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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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НЕБЕСНАТА МЕХАНИКА В ТВОРЧЕСТВОТО НА КИРИЛ ПОПОВ

ЕМИЛ ХОРОЗОВ

We trace the role of selestial mechanics in the research of the prominent Bulgarian mathematician and physisist Kiril Popov.

През цялата история на математиката, а дори в историята на науката, небесната механика (заедно с астрономията) е играла фундаментална роля. Краят на XIX и началото на XX век са обвеяни с особена романтика – тогава излизат епохалните работи на Поанкаре. Кирил Попов е имал щастието, но преди всичко усета, да се насочи в тази област. Неговите научни занимания започват с астрономия. Много скоро той обръща внимание и на теорията, изучавайки самостоятелно класическия учебник по механика на Апел. Интереса си към астрономията той дължи на своя професор М. Бъчваров, но изглежда, че теоретичните му занимания са основно плод на собствени търсения. Като следствие от неподходящ преподавател още като студент Попов сам изучава трудове на Тисеран. (Последният е предшественик на Поанкаре като един от най-големите специалисти по небесна механика, а и ръководител на катедрата.) Самият Попов се оплаква от бавния темп на математическата подготовка в тогавашния Физико-математически факултет. „Математиката, пише Попов, живееше интензивен живот, за който ние нямахме представа." Тогавашното списание на Физико-математическото дружество дълго време коментира откритието на 81-ата забележителна точка в триъгълника в момент, когато големите проблеми на физиката разтърсват научния свят. Като учител той започва издаването на „Библиотека от значителни доклади в международни конгреси", за която превежда Х. Херц, Освалд, готви се да преведе и Поанкаре. С няколко думи, Кирил Попов проявява рано своята търсеща натура. Изучавайки самостоятелно серия класически трудове на Поанкаре, Дарбу, Тисеран и др., той се подготвя така, че след известно време един от най-добрите университети по онова време – Сорбоната – решава, че може да признае неговото образование за достатъчно, за да започне докторантура. Добре е известно, че там научен ръководител му става Поанкаре. Поанкаре му предоставя сам да си избере тема, нещо нетипично за знаменития математик, който в случая веднага одобрява темата. И темата е известна в нашия научен свят – движението на малката планета Хекуба. А с какво е интересна теорията на движението на тази планета, може би не всички знаят. Поне аз до преди занимaнията си със съчиненията на Поанкаре съм считал темата за екзотична.

Нека започна с математическия модел, т.е. избирането на подходящи диференциални уравнения. Малките, както и големите, планети се движат главно под въздействието на Слънцето по кеплерови елипси, т.е. за малки интервали от време въздействията от други небесни тела могат да се пренебрегнат без значителна загуба на точност. За големи интервали от време – от порядъка на години – този модел не е достатъчен; към влиянието на Слънцето трябва да се добави и влиянието на някои други планети. В повечето случаи е най-важно е да се отчете притеглянето на Юпитер. Ако приемем масата на Слънцето за единица, то тази на Юпитер е от порядъка на 1/1000. След Поанкаре бихме казали, че това е малкият параметър на системата. От своя страна малките планети не оказват практически никакво влияние върху движението и на Слънцето, и на Юпитер. Накрая, може да се предположи, че Юпитер се движи само под влиянието на Слънцето, т.е. по кеплерова орбита. Това приближение е достатъчно, тъй като неточностите ще се отразят на малката планета от по-висок порядък. С тези данни на основата на втория закон на Нютон и закона за всемирното привличане лесно се съставят уравненията на движение. Този модел е основен за небесната механика и се нарича ограничена задача за трите тела. Пренебрегвайки Юпитер, астероидът се движи по кеплерова елипса. Това може да се вземе за начално приближение.

След определяне на начално приближение, т.е. елементите на кеплеровата орбита, следващите приближения се търсят във вид на редове, чиито коефициенти се пресмятат чрез вече намерените само с аритметични действия. В методите, предложени от Льоверие, с които той е открил планетата Нептун, се налага да се дели на линейни комбинации с цели коефициенти от средните движения на планетите – в случая Юпитер и малката планета. Пояснявам, че средното движение е нещо като честота. В случая на Юпитер и Хекуба тези средни движения се отнасят приблизително както 2:1, т.е. има резонанс от най-ниския порядък. Следователно методите на Льоверие са неприложими за пресмятане още на следващото приближение.

Тук се появява една от основните трудности на небесната механика и въобще в пертурбационните задачи на динамичните системи. Става въпрос за малките знаменатели. Следващото поколение астрономи и математици изнамират по-фини средства, които позволяват да се работи с малките знаменатели.

Това е станало в работите на Линдщед, Делоне, Гилден и др., още преди Поанкаре, но след това тези средства са силно опростени и най-важното – идейно изяснени от Поанкаре. Те са основани на канонични смени на променливите, които придават на уравненията по-удобен за работа вид, запазвайки тяхната хамилтонова форма. „Най-добрият пример за приложение на теорията на Делоне, казва Поанкаре в своите *знаменити лекции по небесна механика*, четени многократно в Сорбоната, е приложението му към планетата Хекуба." С други думи, темата на дисертацията на Попов съвсем не е въпрос с ограничен интерес. Напротив, следвайки сегашните указания на ВАК, трябва да кажем, че темата на дисертацията е изключително актуална, но за разлика от общия случай в днешно време това дори би било вярно.

С движението на Хекуба са се занимавали няколко астрономи и математици – сред тях са астрономът Симонен и самият Поанкаре, който, общо взето, следва работите на Симонен в своите лекции. Всъщност Поанкаре само скицира идеите и оставя дългите и съвсем не прости преобразования на читателя. За числена реализация и дума не може да става. Когато Кирил Попов се заема да осъществи на практика скицата, предложена от Поанкаре, се оказва, че числените резултати се отличават значително от действителните наблюдения. Кирил Попов разполага с точни наблюдения за периода от 1869 до 1901 и те сочат отклонение на пресметнатите от действителните резултати с около 2 градуса на година. Следователно моделът на Поанкаре има недостатъци. И наистина той е пренебрегнал някои членове в уравненията, без да има достатъчно основания за това. Впрочем това е посочено от самия Поанкаре в лекциите и съвсем не са истина легендите, че Попов е поправил грешка на Поанкаре.

Кирил Попов се справя отлично със задачата. Той избира по-точни уравнения, а след това прилага теорията на Поанкаре (наречена метод на Делоне). В много отношения това изглежда рутинна задача и лично аз от сегашна гледна точка я считам за рутинна. Но понеже се страхувам, че мнението ми може да изглежда високомерно, ще поясня, като си послужа със спомени на самия Кирил Попов. В един разговор с него директорът на Парижката обсерватория Ернст Есклангон изказва учудването си, че капиталните трудове на Поанкаре оказват слабо влияние на математиците. Цитирам мнението на Кирил Попов: „Трудовете на Поанкаре са по-скоро идейни. В тях не намираме тази техническа разработка, която улеснява тяхното приложение от по-слаби математици." Нашият сънародник е съумял да ги разбере и да направи някои от тях по-достъпни. Той е трябвало да изучи методи, създадени само 10-15 години по-рано и за които Поанкаре е получил всички възможни почести по онова време, т.е. това са методи от самите върхове на математиката. Вместо да се занимава със странични и никого не интересуващи въпроси, Кирил Попов с първите си научни занимания се хвърля в голямата наука.

Бихме могли да направим паралел със сегашното състояние на науката у нас...

Да се върнем към дисертацията. Освен дълбоката теория, нужна за атакуване на задачата, Попов владее до съвършенство и методите за числено пресмятане. Заедно с това умее отлично да се справя с данни от астрономията – да не забравяме, че той е бил преди това астроном със стаж в някои от най-добрите обсерватории в света. Всичко това предопределя високото научно ниво на дисертацията. Числените резултати съвпадат с наблюденията с висока точност. Неговият първи научен ръководител Поанкаре е бил починал няколко месеца по-рано, но останалите колеги на знаменития учен високо я оценяват. За председател на журито е назначен на мястото на Поанкаре Пол Апел (или може би сам се е назначил – по това време той е декан на факултета на науките). Присъства и Жак Адамар, който преди защитата има дълга беседа с докторанта. Нека напомня и сложните обстоятелства, при които се провежда защитата. През септември 1912 г. Кирил Попов получава мобилизационна заповед за армията – наближава Балканската война. Факултетът откликва на трудната ситуация на българина. Някои от формалностите се нарушават. Тогава още е било ваканционно време. Деканът Пол Апел и секретарят на факултета намират останалите членове на журито, които също се съгласяват да прекъснат почивката си. За три дни защитата е организирана и проведена. И веднага след това Попов заминава на фронта. Вече там той получава вестникарски изрезки на френски, немски и английски, отразяващи (по вестникарски) защитата. В една от тях се намира добре известното на повечето математици у нас изразително съобщение: „Сорбоната се мобилизира през това ваканционно време, тъй като България мобилизира своите войски.“

Работата на Кирил Попов върху дисертацията се отразява върху цялото му творчество. Струва ми се, че най-успешната област в неговата кариера е външната балистика. За нея има друг лектор. Но и там, въпреки че съм чувал и други мнения, влиянието на небесната механика е очевидно. Достатъчно е да си припомним заглавието на неговите високо оценени лекции: „Методите на Поанкаре за интегриране и общия проблем на външната балистика“. Не е случайно, че предговорът към лекциите по балистика на Попов, четени в Сорбоната, е написан от Емил Пикар и съвсем не от военните.

Вероятно най-важното следствие от заниманията по дисертацията е утвърдените умение и стремеж на Попов да търси и намира дълбоки проблеми от естествознанието. Ще коментирам накратко две съчинения отново по ограничената задача за трите тела.

Добре е известно каква тежест придава Поанкаре на периодичните решения. „Те са единственият жалон, по който можем да проникнем в област, считана по-рано за недостъпна“ – пише знаменитият учен в основното си съчинение „Нови методи в небесната механика“. В работа, публикувана в *Buletin astronomique*, нашият сънародник се заема със задача „да възстанови престижа на теоремата на Поанкаре“, твърдяща, че периодичните решения се раждат и изчезват по двойки както корените на алгебричните уравнения. По-късно Уинтнер (това е авторът на една доста популярна и досега монография по небесна механика) се опитва да опровергае твърдението. И действително, за по-общи уравнения това очевидно не е вярно – например бифуркацията на Андронов-Хопф. Всъщност Поанкаре изказва такова нещо, но в контекст, доста различен от това, с което се занимават Уинтнер и Попов. Статията на Попов се занимава с изключително трудни качествени въпроси от небесната механика. Аз не съм уверен в математическата ѝ прецизност – там има трудности дори в дефинициите. Въпреки това работата впечатлява с не така често срещания сега стремеж за занимания с естествени дълбоки проблеми. За тези, които искат все пак да чуят някакви обяснения, ще припомня знаменитата хипотеза на Поанкаре, че периодичните движения в механични системи в общо положение са навсякъде гъсти. Така че приемайки хипотезата за вярна (за което има достатъчно основания), не е съвсем ясно какво изчезва. Разбира се, Попов не се е заблудил на това елементарно място. Просто искам да кажа, че трябва по-внимателно да се подходи. Има различни семейства от периодични траектории – на Хил, на Ляпунов, на три вида на Поанкаре – и човек трябва да се ограничи с някои от тях. Изобщо това е типично трудна качествена задача за структурата на решенията в механични задачи. Този вид задачи и сега не умеем да решаваме. Сред средствата, с които си служи Попов, ще отбележа прочутата регуляризация на решенията. С нейна помощ Зундман получава развитие в ред върху цялата ос на решенията, отговарящи на сблъскване на две планети – така нареченото „решение на задачата за трите тела“ , публикувано през 1913 г., и направило сензация в научния свят. В друга работа в *Mathematische annalen* Попов изследва и самите решения на Зундман. Той показва, че макар да има повече решения с разлика на координатите, равна на нула, (т.е. – сблъскване), само решенията на Зундман отговарят на реални движения. И макар първата част на твърдението да е известна (принадлежи на Шази и е публикувана почти веднага след статията на Зундман), и тя има стойност – направена е с други средства.

Надявам се, настрана от краткия обзор на работите на Попов по небесна механика, да е станало ясно какво още се опитвам да кажа с този доклад. Нека докосването до творчеството на един от най-големите български учени е повод да огледаме и себе си.

