systems. But Lex II is not unconditionally true: it is true for some systems of reference (inertial according to Newton) and untrue for other ones.

As regards Ax 2 N, it is completely wanting among Newton's *axiomata sive leges motus* [30, p. 129]. Undoubtedly, Newton knew and used it. Nevertheless, he thought wrongfully that it is an immediate corollary from Lex II. It is not. The erroneous belief that it is represents a prejudice shared even by modern authors of text-books, treatises and monographs on analytical mechanics. Its analysis is put off until later.

8. For an axiomatical definition of V see, for instance, [86].

9. For the sake of simplicity the definitional domain of (1) is hypothesized here to be R rather than some appropriate subset of R.

10. An *affine Cartesian system of reference* α is defined as the set

$$\alpha \text{ sgn} : \{\overrightarrow{a}_\nu\}_{\nu=1}^{3}, \tag{1}$$

where

$$\overrightarrow{a}_\nu \text{ sgn} : (\boldsymbol{a}_\nu, \boldsymbol{A}_\nu) \qquad (\nu = 1, 2, 3), \tag{2}$$

provided

$$\boldsymbol{a}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3), \tag{3}$$

$$\boldsymbol{A}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3) \tag{4}$$

are given vector functions with

$$\boldsymbol{a}_1(t) \times \boldsymbol{a}_2(t) \cdot \boldsymbol{a}_3(t) \neq 0 \qquad (t \in R), \tag{5}$$

$$\boldsymbol{a}_\mu(t)\boldsymbol{A}_\nu(t) + \boldsymbol{a}_\nu(t)\boldsymbol{A}_\mu(t) = 0 \qquad (t \in R) \tag{6}$$

($\mu, \nu = 1, 2, 3$). The arrows (2) are called the *axes* of α and the vectors $\boldsymbol{a}_\nu$ are called the *axis vectors* of $\overrightarrow{a}_\nu$ ($\nu = 1, 2, 3$) respectively.

By virtue of (5), (6) the system of vector equations

$$\boldsymbol{a} \times \boldsymbol{a}_\nu = \boldsymbol{A}_\nu \qquad (\nu = 1, 2, 3) \tag{7}$$

has exactly one solution

$$\boldsymbol{a} : R \longrightarrow V, \tag{8}$$

namely

$$\boldsymbol{a} = \frac{1}{2}\sum_{\nu=1}^{3} \boldsymbol{a}_\nu^{-1} \times \boldsymbol{A}_\nu, \tag{9}$$

provided

$$\boldsymbol{a}_\nu^{-1} \text{ sgn} : \frac{\boldsymbol{a}_{\nu+1} \times \boldsymbol{a}_{\nu+2}}{\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3} \qquad (\nu = 1,\ 2,\ 3) \tag{10}$$

with

$$\boldsymbol{a}_{\nu+3} \text{ sgn} :\ \boldsymbol{a}_\nu \qquad (\nu = 1,\ 2) \tag{11}$$

are the *reciprocal vectors* of the reper (3). The function (8), defined by (9), is called the *origin* of α. It is the intersecting point of the three axes of α.

The system of reference (1) is called *rigid* if

$$\frac{d}{dt}(\boldsymbol{a}_\mu(t)\boldsymbol{a}_\nu(t)) = 0 \qquad (t \in R) \tag{12}$$

$(\mu, \nu = 1,\ 2,\ 3)$. It is called *orthonormal* if

$$\boldsymbol{a}_\mu(t)\boldsymbol{a}_\nu(t) = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1,\ 2,\ 3) \tag{13}$$

$(t \in R)$ and *right-hand oriented* if

$$\boldsymbol{a}_1(t) \times \boldsymbol{a}_2(t) \cdot \boldsymbol{a}_3(t) > 0 \qquad (t \in R). \tag{14}$$

11. The system of reference α being defined as above and the functions (3), (4) being differentiable, let

$$\boldsymbol{p} :\ R \longrightarrow V \tag{15}$$

be any differentiable function. Then

$$\boldsymbol{p} = \sum_{\nu=1}^{3} (\boldsymbol{p}\boldsymbol{a}_\nu^{-1})\boldsymbol{a}_\nu \tag{16}$$

and the function

$$\frac{d_\alpha \boldsymbol{p}}{dt} \text{ sgn} :\ \sum_{\nu=1}^{3} \left(\frac{d}{dt}(\boldsymbol{p}\boldsymbol{a}_\nu^{-1})\right) \boldsymbol{a}_\nu \tag{17}$$

is called the *derivative of* $\boldsymbol{p}$ *with respect to* α or the *local* (*with respect to* α) *derivative of* $\boldsymbol{p}$. If the context permits no collision of notations, a simpler symbolics is used.

So for instance the derivatives with respect to $Oxyz$ are usually denoted by $\frac{d}{dt}$ instead of $\frac{d_\alpha}{dt}$. In the special case when t represents time, dots are traditionally used, for instance

$$\dot{p} = \frac{dp}{dt}. \tag{18}$$

12. Examples of 1-dimensional rigid bodies are the so-called rings, wires, rods, etc.; examples of 2-dimensional rigid bodies are the so-called discs, lamellae, plates, slabs, etc.; with the exception of these extravagant samples, all "normal" rigid bodies are 3-dimensional.Naturally, the dimensions of rigid bodies are described strictly in the mathematical definition of the rigid body concept in any particular case. As regards this part of the exposition, an appeal is made to the reader's own experience in analytical mechanics.

13. Over the whole space V, as a matter of fact, in the mathematical definition of the rigid body concept. At that, it is supposed that $\varkappa = 0$ outside the "geometrical borders" ,or the "delineations" of the particular rigid body under consideration.

14. "It is clear enough that in statics the equilibrium of moments is not insured by the equilibrium of forces, nor *vice versa*. In dynamics, the principle of moment of momentum developed late, and much of the earlier work concerning it gives the impression that the two principles were somehow hoped to be equivalent, so that there would be but a single law of motion. This illusion is fostered in the teaching of mechanics by physicists today ... The law of moment of momentum is subtle, often misunderstood even today" [1, p. 128 - 129].

15. Strange though it may seem, the simple corollaries (47) and (48) from Ax 1 N and Ax 2 N respectively concerning S_n, nowadays known as Newton's laws of *momentum* and of *kinetical moment* respectively for a system of finite number of discrete mass-points, are nowhere to be found in Newton's *Principia*. They have been discovered by Euler about half a century after the publication of Newton's work.

16. Any system of reference, which is moving rectilinearly and uniformly without rotation with respect to an inertial system of reference, is an inertial system of reference itself.

17. Карл Максимович Бэр (1792 — 1876), Russian natural scientist. In 1857 he explained the erosion of the right (left) banks of rivers flowing in the directions of the meridians in the northern (southern) hemisphere by means of the Earth revolution.

18. We beg the reader's pardon for citing for the second time this position of Truesdell. We shall, however, never get tired in repeating it over and over again, as long as some mechanician's mental constitutions make it so timely and topical.

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