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ВЪРХУ ИЗСЛЕДВАНИЯТА НА КИРИЛ ПОПОВ ПО БАЛИСТИКА [1]

ЛЮБОМИР ЛИЛОВ

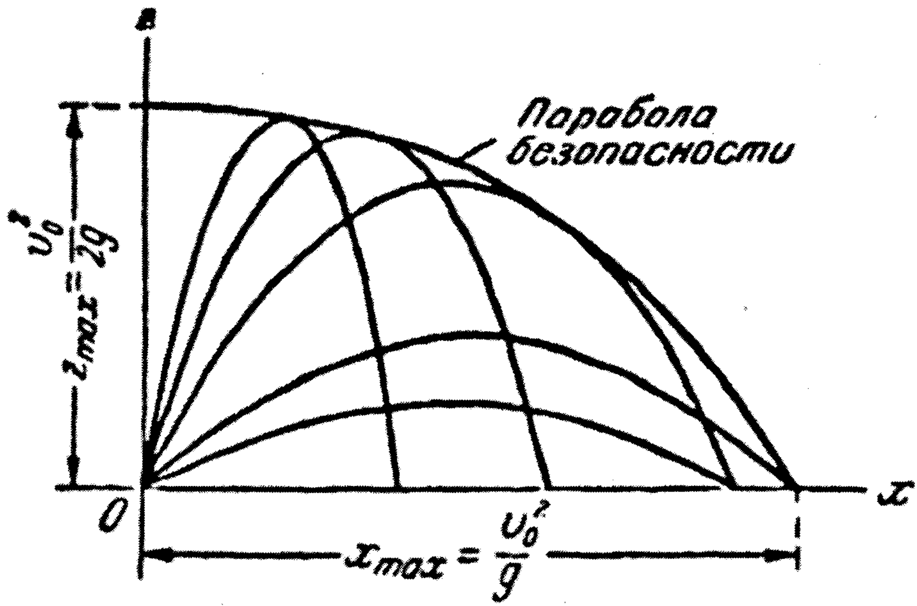
A short survey on the main results of Кирил Popoff in the field of ballistics is presented. Special attention is given to the affine transformations of the gravity center trajectories and the shell rotations around its gravity center.

В изложението се използват минимален брой формули и ще се даде представа за основните идеи и постижения на Кирил Попов в областта на външната балистика. Това вероятно ще лиши изложението от определена строгост, но ще го направи по-достъпно за неспециалистите.

Доклада си ще започна с едно приложно изследване на Кирил Попов, което показва свежия му поглед върху отдавна решени и рутинни задачи и тази нова интерпретация на познати резултати неочаквано води до създаване на по-ефективни методи за изследване, а в много случаи и до ново знание. Задачата е конструктивно да се установят траекториите на артилерийски снаряд по данни от стрелби на полигона – една задача, която има без съмнение основно значение за военните. Известен резултат от механиката е, че масовият център на един снаряд се движи като материална точка, на която действат всички приложени към точките на снаряда външни сили. В силно идеализирания случай на отсъствие на съпротивление единствената действаща сила е силата на тежестта. Да напомня добре известното решение на задачата за движение на точка в хомогенното поле на силата на тежестта при отсъствие на съпротивление. Тази задача е задължителен пример при преподаването на

[1] Доклад, изнесен на честването на 125-годишнината от рождението му.

раздела „Динамика на точка“ в курса по механика. Точката, изстреляна под ъгъл α към хоризонта с начална скорост v_0, описва парабола като при зададена начална скорост височината на траекторията е най-голяма при $\alpha = \pi/2$, а далечината на полета – при $\alpha = \pi/4$ (фиг.1). Менейки α от 0 до $\pi/2$, получаваме семейство траектории в първи квадрант на координатната система Oxy, което семейство има като еволвента отново парабола, минаваща през най-високата и най-далечната точка, достижими със скорост v_0 – така наречената парабола на безопасността.

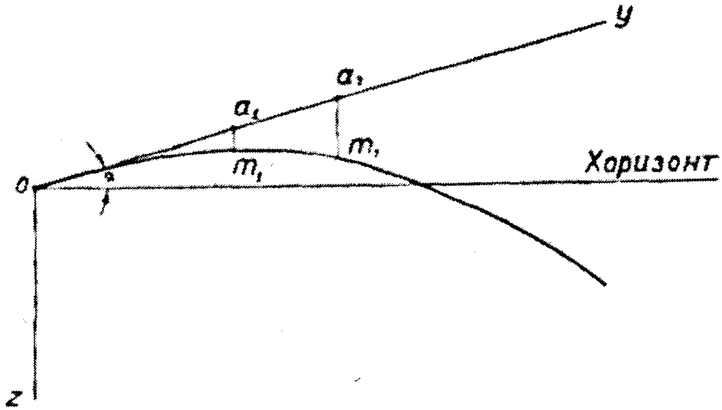


Фиг. 1

Кирил Попов прави интересна интерпретация на това движение, която след това му дава възможност да разгледа истинското движение на снаряда в съпротивителна среда. За простота на разглежданията да приемем навсякъде по-нататък, че масата на материалната точка е единица. Да отнесем движението на точката към координатна система, чиято ос Oy съвпада с началната скорост v_0 на точката, която сключва ъгъл α с хоризонта, а оста Oz е насочена надолу към центъра на Земята (фиг. 2).

При този избор на координатната система движението на точката може да се разглежда като геометричен сбор от движение на точката по оста Oy с постоянна скорост v_0

$$y = v_0 t$$



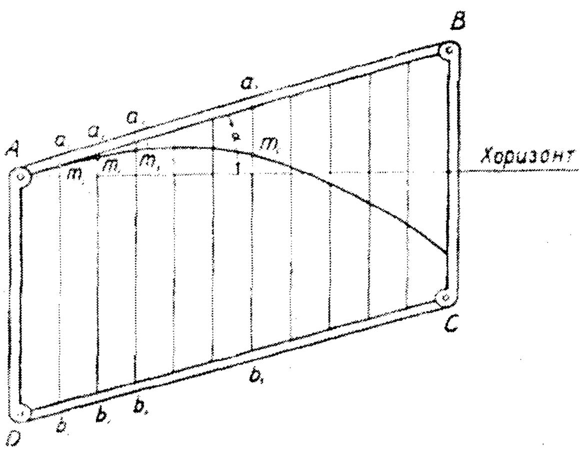
Фиг. 2

и движение по оста Oz, с начална скорост нула и ускорение g:

$$z = g\frac{t^2}{2}.$$

Удобството на така въведената от Кирил Попов координатна система е, че тези два израза са независими от ъгъла α, който началната скорост сключва с хоризонта, и зависят само от времето t, от g и от v_0, които са постоянни. Това дава възможност траекториите на точката при различните ъгли α да могат да се изведат една от друга чрез една проста трансформация.

Да си изберем една деформируема рамка с постоянни, успоредни страни, ъглите между които могат да се менят и да вземат различни стойности (фиг.3).



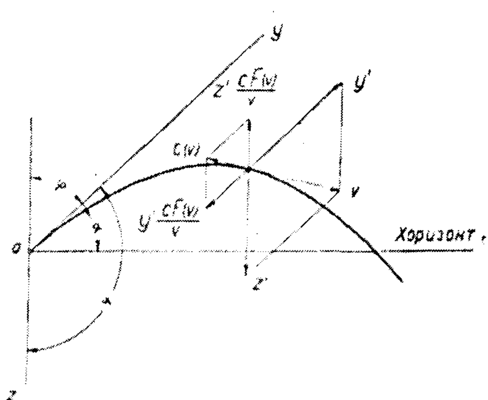
Фиг. 3

Да отбележим върху рамото AB на тази рамка точките a_1, a_2, a_3,... на разстояние v_0, $2v_0$, $3v_0$,... от началото A, а също и точките b_1, b_2, b_3,... върху рамото CD на същите разстояния от D. Да опънем между съответните точки a_i, b_i нишки a_ib_i и върху тях да нанесем точките m_1, m_2, m_3,... на разстояния $\frac{g}{2}$, $g\frac{2^2}{2}$, $g\frac{3^2}{2}$,... от a_1, a_2, a_3,... При различните ъгли, които рамото AB сключва с хоризонта, точките m_1, m_2, m_3,... ще се редят по съответните траектории на точката m, хвърлена от върха A с начална скорост $\mathbf{v}_0$, която по посока съвпада с рамото AB. При деформациите на рамката, като меним ъгъла BAD, ще се получат всички траектории, които отговарят на началната скорост $\mathbf{v}_0$ и на ъгли на хвърлянето $\angle BAD$.

Кирил Попов си задава въпроса, доколко и при какви условия това афинно свойство на траекториите в безвъздушно пространство да се извеждат една от друга чрез описаната деформируема рамка се запазва при движението на материалната точка във въздуха, т.е. при движение в съпротивителна среда.

На пръв поглед самото поставяне на въпроса изглежда абсурдно, тъй като, на една и съща начална скорост и при един и същи ъгъл на изстрела траекториите в една съпротивителна среда по размер коренно се отличават от траекториите в безвъздушното пространство, особено ако съпротивлението на средата, нейната плътност, е много голямо. Но въпреки това много важни свойства на траекториите в безвъздушното пространство се запазват при траекториите на артилерийския снаряд във въздуха, свойства, които могат да се използват при артилерийските стрелби.

Да допуснем сега, че материалната точка, с маса единица, е изстреляна с начална скорост $\mathbf{v}_0$, която сключва ъгъл α с хоризонта на отвора на оръдието.



Фиг. 4

Да изберем и в този случай отвора на оръдието за начало на координатната система, оста Oy на която съвпада с началната скорост $\mathbf{v}_0$, а оста Oz е насочена по вертикалата към центъра на Земята (фиг. 4).

Да означим с m положението на материалната точка върху нейната траектория в момента t и с v — нейната скорост в този момент, скорост, насочена по тангентата на траекторията. Да означим с $F(v)$ съпротивлението на средата при движението на точката, съпротивление отнесено към единица маса и насочено по тангентата на траекторията, в обратна посока на скоростта, съпротивление, което е функция на скоростта. Да означим с y' и z' компонентите на скоростта по координатните оси. Компонентите на съпротивлението по осите Oy и Oz ще бъдат, вземайки предвид чертежа на фиг. 4, съответно

$$y'\frac{F(v)}{v} \quad \text{и} \quad z'\frac{F(v)}{v}\,. \tag{1}$$

Диференциалните уравнения на движението спрямо избраната координатна система ще бъдат следователно

$$\begin{aligned} \tfrac{d^2y}{dt^2} &= -y'\tfrac{F(v)}{v} = -y'f(v), \\ \tfrac{d^2z}{dt^2} &= g - z'\tfrac{F(v)}{v} = g - z'f(v), \end{aligned} \tag{2}$$

при начални условия $t = 0,\ y = z = 0,\ y' = v_0,\ z' = 0$. Тук

$$\begin{aligned} v^2 &= y'^2 + z'^2 - 2y'z'\sin\alpha \\ &= (y' + z')^2 - 4y'z'\sin^2\tfrac{\psi}{2} \\ &= (y' - z')^2 + 4y'z'\sin^2\tfrac{\varphi}{2}, \end{aligned}$$

където $\psi = \pi/2 + \alpha$, $\varphi = \pi/2 - \alpha$.

Оттук се вижда, че интегралите на уравненията (2) при началните условия $t = 0$, $y = z = 0$, $y' = v_0$, $z' = 0$ зависят от ъгъла α само посредством v и ще бъдат независими от α, ако

$$\frac{F(v)}{v}$$

се редуцира на една константа k, т. е. ако

$$F(v) = kv.$$

Това означава линейно съпротивление на средата, което е типично за скорости от порядъка до 0,36 км/ч, каквато скорост значително се надхвърля от артилерийския снаряд. Кирил Попов разглежда обаче отначало този случай и получените при неговото изследване резултати обобщава по-късно за реалното движение на снаряда. В този случай каквато и да е стойността на константата k, при една и съща начална скорост v_0, както координатите y и z, тъй и компонентите на скоростта y' и z' по осите ще бъдат независими от ъгъла α, който началната скорост v_0 сключва с хоризонта, и ще зависят изключително от k и от времето t. Тъй ще имаме

$$y = y(t, v_0, k), \qquad y' = y'(t, v_0, k),$$
$$z = z(t, v_0, k), \qquad z' = z(t, v_0, k), \tag{3}$$

където производните са взети по времето t.

При $F(v) = kv$ диференциалните уравнения (2) приемат вида

$$\frac{d^2y}{dt^2} = -ky',$$
$$\frac{d^2z}{dt^2} = g - kz'. \tag{4}$$

Интегрирани при начални условия

$$t = 0, \qquad y = z = 0, \qquad y' = v_0, \qquad z' = 0,$$

те дават

$$y = \frac{v_0}{k}\left(1 - e^{-kt}\right), \qquad y' = v_0 e^{-kt}$$
$$z = \frac{g}{k}\left(t - \frac{1}{k} + \frac{e^{-kt}}{k}\right), \quad z' = \frac{g}{k}\left(1 - e^{-kt}\right). \tag{5}$$

Лесно е да се види, че при $k = 0$ тези изрази преминават в съответните изрази за безвъздушното пространство. От формулите (5) за $t \to +\infty$ получаваме

$$\lim_{t \to +\infty} y = \frac{v_0}{k}, \qquad \lim_{t \to -\infty} y' = 0$$

$$\lim_{t \to +\infty} z = +\infty, \qquad \lim_{t \to -\infty} z' = 0.$$

Следователно, както и да избираме константата $k \neq 0$, всяка траектория при $F(v) = kv$ асимптотично се приближава до една вертикална права, уравнението в избраната координатна система на която е

$$y = \frac{v_0}{k}.$$

Да се обърнем сега към нашия паралелограм и по рамената AB и DC от A и D на разстояния

$$y = \frac{v_0}{k}\left(1 - e^{-kt}\right), \text{ при } \quad t = 1,\ 2,\ 3,\ \ldots$$

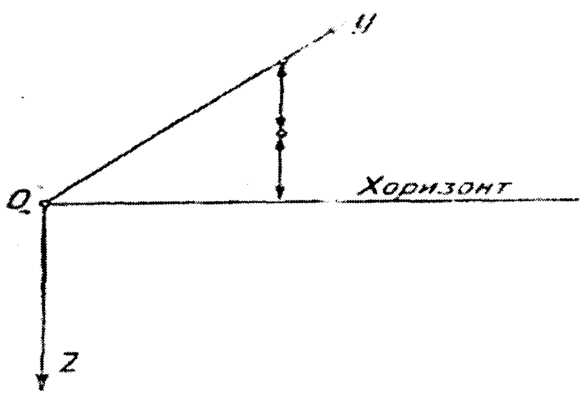
нанесем съответно точките $a_1, a_2, a_3, \ldots$ и $b_1, b_2, b_3, \ldots$ Да съединим съответните точки a_i, b_i с нишки a_ib_i, по които от точките a_1, a_2, $a_3, \ldots$ на разстояния

$$z = \frac{g}{k}\left(t - \frac{1}{k} + \frac{e^{-kt}}{k}\right) \text{ при } \quad t = 1,\ 2,\ 3,\ \ldots$$

да нанесем точките $m_1, m_2, m_3, \ldots$ При всяко положение на рамката тези точки ще се нареждат по една траектория, която отговаря на началната скорост v_0, при $\alpha = \angle BAD - \pi/2$. При деформацията на рамката кривата $Am_1, m_2, m_3, \ldots$ ще съвпада последователно с всички траектории на фамилията траектории, характеризирани с параметрите v_0 и k. Пресмятането на една от тези траектории води до познаването на всички останали. По аналогичен начин, когато вместо y и z вземем y' и z', ще получим и ходографа на скоростите.

С други думи, и в случая на съпротивление (но само пропорционално на скоростта) установените при движение в несъпротивителна среда афинни свойства на траекториите остават в сила и могат да се получават една от друга с помощта на описания от акад. Кирил Попов успоредник.

Как сега от тези резултати Кирил Попов минава към реалния случай, когато законът на съпротивление може да има много по-сложен вид? Интуитивната му догадка е следната. Тъй като стойностите на дадена холоморфна функция $F(v)$ в даден интервал върху положителната ос на комплексната равнина v може да се заключат между стойностите на функциите kv при същия интервал за v, които отговарят на две близки стойности на k, би могло да се допусне, че установеното свойство на траекториите при $F(v) = kv$ с голямо приближение ще бъде практически удовлетворено и при всеки физически допустим закон на съпротивление $F(v)$ на въздуха. Това досещане той обосновава строго след това по два различни начина, при това като взема предвид и обстоятелството, че в най-общия случай съпротивлението на средата зависи и от нейната плътност, в дадения случай от плътността на атмосферния слой, през който снарядът преминава, а тази плътност намалява с височината h над хоризонта, височина, която при нашия избор на координатната система е $h = y \sin\alpha - z$ (фиг. 5).



Фиг. 5

Няма да се спирам на строгата математическа обосновка на този резултат, а направо ще премина върху практическото му използване и ще покажа как установените афинни свойства на траекториите при най-общ случай на закон на съпротивлението могат да се използват за получаване по конструктивен начин на траекториите във въздуха по данни от стрелба на полигона..

Да допуснем, че за дадена стойност α на ъгъла на изстрела сме установили на полигона разстоянието d от отвора на оръдието до мястото на падането на снаряда в хоризонта на оръдието и интервала време t от момента на изстрела до момента на попадението. С това ние имаме всички данни за определяне на координатите на точката на попадението в координатната система, при която осите Oy и Oz сключват ъгъл $\alpha + \frac{\pi}{2}$.

Но тази точка е същевременно точка от траекторията, която отговаря на дадения ъгъл на изстрела α с начална скорост v_0, познаването на която в дадения случай не е необходимо. Така ние имаме

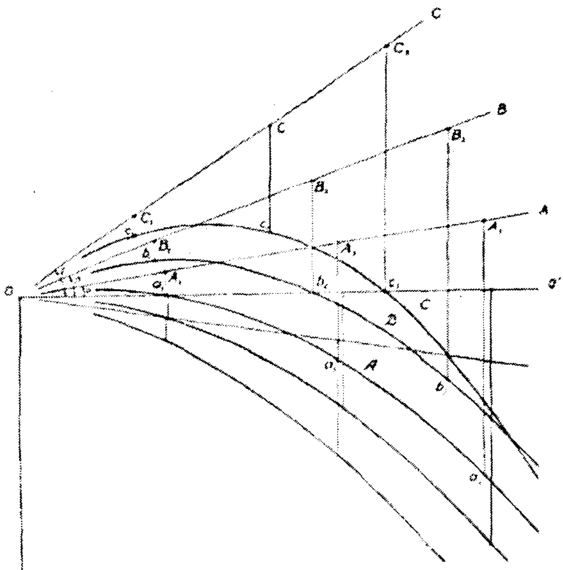
$$z(t) = d\ tg\alpha, \qquad y(t) = \frac{d}{\cos\alpha}.$$

Познаваме ли координатите за даден момент t на една точка върху дадена траектория α, с това ние познаваме координатите в съответната координатна система на съответната точка върху траекторията от фамилията траектории, които отговарят на същата начална скорост при най-различни ъгли α на изстрела. Една точка върху дадена траектория води до познаването на съответните точки върху цялата фамилия траектории v_0.

Методът, който Кирил Попов предлага, се състои в това: да се определят точно разстоянията на попадението в хоризонта на оръдието за ред траектории, които отговарят на една редица $\alpha, \beta, \gamma, ...$ на ъгли на изстрела. Всяка една такава точка води чрез прости конструкции до съответните точки на избраната редица траектории.

За тази цел си избираме на полигона една добре нивелирана хоризонтална ивица, по която се извършват всички опитни стрелби. Стреля се с едно и също оръдие, при един и същ снаряд и при един и същи заряд. Както пише Кирил Попов „ С най-голямо старание и точност се определят както ъгъла на изстрела, тъй също и попадението в хоризонта на отвора на оръдието." Определянето на времето не е необходимо, ако задачата се свежда само до определяне формата на траекториите във въздуха. То обаче е необходимо, ако искаме да знаем и момента, в който снарядът достига дадена точка на съответната траектория, както това е случаят при снаряди, които трябва да експлодират в дадена точка на траекторията.

Да означим с $\alpha, \beta, \gamma, \delta, ...$ предварително избраните ъгли на изстрела, с A, B, C, D,... – съответните траектории, и с $a_1, b_2, c_3, d_4, ...$ – точките в хоризонта на тези траектории, определени и измерени на полигона (фиг. 6).



Фиг. 6

На един добре опънат чертежен лист означаваме отвора O на оръдието, хоризонта OO' и правите OA, OB, OC, $OD, \ldots$, които сключват с хоризонта ъгли $\alpha, \beta, \gamma, \delta, \ldots$. Върху правата OO', в избран мащаб нанасяме точките $a_1, b_2, c_3, d_4, \ldots$, точки от съответните траектории, и на разстояние от O съответно на наблюдаваните и измерени разстояния на полигона.

През точката a_1, която е точка в хоризонта на траекторията A, прекарваме правата a_1A_1, успоредна на оста OZ, до пресичането ѝ с правата OA. По такъв начин получаваме координатите $y_1 = OA_1$, $z_1 = a_1A_1$ на точката a_1 от траекторията A в координатната система AOZ. За да получим съответните точки върху траекториите B, C, D, нанасяме върху правите OB, OC, OD точките B_1, C_1, D_1 на разстояние от O, равно на OA_1, и по правите, теглени от тези точки успоредно на оста OZ, нанасяме надолу точките $b_1, c_1, d_1, \ldots$ на разстояние a_1A_1. Така получените точки $b_1, c_1, d_1, \ldots$ са точки от траекториите B, C, D, които отговарят на точката a_1 от A.

За да получим точки на траекториите, които отговарят на точката b_2 от траекторията B, прекарваме през b_2 права, успоредна на OZ, до пресичането ѝ с правата OB. Да означим с B_2 съответната точка на пресичането. Така получаваме в координатната система BOZ координатите на точката b_2 от траекторията B

$$y_2 = OB_2, \; z_2 = B_2b_2,$$

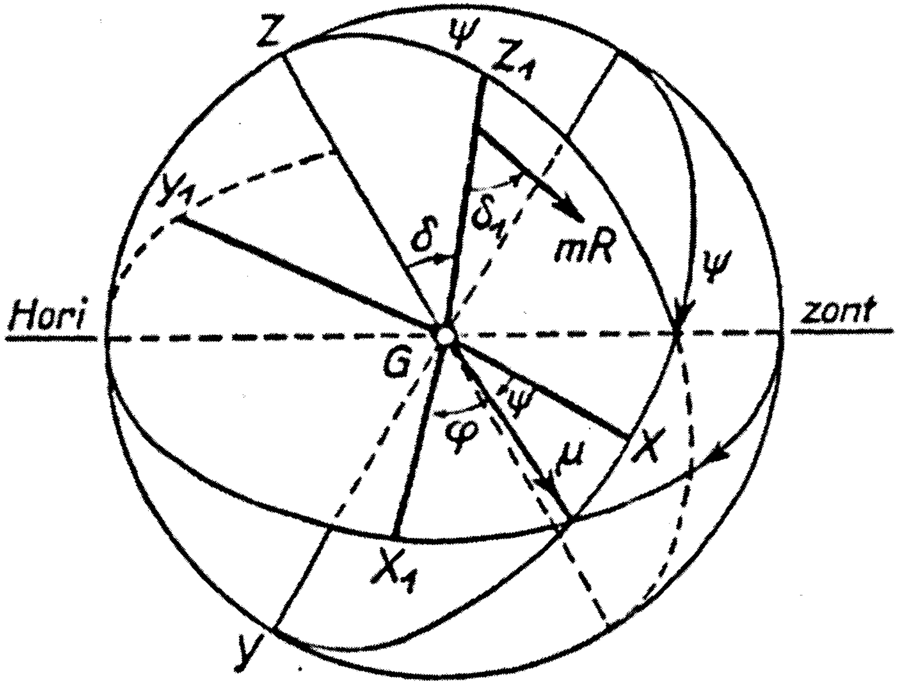
които са равни на координатите в съответните координатни системи на съответните точки $a_2, c_2, d_2, ...$ върху траекториите A, C, D. За да получим тези точки, върху правите OA, OC, OD,... и от началото O нанасяме точките $A_2, C_2, D_2, ...$ на разстояние y_2, през които прекарваме прави, успоредни на OZ и върху които от $A_2, C_2, D_2, ...$ нанасяме надолу на разстояние z_2 точките $a_2, c_2, d_2, ...$, които са точки от траекториите A, C, D, съответни на точката b_2 от траекторията B.

По същия начин постъпваме с точката c_3 от траекторията C и т. н. и т. н.

По този начин по данни от стрелбите на полигона ние можем да възстановим точка по точка фамилията траектории, които отговарят на една и съща начална скорост v_0.

На една международна конференция по астронавтика във Варна през 1964 г., както отбелязва синът на Кирил Попов – Борис Попов, в спомени за баща си, Кирил Попов разказва на присъствалия на тази конференция виден американски учен от унгарски произход проф. Теодор фон Карман за конструирания от него успоредник с опънати между горната и долната страна нишки, чрез който може да се намери всяка възможна траектория на снаряд, ако се познава една. Проф. Карман е останал във възторг от този прост уред и е отбелязал, че може би този принцип е по-съвършен от методите, с които се определят траекториите на снарядите с помощта на компютри.•

Изложеният метод е включен в монографията на Кирил Попов „Основни проблеми на външната балистика в светлината на съвремената математика" ("Die Hauptprobleme der äusseren Balistik im Lichte der modernen Mathematik"), издадена в Лайпциг през 1954 г., която включва основните му изследвания по балистика. Тази монография е възникнала от лекциите, които Кирил Попов е чел в различни години по покана в Сорбоната, в университетите на Берлин, Мюнхен, Хамбург, Рим, в Института по аеродинамика „Кайзер Вилхелм" в Гьотинген, в Школата по приложение на артилерията в Торино и в Училището по морска артилерия в Париж. Част от изложения в книгата материал е удостоен с наградата „Монтийон" по механика за 1926 г. от Парижката академия на науките. Лекциите и публикациите му по балистика донасят на Кирил Попов световна известност и той е канен в много европейски университети. Оскъдното време не ми дава никаква възможност да се спра по-подробно на изложените в монографията резултати. Ще направя кратък коментар само по постановката на задачата за движение на снаряд и на естеството на трудностите, които трябва да се преодолеят при нейното решаване. Най-трудната за изследване част от задачата за движение на снаряд е задачата за движението му около масовия му център – защото вече става дума за движение на тяло, а не на точка, каквато е масовият център. Двете движения са свързани и движението около масовия център влияе по решаващ начин върху основните параметри на стрелбата и преди всичко върху точността на попаденията.



Фиг. 7

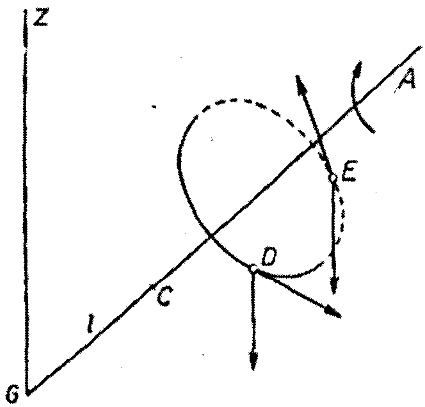
Със снаряда е неизменно свързана координатната система $Gx_1y_1z_1$, като G е масовият център на снаряда, а z_1 е оста на симетрия на снаряда (фиг. 7). Тази координатна система се движи по отношение на координатната система $Gxyz$, като се предполага, че оста Gz е по тангентата, оста Gy по главната нормала и оста Gx по бинормалата към траекторията на масовия център. Теоремата, определяща движението около масовия център, е теоремата за кинетичния момент, която гласи, че производната в инерциалното пространство на кинетичния момент на снаряда, пресметнат по отношение на масовия център, е равна на главния на момент на всички външни сили, действащи на снаряда спрямо масовия му център. Проектирано в системата $Gx_1y_1z_1$, това векторно равенство води до системата диференциални уравнения

$$
\begin{aligned}
&B\dot{p} + (C - B)\,qr = L \\
&B\dot{q} + (C - B)\,pr = M \\
&C\dot{r} = N,
\end{aligned}
$$

където p, q, r са компонентите на ъгловата скорост на снаряда, L, M, N – компонентите на главния момент на действащите външни сили, а B и C – инерчните моменти на снаряда за осите x_1 (y_1) и z_1.

Предполага се, че траекторията на масовия център на снаряда е равнинна крива; следователно реперът $Gxyz$ се върти само около бинормалата Gx. Положението на снаряда се определя от ъглите на Ойлер ψ, δ, φ.

Какви сили действат на снаряда, т.е. какви са величините L, M, N? Преди всичко това са силата на тежестта mg, която не оказва обаче влияние на движението около масовия център, защото нейният момент спрямо него е нулев, и силата на съпротивление $mR(v)$, която лежи в равнината на съпротивление (zGz_1) и сключва ъгъл δ_1 с оста на симетрия. И ъгълът δ_1, и точката на приложение на силата на съпротивление са неизвестни величини. Изобщо аеродинамичното въздействие е много сложно и освен силата на съпротивление се пораждат и странични сили на триене поради придадената на снаряда при изстрелването му голяма ъглова скорост около оста му на симетрия (фиг. 8).



Фиг. 8

Въртенето около оста на симетрия и промяната на нейното положение в пространството поражда и други ефекти като например ефекта на Магнус, завихряния, които също трябва да се отчитат. За да се отчетат всички тези явления и да може да се изгради представа за естеството на тези сили, трябва да се използват експериментални методи. Това обаче при типичните за балистиката скорости е необикновено сложна, практически неразрешима задача. Това налага да се строят хипотези, които само след сравняване на резултатите от стрелбата с математическите резултати въз основа на приетата хипотеза могат да бъдат потвърдени или опровергани. За това обаче е безусловно необходимо математическият апарат да позволява сигурни заключения въз основа на приетите хипотези, без да ги фалшифицира. Така става ясна важната роля на математическите методи: те трябва да допълнят оскъдните аеродинамични познания и да дадат възможност приетите аеродинамични хипотези да бъдат оцененени.

Точно тук е заслугата на Кирил Попов. Той привлича за задачите на балистиката математическите методи, представени в работите на Поанкаре, Пикар, Бендиксон и Сандман, и ги доразвива. Вдъхновение за него е примерът на Поанкаре, който е бил негов преподавател и който го насочва в избора му на тема за докторската му дисертация. В книгата си „Нови методи на небесната механика" Поанкаре, използвайки теорията на конформните изображения и теорията на аналитичното продължение, получава важни резултати в небесната механика. Кирил Попов решава, както пише в предговора на книгата си, че „...времето е подходящо и балистичните теории да бъдат разгледани от тази гледна точка". Тук, разбира се, трудностите са от друго естество. Докато в небесната механика силите са централни и притежават потенциал, тук аеродинамичното съпротивление има съвсем друга природа, но независимо от това споменатите теории довеждат до нови резултати, отличаващи се освен с дълбочина също така и с елегантност.

Направеният кратък обзор на изследванията по балистика на Кирил Попов показва тяхната оригиналност и дълбочина. Те са получили световно признание и така са способствали много за издигане авторитета на българската наука.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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КИРИЛ ПОПОВ – ПИОНЕР НА БЪЛГАРСКАТА АСТРОНОМИЯ

ВАЛЕРИ ГОЛЕВ

Some details of the early history and early timeline of modern Bulgarian astronomy are reviewed. The role of prof. Kiril Popov, member of Academy and a great Bulgarian mathematician, in the development of modern astronomy in Bulgaria is discussed.

„През всичкото време на моята специализация в странство аз успешно се подготвях по астрономия, като че ми предстоеше да управлявам най-голямата обсерватория в света.“

През настоящата година отбелязваме 125 години от рождението на един от най-големите учени на България – големият математик акад. Кирил Попов. Тук ще направим опит да припомним ролята акад. Кирил Попов, която той изиграва в развитието на астрономията в България – роля, която дълги години остава скрита в сянката на неговите световни постижения в различни области на математиката. Преди век тогавашното Висше училище със закон е преименувано в университет и издава своя първи годишник през 1905 г. В този труден за намиране сега първи том читателят ще открие и първата научна публикация по астрономия, отпечатана в България. Тя излиза под перото на току-що назначения за асистент по астрономия млад учен Кирил Попов. Пак тогава излиза и първият том на списанието на българското Физико-математическо дружество, където подписът на Кирил Попов стои под три публикации. Всичко това не ни оставя безразлични, защото хвърля светлина върху някои дребни на пръв поглед, но важни за нас, астрономите, факти от началната история на нашата астрономия.

Дейността на Кирил Попов като математик е всепризната и добре известна. Знаменит ученик (един от малкото) на великия Поанкаре и световно известен учен, Кирил Попов сам се е смятал колкото за математик, толкова и за астроном (вж. по този повод неговата „Автобиография“ [1]). Неговите широко известни постижения в различни области от математиката и особено във външната балистика и математичните основи на небесната механика (които тук няма да коментираме), както и неговото израстване като водещ математик в Софийския университет оставят в сянка за обществото ни значителния принос, който той има за развитието на катедрата по астрономия през първата третина на миналия век.

А тя е уникална, защото:

1) Кирил Попов е *първият* доктор по астрономия (небесна механика) в България.

2) На Кирил Попов дължим и *първата* научна публикация по астрономия в България [2] (на нея ще се спрем по-долу).

3) Негови са и едни от *първите* у нас научнопопулярни публикации по астрономия - вж. напр. [3] и [4].

4) Кирил Попов провежда и публикува единствените в България наблюдения на Халеевата комета от Университетската астрономическа обсерватория – вж. K. Popoff, "Observation de la comete Halley faites a l'Observatoire de Sofia", Comptes rendus de l'Academie des Sciences de Paris, t.157, 1395-1397. Това са *първите* стойностни професионални астрономични наблюдения, проведени в България.

5) Около 1/3 от научните му публикации са посветени на астрономията (вж. собствения му списък, публикуван в [1]).

6) Негова е и грижата за развитието на Катедрата по астрономия малко преди и непосредствено след смъртта на проф. Бъчеваров.

Нека отначало проследим накратко основните дати в ранното развитие на българската астрономия, което е свързано изцяло и само с Катедрата по астрономия в Софийския университет.

1. ВРЕМЕВА ЛИНИЯ НА НАЧАЛНОТО РАЗВИТИЕ НА БЪЛГАРСКАТА АСТРОНОМИЯ

Времевата линия на астрономията в България започва с проф. Марин Бъчеваров (роден през 1859 г. в Горна Оряховица и завършил астрономия в Московския университет), който работи университета от 1890 г. и до смъртта си през 1926 г. Астрономията е създадена от него като дисциплина във Физико-математическия факултет на все още наричаното тогава Висше училище. М. Бъчеваров започва да чете първия курс по астрономия през пролетта на 1892 г. Той създава Катедрата по астрономия и Астрономическата обсерватория на университета през 1894 г., както и несъществуващия сега астрономически институт към университета през 1910 г. [5,6].

От 1.IX.1901 г. до 1.I.1904 г. в обсерваторията работи първият асистент по астрономия. Това е бъдещият редовен професор по геодезия и културна техника в Агрономо-лесовъдския факултет Йордан Ковачев (роден през 1875 г. в Кюстендил). Творчеството му е изключително плодотворно и включва над 150 работи в областта на астрономията и геодезията, повечето от които са научнопопулярни статии и книги [6].

От септември 1904 г. до 1914 г. асистент в катедрата е шуменецът Кирил Попов (роден през 1880 г.).

Подробностите от биографията му са известни. В периода на университетската криза от 11.VI.1907 до 31.I.1908 по линия на „алтернативния университет" за извънреден професор по астрономия е назначен дотогавашният асистент по математика Никола Стоянов (роден през 1874 г. в Дойран). Той обаче не е чел лекции, а е бил командирован във Франция и Германия. Два пъти той е назначаван за редовен доцент по астрономия – през 1920 г. и 1926 г., но не е заемал длъжността (вероятно поради високите постове, които тогава е заемал в българската банкова система).

От 20.IX.1924 г. до деня на преждевременната си смърт – 18.V.1927 г., в Катедрата по астрономия е асистент Венцеслав Черноколев (роден през 1896 г. в Кюстендил), завършил с „лисансие де сианс" в Париж през октомври 1923 г. [6]. Неговата личност е малко известна на нашата астрономическа колегия.

След смъртта на проф. Бъчеваров за Катедрата по астрономия започва да се грижи Кирил Попов. От 29.VI.1927 г. в катедрата е назначен за асистент роденият в Струга през 1894 г. Димитър Дудулов. А от 1.XII.1928 г. като редовен доцент катедрата поема възпитаникът на Кирил Попов и бъдещ академик Никола Бонев (роден през 1898 г. в Стара Загора).

2. ПЪРВАТА НАУЧНА ПУБЛИКАЦИЯ ПО АСТРОНОМИЯ В БЪЛГАРИЯ

Първата съвременна научна публикация с оригинален принос по астрономия, написана на български и отпечатана в България, е на Кирил Попов, тогава асистент по астрономия. Тя се нарича „Една метода за определяне хелиографичното положение на слънчевите петна" и е посветена на наблюденията на Слънцето [2] (фиг. 1). Това е и неговият пръв научен труд и появата му не е случайна. По време на следването си от 1898 г. до 1902 г. по поръчение на проф. Бъчеваров Кирил Попов *всеки божи ден* без ваканциите, когато времето е позволявало, в 2 часа след пладне е извършвал (както пише самият той в [1]) „систематично наблюдение и зарегистриране на броя, големината и положението на слънчевите петна". Натрупаният при тези наблюдения опит е синтезиран в новата му идея за определяне на хелиографското положение на слънчевите петна, изложена в посочената публикация.

Впрочем активността на Кирил Попов като студент не се е състояла само в извършването на тези без съмнение отговорни наблюдения (питам се дали бихме могли да намерим сега студент-първокурсник, на когото да разчитаме, че ще свърши подобна работа за 4 години напред...). Както той пише, „По инициатива на студентите от нашия курс образувахме при факултета Студентско астрономическо общество, уставът на което бе одобрен от факултетния съвет и в което се изнасяха реферати из областта на астрономията. Бяха организирани наблюдения над падащите звезди: Леониди, Персеиди и др." [1]. Любопитно ще е за читателя да узнае, че този ПЪРВИ студентски кръжок по астрономия още функционира към университетската астрономическа обсерватория!

Една метода за опрѣдѣляне хелиографичното положение на слънчевитѣ петна.[1)]

При работата съ тази метода трѣбва отнапрѣдъ да ни сж познати елементитѣ на слънчевата ротация, а именно: дължината на вжзела и наклонътъ на слънчевия екваторъ къмъ еклиптиката.

Съ хелиометра, или съ другъ микрометръ, опрѣдѣляме видимия диаметъръ на слънчевия дискъ, позиционния жгълъ (Θ) на петното спрѣмо слънчевия центъръ и геоцентричното му отстояние отъ центра на слънчевия дискъ; знаемъ ли това — опрѣдѣляме и хелиоцентричното отстояние ρ (= as) на петното до пробода на слънчевата повърхнина съ земния радиусъ-векторъ.

Нека a (фиг. I.) ни прѣдставя слънчевото петно, EQ — слънчевия екваторъ и p — полюса на този екваторъ. Нека NS е прѣсѣчката на слънчевата повърхнина съ деклинационния кржгъ, който минава прѣзъ центра на слънцето. *Тази равнина* сѣче слънчевата повърхнина въ голѣмъ кржгъ, който се

Фиг. 1. Началната страница на първата съвременна научна публикация с оригинален принос по астрономия в България [2], написана от Кирил Попов.

3. АСИСТЕНТСКИ ГОДИНИ

От 1904 г. Кирил Попов е асистент по астрономия в Софийския университет. Още преди началото на учебната 1904/5 г. той започва усилено да подготвя упражненията, които трябвало да води със студентите от четвърта година. „Бях на 24 години, а те почти всички бяха по-възрастни от мен и не бяха новаци. Една част от упражненията бяха теоретични, а друга – работа с инструменти. Теоретичните задачи се даваха през време на лекциите от проф. Бъчеваров. В това отношение бях спокоен. Но работата с инструментите не трябваше да има само демонстративен характер. Трябваше студентът да почувства, че върши истинска научна работа... Имах пълната свобода при моята практическа работа със студентите и работата ми морално ме удовлетворяваше...

През първите две години щудирах основно „Аналитична механика“ на Пол Апел, надзърнах в „Небесната механика“ на Ресал, изучих два тома от „Небесната механика“ на Тисран, запознах се основно с някои от съчиненията по математическа физика на Поанкаре и с „Теория на вероятностите“ на Бертран. По този начин бях готов да започна разработката на една докторска теза. Интересите бяха вече събудени и аз продължавах в тази насока и в следните години...“ [1].

През 1906 г. Кирил Попов е изпратен от проф. Бъчеваров на специализация в Германия и Франция. В Германия посещава обсерваториите в Мюнхен и Хайделберг, където се запознава с идеите на Поанкаре за трите тела и с неговата теория на пертурбациите. От 1907 г. е в обсерваторията в Ница, където активно наблюдава астероиди. През есента на същата година е в Париж, където слуша в Сорбоната лекции. Оттам прави научни командировки в обсерваториите в Гринуич (едва ли е имало тогава по-прочута от нея) и в Страсбург – тогава все още немски град. В Париж получава от Поанкаре като тема за своята докторска дисертация задача да създаде теория на движението на малката планета Хекуба. Това е задача за трите тела (движението на този астероид се определя от Слънцето и Юпитер). Много е любопитен един пасаж, споменат от акад. Попов по този повод: „... Струва ми се, че аз съм първото лице от България, за което Парижкия университет зачете документите на Софийския за еквивалентни с парижките по отношение на докторския изпит“ [1]. Това се е дължало на безупречните му астрономични наблюдения, част от които вече били публикувани и изградили на автора си солиден авторитет.

Кирил Попов започва работа върху движението на Хекуба въз основа на теорията на Поанкаре, но се оказва, че получените резултати не съвпадат с наблюденията. И тогава, за да съгласува теорията с наблюденията, Кирил Попов коригира по подходящ начин пертурбационата функция на Поанкаре. Това е неговата главна заслуга за развитието на астрономията. Той завършва своя научен труд след 4 години. Защитата му е насрочена през есента на 1912 г. в Сорбоната на Париж . Но същевременно е обявена Балканската война и Кирил Попов е мобилизиран. Тогава по изключение журито е събрано въпреки ваканционния период още на 12.09.1912 г. Кирил Попов защитава блестящо своята прочута докторска дисертация, озаглавена „Sur le mouvement de 108 Hecube“ и веднага след защитата се завръща в България, където взема участие във войната.

През 1914 г. Кирил Попов е избран за доцент по основи за висшата математика. От този момент нататък започва възходът му като математик. Но макар и вече известен математик, Кирил Попов остава верен на любовта си към астрономията и в периода между двете световни войни чете поне 15 пъти лекции по небесна механика в Сорбоната, в Берлин и Хайделберг [1].

4. АСТЕРОИДЪТ ЦВЕТАНА (785 ZWETANA)

Първото име на астероид, свързано с България, дължим на авторитета на Кирил Попов, който той си изгражда в средите на европейската астрономия. Става дума за името на малката планета Цветана (785 Zwetana), открита на 30 март 1914 в Хайделберг от A. Massinger.

Обстоятелствата около това име са свързани с дружбата на Кирил Попов със станалия по-късно директор на Astronomisches Rechen-Institut в Хайделберг A. Kopff - негов личен приятел от хайделбергските му години, който е ръководил и докторската теза на Никола Бонев. В спомените на Кирил Попов четем: „Като директор на Astronomisches Rechen-Institut, една от задачите на който е да координира и редактира резултатите от наблюденията в различните обсерватории, нему и на института се падаше задачата между другото да идентифицира наблюдаваните астероиди с вече познатите и да кръщава новите със съответни имена... Един ден той ми каза, че е кръстил една от новооткритите малки планети на името на дъщеря ми Цветана, която (планета – б.а.) днес се носи из междупланетното пространство и всяка година се дават в астрономичните алманаси данни за нейното положение между звездите. В едно писмо до дъщеря ми той пише, че планетата Цветана е именно на дъщеря ми Цветана, а не на друга Цветана“ [1]. Тази голяма чест, оказана на Кирил Попов, се развива на фона на политическите събития в Германия през 30-те години: „Беше във времето на Хитлер. Един ден Kopff ми се оплака, че всички по-издигнати лица от партията на Хитлер отивали при него с пожелания да видят името си свързано с някоя новооткрита планета. Институтът имаше международен характер. За да не компрометира този му характер, той по един или друг начин избягваше да удовлетворява желанията на видните партийци. Даже името на Хитлер не се среща между малките планети.“ [1].

5. ПЪРВОКЛАСЕН УЧЕН И НАУЧЕН РЪКОВОДИТЕЛ

Почувствал грижата на своя ръководител проф. Бъчеваров, след време Кирил Попов осигурява възможността на своя асистент в Катедрата по висша математика Никола Бонев да специализира във Франция и Германия и след защита на докторска дисертация в Потсдам да поеме като доцент останалата без титуляр Катедра по астрономия. В спомените си Кирил Попов пише: „След първите парижки лекции бях поканен да гостувам в Берлинския университет. ... На встъпителната лекция присъства почти целият факултет. Тук за втори път се срещнах с проф. Айнщайн. ... Преди да започна лекцията, имах продължителен разговор с проф. Лудендорф (директор на Астрономическата обсерватория – б.а.), който се заинтересува от състоянието на астрономическата наука в България. Казах му между другото, че в този момент като стажант в Парижката обсерватория работи моят асистент Никола Бонев. ... Лудендорф ми съобщи, че той разполага в момента с кредит за един научен сътрудник и че е готов да приеме Н. Бонев. ... Тъй той можа да прекара две години в Берлин, да подготви докторска теза и да се завърне подготвен да заеме Катедрата по астрономия в Софийския университет (която след смъртта на проф. Марин Бъчеваров беше останала свободна и аз четях временно лекции по небесна механика). ... Освен мен Бонев е може би професорът от университета, който е имал най-много време (пълни четири години) да се готви в чужбина за катедрата, която заема" [1].

Акад. Наджаков в спомените си, написани по повод 90-тата годишнина на Софийския университет, казва следното: „Кирил Попов, който се готвеше за доцент по астрономия ... се ориентира (през 1914 г. – б.а.) към обявената вакантна доцентура по диференциално и интегрално смятане, където (после – б.а.) стана професор. При Кирил Попов, който беше първокласен научен ръководител, израснаха най-много и най-бързо млади научни кадри" [7]. Сред тези кадри акад. Наджаков посочва двама бъдещи академици – математика Никола Обрешков и астронома Никола Бонев. Наджаков пише тези редове с известна тъга и благородна завист към младите учени под крилото на Кирил Попов на фона на неговото собствено развитие, преминало при съвсем друг колегиален „климат" .

Н. Сретенова наскоро публикува някои документи от архива на Кирил Попов, в един от които, писан в края на 50-те години, може да се прочете следното: „При моята катедра като асистенти са се формирали следните професори: акад. Никола Обрешков, член-кореспондент на БАН проф. Никола Бонев, проф. Георги Брадистилов и доцентът Ярослав Тагамлицки" [8]. Трудно може да се намери в България друго толкова ярко съзвездие от таланти, групирани около своя учител и ръководител. По мое убеждение това е било възможно не на последно място и поради високоморалния възглед на Кирил Попов за това какво значи да се грижиш за научното израстване на по-младите си колеги – „Като съм се грижил за другите, аз не съм забравял, че не бива да се кича с чужди успехи, и гледах аз лично да имам един научен актив, който надхвърля обикновените изисквания, като публикувам работи, които задоволяват, мерени с европейски мащаб" [1].

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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100 ГОДИНИ ОТ РОЖДЕНИЕТО НА БЛАГОВЕСТ ДОЛАПЧИЕВ

ЕВГЕНИ ХРИСТОВ

The main facts and dates in the biography of the well-known bulgarian scientist B. *Dolaptschieff are given.*

Няма да скрия, че изправяйки се в този ден пред тази аудитория, се вълнувам повече от обичайното: защо – може би ще стане ясно в края на това изложение, където ще се постарая да отговоря на въпроса, кой е професор Долапчиев за моето поколение и за мене лично.

За научните интереси и приноси на проф. Долапчиев ще говорят моите колеги чл.-кор. Ст. Радев и проф. Л. Лилов. Моята задача тук е да очертая отначало в дати живота на проф. Долапчиев. Много от изброените по-долу факти са взети от личния му архив: „Долапчиев за Долапчиев" (Годишник на СУ, том 88, 1994), който ще цитирам често, с оглед максимална достоверност на изложението. Считам, че това е и по-интересно, защото само двама български математици – Кирил Попов и Благовест Долапчиев – са оставили автобиографични спомени.

Роден е точно преди 100 години на този ден (16 декември 1905 г.) в гр. Сливен. Той е петото и последно дете в семейството на Иван Димитров Ганев – Долапчията и Съба Драганова Гагова (долапчия е турска дума, на български – тепавичар, с който занаят бащата е изхранвал многочисленото си семейство.) Радостта на бащата след три дъщери да има син Драгомир е сигурно документирана, а за това, че след година и половина има още един син е засега известно само, че се казва Благовест Иванов Долапчиев.

За детството си впоследствие той пише затрогващо: „Беше към края на Европейската война, но тя още не бе свършила. Още зъзнехме, още гладувахме и мизерствахме.... Веднаж, минавайки край една фурна, където още се виждаха няколко невзети хляба, брат ми силно ме удиви, като каза: Иде ми да грабна един хляб – толкова съм гладен!“

Гимназиалното си образование завършва през 1924 г. в родния си град. Не е ясно както от собствените му архиви, така и от спомените на близките му кое събужда любовта му към математиката в ученическите години, за да обясним защо на предложението на (цитирам) „един възрастен висок, слаб и благ професор Иван Ценов“ , декан на Физико-математическия факултет:

„Помислихте ли добре? Вие можете да запишете и медицина, и химия – балът Ви позволява.“

Отговорих: „Не, избрал съм математика. Друго не бих записал!“

„Добре, добре“ , стана, ръкува се с мен и ми пожела на „Добър час“ ! Това бе на 14 октомври 1925 г. Не можех да зная, че след четири години, точно по същото време ще стана негов асистент.“

Любимият предмет на ученика Бл. Долапчиев е стенографията, овладял я е перфектно, което впоследствие му е помогнало да се издържа и като студент. Пише: „Следването ми в Университета, работата ми в Народното събрание напълно заеха всичките ми мисли. Едното беше живот, в който бях не само пасивен наблюдател, но и активен участник; другото – зрелище от най–висока класа. “ *(А сега не възприемаме ли Народното събрание като зрелище от класа?!)*

Младият Долапчиев няма средства да започне следването си веднага след завършване на средното си образование. Това го принуждава да работи през 1924-25 г. като учител по стенография в Самоков. Уроците на младия учител минават успешно дотолкова, че на учителски съвет учителят по български език протестирал, че Долапчиев развивал стенографията за сметка на останалите учебни предмети. Учениците учели и говорели само за стенография. В часовете по български език, по история, по математика те се занимавали само със стенография. Директорът възразил: „Развивайте и вие вашите предмети за сметка на стенографията.“ и вдига заседанието. Може би още този цитат дава отговор на въпроса, защо харесвахме лекциите на проф. Долапчиев.

Постъпва във Физико-математическия факултет на Софийския университет във времето, когато Факултетът е вече с изграден авторитет. Ето как студентът Долапчиев вижда своите учители: „Вече бяхме опознали нашите преподаватели и бяхме почнали задружен живот с професорите Табаков и Обрешков. Проф. Димитър Табаков ни четеше аналитична геометрия. Четеше увлекателно, дори със страст.

Ясно ли е?– обръщаше се той към аудиторията, подсмихвайки се и започвайки поредния турски анекдот:

Тя вашата прилича на ибрахимовата, дето си мислел, че всичко му е ясно и всичко разбира. „Връща се Ибрахим след 5-6 години следване по математика в Сорбоната. – Е, разбираше ли всичко дето ви го преподаваха там? – Да, ефендилер! Всичко разбрах, само едно нещо и досега не мога да разбера: защо всички професори в лекциите по математика говореха за две врати: дьо капи, дьо капи! Толкоз разбрал Ибрахим математика, колкото научил и френски език (за успокоение на тези, които не са разбрали анекдота ще кажа, че и Долапчиев отбелязва „късно схванахме тази шега на Табаков“).

Съвършено друг бе проф. Обрешков. Наднормено висок, с черна коса, млад (едва навършил 32 години), а редовен професор. Той никога не напускаше спокойното си състояние със снизходително - срамежлива усмивка. Четеше висша алгебра и развиваше всичко просто и леко, като да разказваше съвсем елементарни неща. Проф. Обрешков беше математик от международна класа.

Друг наш професор, който си беше извоювал международно име, беше проф. Кирил Попов – астроном, балистик, преподаващ диференциално и интегрално смятане. Той беше професор - артист: Когато преподаваше някое трудно място, си хапеше кокалчетата на пръстите, въртеше се на токове ту към дъската, ту към нас, а очите му – малки, черни, живи – ни завладяваха. Той също си служеше с duховитости, но от висшия математически елит сред който се движеше в Европа (Поанкаре, Пенлеве, Адамар – все крупни светила).“

През 1929 г. Долапчиев завършва с награда за отличен успех и през същата година е назначен за асистент към Катедрата по аналитична механика. Пише: „Трима асистенти на шест катедри (заедно с Долапчиев асистенти са Б. Петканчин и Г. Брадистилов), като дисциплините, по които водят упражнения, достигат шест. Имахме голямо самочувствие като асистенти в Университета. И може би това се дължеше тъкмо на това, че водехме така разнообразна учебна работа!“

През 1932 г. е уволнен по финансови причини, като критерият за уволнение е последният постъпил асистент: Долапчиев е най-младият от тримата асистенти. Пише: „Така беше почти навсякъде. Какво щях да правя например аз, когато след втората година на моето асистенство бях уволнен по бюджетни причини. Постъпих отново стенограф в Народното събрание, макар че бях вишист-математик, за когото нямаше и учителско място.“

Възстановен е на 1 юни 1933 г.

На 29 години (1935) заминава за Гьотинген с Хумболтова стипендия. Поетично звучи: „Град-градина, Меката на математиката, града на Гаус и Вебер, Клайн и Хилберт, в който е следвал проф. К. Гълъбов и специализирал акад. Чакалов.“ А иначе: „Цял семестър освен Guten Tag и Aufwiedersееn не проговорих немски. Нямаше с кого. Хазяите ми се оказаха темерути.“

Специализира при световно известния учен Лудвиг Прандъл – създател на теорията на граничния слой, където започва докторската си дисертация.

Любопитно е да се отбележи, че докторантите и специализантите по аеродинамика са минавали курс по безмоторно летене, за да почувстват непосредствено върху себе си подемната сила на самолетното крило. Още с първия си полет докторантът Долапчиев успява да приземи вертикално безмоторника и по чудо остава жив.

Стипендията свършва през 1937 г. и по тази причина самата защита е в София на академичен изпит при професорите Чакалов, Обрешков и Стоянов. Темата е „Принос към стабилитета на кармановите вихрови улици и траектории на отделните вихри“ . Това е първата дисертация, за която в Софийския университет е присъдена научната степен „доктор на математическите науки“ . През 1943 г. е избран за доцент в катедрата по аналитична механика, през 1947 г. е повишен в извънреден професор, през 1951 – за редовен професор и ръководител на катедрата, какъвто остава до края на живота си (3.02.1974 г.). През 1967 г. е избран за член-кореспондент на БАН по аналитична механика и т.н.

И още искам днес да подчертая, че зад изброяването на тези дати, което може да продължи, имаше един забележителен човек. Казвано е и от други, повтарям го и аз. „Долапчиев бе велик пример за висши житейски принципи и граждански добродетели. Всяка несправедливост предизвикваше у него почти физическа болка. Той имаше алергия към лицемерието и фалша, към конюнктурната адаптация и политика на момента, към административната кариера и властохолизма – и беше от различна кръвна група с носителите на тези вируси. Той беше доброжелателен и толерантен. Чувствата на завист и ревност му бяха абсолютно чужди и той се радваше на успехите на другите както на своите собствени.“ (Написано е от проф.Чобанов в книгата му „Благовест Долапчиев“ (1993).)

И за да допълня казаното, ще се постарая накратко да очертая ролята на проф. Долапчиев за моето поколение. Една задача се решава лесно, когато знаете отговора, но отговорът е на последната страница! А в далечната есен на 1960 г. ние бяхме само студенти, записали да следват математика и само математика, само и само да отърват две години военна служба!

Първи курс минаваше под знака на лекциите по диференциално и интегрално смятане на проф. Я. Тагамлицки. Разказваше увлекателно и с поразителната строгост, която не се вписваше в нашите ученически представи за математиката. И най-важното – накара ни да учим много старателно и по време на семестъра, за което немаловажна роля играеше и фактът, че често трябваше от място да отговорим на въпросите му (някои от тях доста хапливи!). На втори курс проф. Долапчиев започна лекциите си по аналитична механика. Това беше друг стил на изложение, друг стил на мислене. Отговаряше на нашето вътрешно желание да усетим, че зад формулите има един реален свят и това пленяваше. Анализът, геометрията и алгебрата се събираха в една удивителна хармония. Готвейки се за този доклад, прелистих отново неговия учебник по аналитична механика, признавам сега ми се видя по-труден, отколкото тогава! Изпита взех с отлична оценка, участвах и с доклади на семинара му. Вероятно това беше причината в края на втори курс да предложи на мен (заедно с колегите К. Кирчев и Л. Лилов) да продължим образованието си в Съветския съюз. Продължих образованието си в Харковския университет, където завърших и аспирантурата. Беше ме изпратил с пожеланието да специализирам квантова механика. Дисертацията ми беше върху уравнението на Шрьодингер.

Върнах се отново вече като асистент в катедрата по аналитична механика. Водех упражнения по ДИС при проф. Тагамлицки и по аналитична механика при проф. Долапчиев. Проф. Тагамлицки събираше екипите всяка седмица и даваше стриктни указания за задачите, които трябваше да решаваме. Признавам, действаше малко потискащо. Ние вече бяхме кандидати на математическите науки и имахме известно самочувствие. При проф. Долапчиев указанията се свеждаха до една среща в началото на семестъра, в която се очертаваше общата схема, към която следваше да се придържаме. Това действаше стимулиращо. Спомням си, че с другия асистент, колегата Ст. Радев, често обсъждахме задачите за упражнения, прибирайки се вкъщи от Факултета през Борисовата градина към Орлов мост. Упражненията по механика имат специфични трудности, а може би и ние нямахме необходимия опит. Компенсирахме го по своеобразен начин. Радев влизаше по средата на упражнението ми и ми пъхаше едно сгънато листче „Забравил си решението на задачите!“ , после аз в неговия час – „Забравил си условията на задачите!“ и обратно. Имаше известен положителен ефект, посещаемостта на студентите се увеличи. Лошото беше, че като руски възпитаници, ние правехме често грешки в превода на математическата терминология. Координатната система с начало О-прим ставаше О-щрих, а с начало О-секонд ставаше О-два щриха, а ние за студентите: асистента „О-щрих“ , асистента „О-два щриха“ . Позволявам си тук да разкажа тези епизоди, защото вярвам, че от по-добрия свят, където отиде проф. Долапчиев, сигурно гледа този свой 100-ен рожден ден с доброжелателно-снизходителната си усмивка, нашепвайки ми „Знам, знам, доста глупости правехте, но и аз не бях светец и слава богу!“

А за себе си сега ще кажа, както Долапчиев пише за учителя си по стенография: „Говореше бавно, а вървеше дори още по бавно, понеже беше в напреднала възраст – прехвърлил шестдесетте.“

Време е да завършвам. Ще цитирам известния руски математик Левитан (не художника), който казваше, че за да станеш научен работник, трябват три условия: първо да имаш определени дадености, например за математика, а по-добре за математика и механика, второто условие е школата и учителите и третото условие е как по-нататък ще използваш първите две. Няма да коментирам доколко отговарям на първото и третото условие, но за второто убедено мога да кажа: започнах като студент във Факултета по математика и механика на Софийския университет и мой любим учител беше проф. Долапчиев и това определи в голяма степен моите научни интереси!

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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ИЗСЛЕДВАНИЯТА НА БЛАГОВЕСТ ДОЛАПЧИЕВ ПО МЕХАНИКА НА ФЛУИДИТЕ

СТЕФАН РАДЕВ

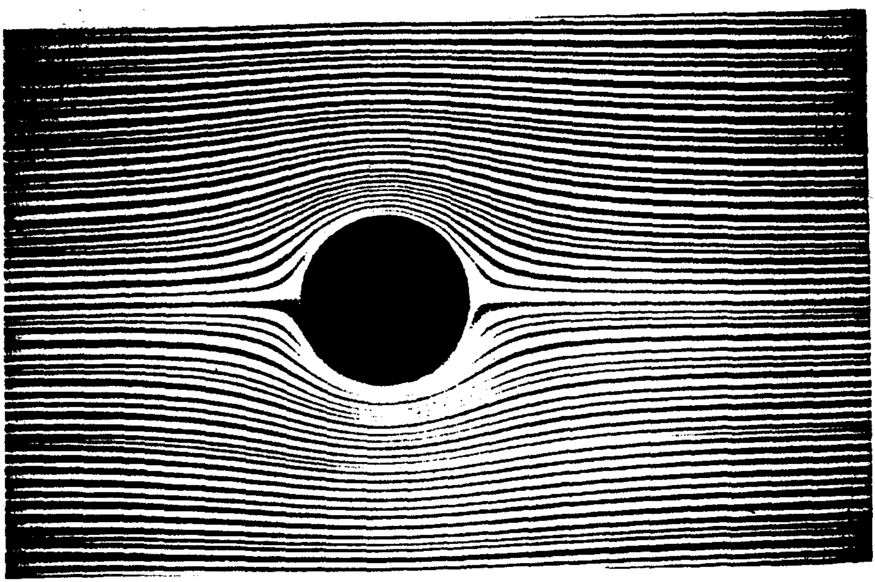
The present lecture concerns Blagovest Dolaptschieff's works on fluid mechanics. The most valuable of his contributions in the field of fluid mechanics are outlined, including the well known Maue-Dolaptschieff condition for the stability of two-parametric vortex streets.

Първото ми впечатление за международната известност на проф. Долапчиев (така винаги съм се обръщал към него) като изтъкнат специалист по механика на флуидите е от студентските ми години в Ленинградския (сега Санктпетербургски) университет. Бях приятно изненадан, че моят научен ръководител Сергей Васильевич Валландер (тогава професор и ръководител на Катедрата по аерохидродинамика в същия университет) се оказа добре запознат с трудовете на Бл. Долапчиев по Кармапови вихрови улици. Заинтригуван, направих справка в университетската библиотека и там намерих дисертацията на Бл. Долапчиев по Кармапови вихрови улици. По-късно като хоноруван асистент на проф. Долапчиев многократно съм имал възможност да разговарям с него на различни теми от механика на флуидите, включително и в областта на Кармановите вихрови улици; на една от темите ще се върна в края на моя доклад.

Активните изследвания на Благовест Долапчиев по механика на флуидите започват по време на специализацията му в Гьотинген под ръководството на проф. Лудвиг Прантъл (L. Prandtl). По това време в Гьотингенския университет Прантъл ръководи един от водещите центрове по механика на флуидите в световен мащаб, в който работят учени, чиито имена са останали завинаги в учебниците по тази дисциплина. С този център по това време единствено може да се сравнява Централният аерохидродинамичен институт в Москва (ЦАГИ). (Ще отбележим, че Долапчиев е бил сред малкото сътрудници в Гьотинген, който еднакво добре е познавал както руските, така и западните изследвания по аерохидродинамика – руското наименование на „механика на флуидите".)

За основните научни направления в този център най-добре можем да научим от самия Долапчиев, от неговата встъпителна лекция като доцент, публикувана в Годишника на Софийския университет (т. LX, год. 1943 - 1944, 55-75) под заглавие „Математически решения в аеродинамиката на летенето". Тя е посветена на състоянието на един от най-първостепенните проблеми по това време – разработването на теорията на крилото и самолетното витло. И в моя доклад, за да открием приносите на Бл. Долапчиев в областта на механиката на флуидите, се налага да се спрем на някои етапи от историята на нейното развитие, както и на някои уводни сведения за нейния математичен апарат.

През 1903 г. братя Райт осъществяват първия полет на двуплощен самолет с двигател с вътрешно горене (с продължителност 59 с). Появата на авиацията налага своя отпечатък върху цялостното развитие на механиката на флуидите през първата половина на двадесети век. В частност теорията на крилото възниква от необходимостта да се пресметне резултантната на силите на взаимодействие между обтичаното тяло и течението на флуида.



Фиг. 1. Безциркулационно (симетрично) обтичане на прав кръгов цилиндър

На фиг. 1 е показана фотография на напречно сечение на флуидно течение около прав кръгов цилиндър. Тъмните ивици съответстват на траекториите на флуидните частици, които започват движението си отляво със скорост, перпендикулярна на образуващите на цилиндъра, след което го заобикалят и продължават движението си надясно. Силите, които действат върху напречното сечение на цилиндъра, най-общо са от два типа: сили на триене между цилиндъра и флуида и сили на налягането. За някои флуиди (като въздуха и водата) силите на вътрешно триене, наричани вискозни, в първо приближение могат да се пренебрегнат и така достигаме до механиката на невискозните (идеалните) флуиди.

За аналитично представяне на течението на фиг. 1 ще въведем отправна координатна система $Oxyz$ с начало в центъра на кръга и ос Oz, перпендикулярна на равнината на фигурата. Тогава на всяка точка от равнината Oxy на течението съответства вектор $\vec{v}(v_x, v_y, v_z)$, задаващ скоростта на флуида в тази точка. За пресмятането на така въведеното поле на скоростите механиката на невискозните флуиди използва следните предположения:

1. Плътността на флуида ρ е постоянна, т.е. изпълнено е уравнението на непрекъснатостта

$$div\vec{v} = 0; \tag{1}$$

2. Течението е стационарно, т.е. не зависи от времето:

$$\frac{\partial}{\partial t} = 0; \tag{2}$$

3. Флуидните частички се движат по един и същ начин в равнини, успоредни на Oxy, или

$$v_z = 0, \ \frac{\partial}{\partial z} = 0; \tag{3}$$

4. Полето на скоростта е безвихрово:

$$rot\vec{v} = \vec{0}. \tag{4}$$

При тези предположения на течението на фиг. 1 може да се съпостави аналитична функция от вида

$$\Phi = \varphi + i\psi = U\left(\zeta + \frac{a^2}{\zeta}\right) + \frac{\Gamma}{2\pi i}\ln\zeta, \tag{5}$$

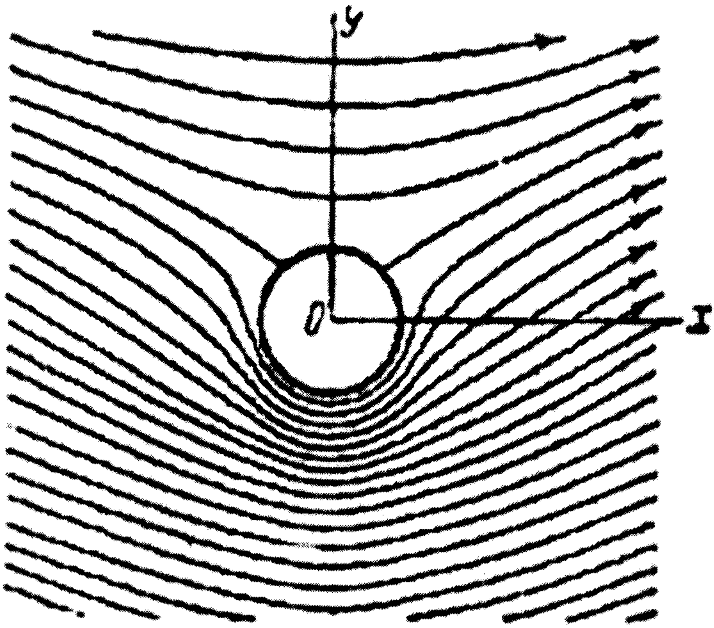
наречена комплексен потенциал на скоростта. В горната формула с $\zeta = x + iy$ е означена произволна точка от равнината на течението, с U-несмутената от присъствието на цилиндъра скорост на течението, а Γ е произволна интеграционна константа, наречена циркулация на скоростта. Траекториите на частиците се получават от условието:

$$\psi = Const, \tag{6}$$

като в разглеждания случай са едновременно силови линии на скоростното поле, наречени още токови линии.

С помощта на Г резултантната сила на налягането, приложена към цилиндъра, може да се представи в следния вид:

$$P_x = 0, P_y = \rho V \Gamma. \tag{7}$$



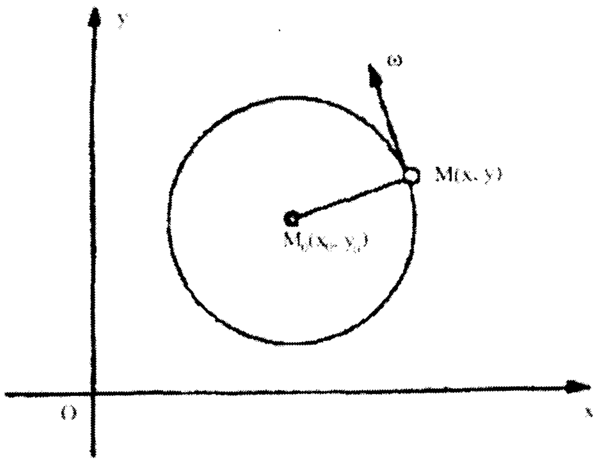
Фиг. 2. Токови линии при обтичане на цилиндър с ненулева циркулация

При $\Gamma = 0$ от (7) получаваме известния парадокс на Даламбер: цилиндърът не изпитва никакво въздействие от страна на флуида. Това е точно случаят, показан на фиг. 1, за който допълнително ще отбележим, че течението е симетрично спрямо оста Ox. При $\Gamma \neq 0$ силата, приложена върху цилиндъра, е перпендикулярна на направлението на течението и се нарича подемна сила.

Пример на течение с $\Gamma \neq 0$ е показан на фиг. 2. Разрушаването на симетрията на течението спрямо оста Ox е индуцирано от последното събираемо в (5), което съответства на комплексния потенциал на точков вихър

$$\Phi = \frac{\Gamma}{2\pi i} \ln(\zeta - \zeta_0), \tag{8}$$

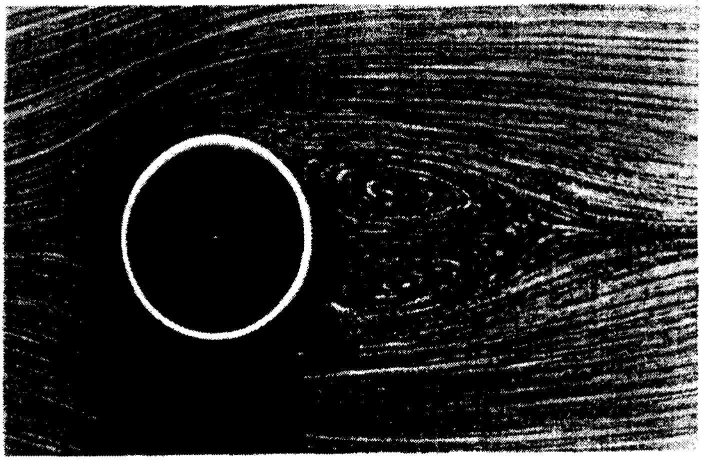
разположен в центъра на кръга ($\zeta_0 = 0$). Действието на точков вихър върху произволна флуидна частичка M, илюстрирано на фиг. 3, се свежда до ротация на частицата около вихровия център M_0.



Фиг. 3. Ротация на флуидна частица около вихров център

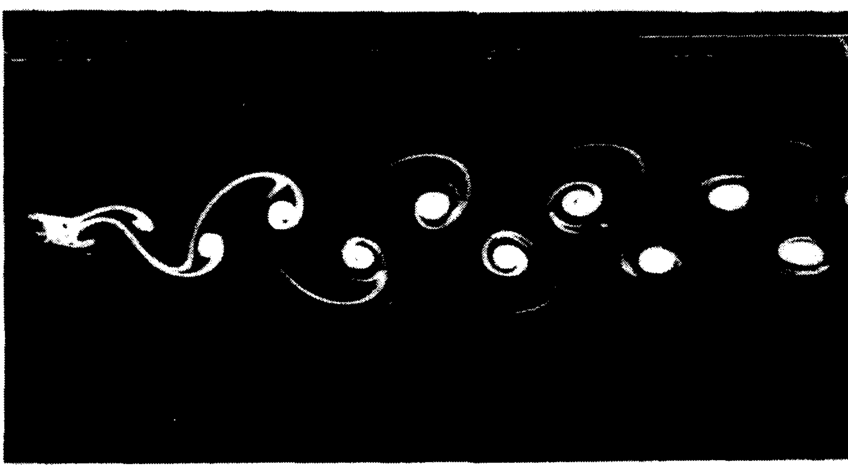
Липсата на тангенциална компонента (срв. (7)) на силата на взаимодействие между цилиндъра и флуида, т.е. на сила на съпротивление, е естествен резултат от пренебрегването на вътрешното триене (вискозитета) на флуида. Това би трябвало да ни наведе на мисълта, че в невискозен флуид сили на съпротивление не могат да се получат. Подобно заключение обаче е само отчасти вярно. То се отнася за онази съставка на съпротивлението, която се дължи на силите на триене между флуида и повърхността на обтичаното тяло (наричано също вискозно съпротивление). Съществува и втора съставка на силата на съпротивление, наречено вихрово, която се индуцира от структури в полето на течението, подобни на точковите вихри. Такива вихрови структури лесно се наблюдават в реални условия в зависимост от големината на несмутената скорост U. Два характерни случая са показани на фиг. 4 и 5.

На фиг. 4 ясно се вижда оформилата се двойка вихри зад цилиндъра, разположени симетрично спрямо средната линия на течението. От кинематична гледна точка те са еквивалентни на центрове на локална ротация на флуида (срв. фиг. 3).



Фиг. 4. Обтичане на кръгов цилиндър от вискозен флуид

На фиг. 5 зад цилиндъра вече се наблюдават две редици вихри, подредени шахматно. Тази вихрова конфигурация е известна като Карманова вихрова

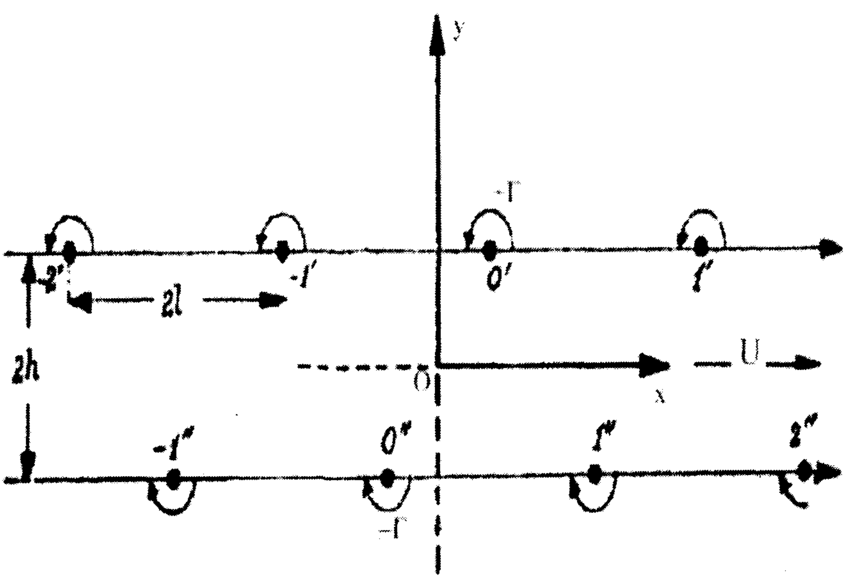


Фиг. 5. Карманова вихрова улица зад кръгов цилиндър

улица. Такива улици можем да наблюдаваме зад потопените в пълноводни реки мостови колони. На тях се дължат чистите тонове, с които „пеят“ телефонните жици, обдухвани от насрещен вятър. На Кармановите вихрови улици ще се върнем отново по-долу, тъй като в тази област са едни от най-съществените приноси на Долапчиев.

Проблемът обаче е там, че подобни вихрови конфигурации от невискозен флуид *не могат да бъдат генерирани. За тяхното обяснение* трябва отново да бъде привлечен вискозитетът на флуида. През 1904 г. споменатият по-горе изтъкнат немски учен Л. Прантъл формулира основополагащата идея, че разликата между вискозното и невискозното течение е концентрирана в *един много тънък слой до повърхността на тялото*, наричан в съвремената литература граничен слой на Прантъл. Именно този слой играе ролята на генератор на вихрите, чието движение може отново да се моделира с методите на механиката на невискозните флуиди.

През 1912 г. Т. Карман (Teodor Karman), впоследствие един от най-известните сътрудници на Прантъл от Гьотингенската школа, предлага една изключително плодотворна схема за пресмятане на съпротивлението, което изпитва кръгов цилиндър, при това, оставайки в рамките на хидромеханиката на невискозните флуиди. Карман заменя реалните вихри от фиг. 5 с точкови вихри (вж. фиг. 6), които подрежда в две успоредни редици с равни по големина, но противоположни по знак циркулации за вихрите от горната и долната редица.

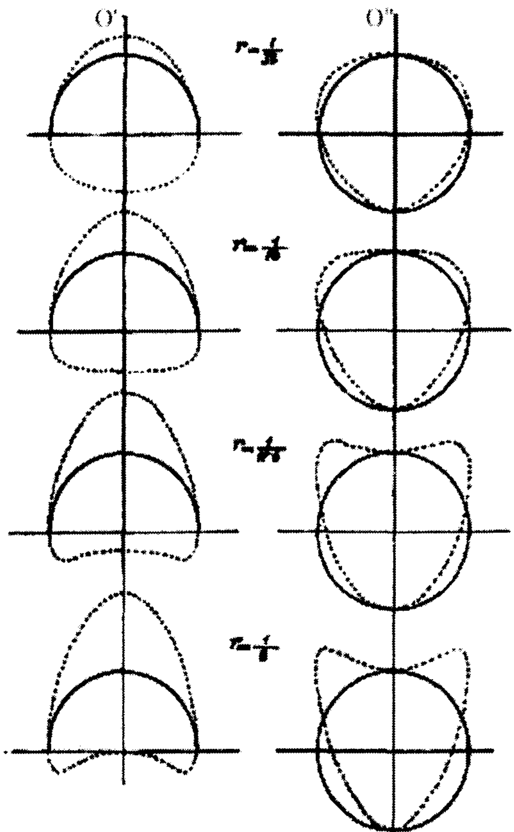


Фиг. 6. Шахматно подредена Карманова улица от точкови вихри

Карман доказва, че при всяка стойност на отношението $2h/2l$ между ширината на улицата и разстоянието между съседните вихри във всяка редица шахматното подреждане осигурява движение на вихровата улица като абсолютно твърдо тяло със скорост, успоредна на направлението на редицата. За такива улици Карман извежда своето знаменито условие за устойчивост по първо приближение, а именно:

$$sh\frac{\pi h}{l} = 1, \tag{9}$$

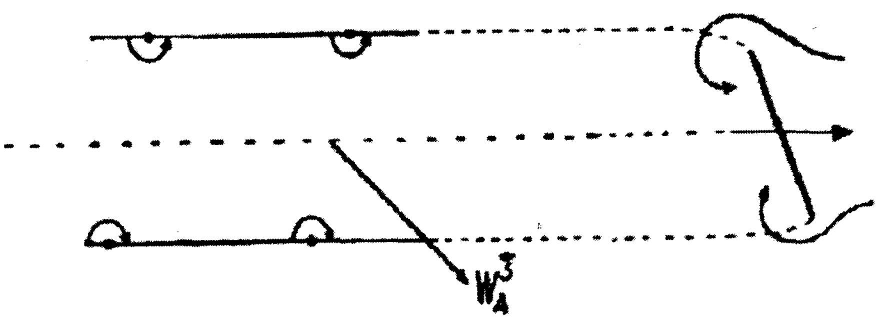
откъдето намираме $h/l \approx 0,281$. С помощта на това условие Карман пресмята вихровото съпротивление на кръгов цилиндър.



Фиг. 7. Траектории по Долапчиев на смутените вихри в устойчива Карманова улица

Темата на докторската дисертация на Бл. Долапчиев е зададена от Прантъл. На младия учен е предложено да изследва траекториите на смутените вихри както за устойчива, така и за неустойчива конфигурация на Карманова вихрова улица. С тази задача Долапчиев се справя успешно, както се вижда от публикацията му „Принос към стабилитета на Karman'овите вихрени улици и траекториите на отделните вихри", отпечатана в Списание на БАН, кн. 57/1938 г.(149 - 211). Фиг. 7 е заимствана от дисертацията на Долапчиев и илюстрира траекториите на двойката смутени вихри O'и O'' от фиг. 6 както в линейно (непрекъснатите полуокръжности и окръжности), така и в нелинейно приближение – пунктирните криви (с точност до квадратични членове).

Съществeният принос на Долапчиев в областта на вихровите улици обаче се състои в обобщението на Кармановото условие за устойчивост върху двупараметрични вихрови улици. По определение (вж. фиг. 8) това са две успоредни вихрови редици, отместени на произволно разстояние d една спрямо друга в надлъжно направление, където $0 < d < l/2$.



Фиг. 8. Двупараметрични вихрови улици

Понастоящем полученото от Долапчиев условие за устойчивост на двупараметрична вихрова улица е известно като условие на Мауе-Долапчиев (Maue-Dolaptchiev) и се представя във вида

$$sh\frac{\pi h}{l} = \sin\frac{\pi d}{l}. \tag{10}$$

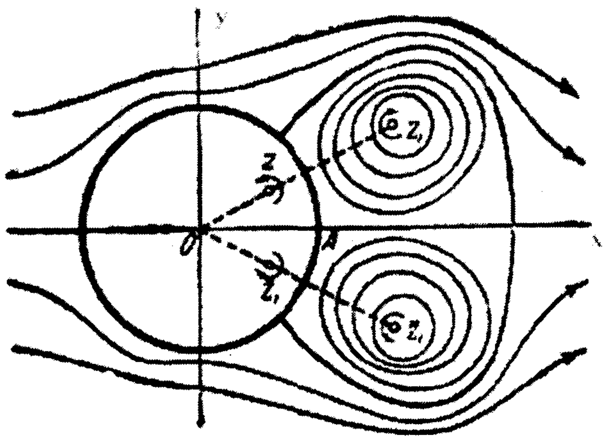
Очевидно при $d = l/2$ горното условие се свежда до това на Карман.

Нашият обзор ще бъде непълен, ако не отбележим статията на Бл. Долапчиев (в съавторство с Бл. Сендов), публикувана в Доклади на АН на СССР (т. 128, 1959, 53 - 56) и посветена на една алтернативна схема за пресмятане на съпротивлението на цилиндър в невискозен флуид. Тази схема е предложена от Фьопл (L. Föppl, 1913) и Рубах (H. Rubach, 1916) и изхожда от конфигурацията

„цилиндър-вихрова двоица", показана на фиг. 4. Комплексният потенциал на това течение може да се запише в следния вид:

$$\Phi = U\left(\zeta + \frac{a^2}{\zeta}\right) + \frac{i\Gamma}{2\pi}\ln\frac{(\zeta - z_1)\left(\zeta - z_1^{-1}\right)}{(\zeta - \bar{z}_1)\left(\zeta - \bar{z}_1^{-1}\right)} \quad . \tag{11}$$

Точковите вихри и токовите линии, съответстващи на този потенциал, са скицирани на фиг. 9.

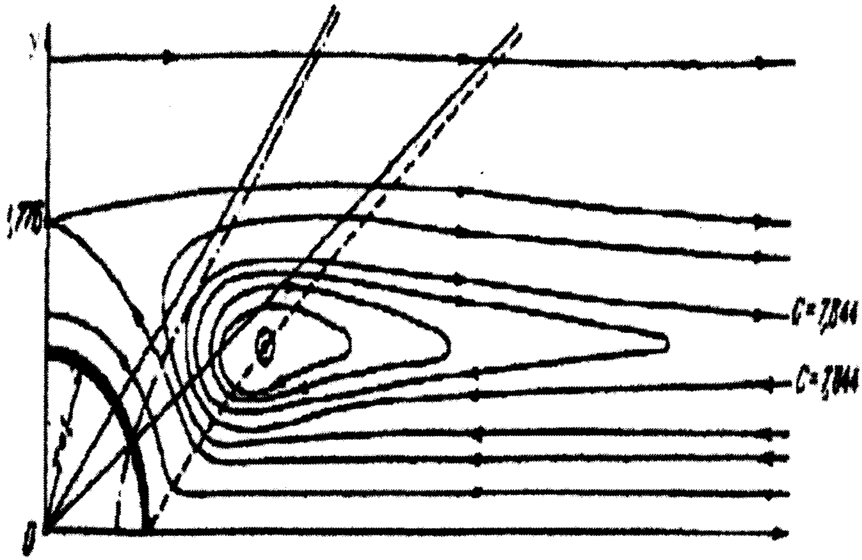


Фиг. 9. Симетрично обтичане на кръгов цилиндър с двоица вихри зад него.

В гореспоменатата публикация Бл. Долапчиев предлага пълно качествено изследване на траекториите на вихрите, резултатите от което са изобразени на фиг. 10.

На горната фигура (симетрична спрямо хоризонталната ос) кривата на Фьопл е показана с пунктир. Тази крива е геометричното място на равновесните (неподвижните) конфигурации на вихровата двоица. При малки смущения на двоицата (но запазващи симетрията на течението) траекториите на вихрите се преобразуват в затворени криви около точките на равновесие. Съществува незатворена критична траектория (номерирана на фиг. 10 с $C = 7,844$), която отделя затворените от незатворените траектории

В заключение ще се спра на последната работа на Долапчиев, написана по времето на тежкото му боледуване и излязла след смъртта му: „Върху една непозната класическа теорема (на SYNGE) в един стар хидродинамичен



Фиг. 10. Крива на Фьопл и траектории на двоица вихри зад кръгов цилиндър.

проблем (на KARMAN)”. В нея той публикува (Годишник на СУ, т. 67, 1972/73, 355 - 362) едно строго математическо доказателство на Синг – изтъкнат учен от Ирландската кралска академия. Доказателството е направено по молба на Долапчиев и се отнася за скоростта, която един точков вихър индуцира върху друг точков вихър. С него беше потвърдено интуитивното приемане, което Долапчиев намираше за неудовлетворително, че всеки вихър действа върху съседен на него вихър по същия начин, както и върху произволна флуидна частица, независимо от обстоятелството, че вихрите са особени точки в полето на течението.

Накрая ще завърша с думите на проф. Чобанов – най-близкият сътрудник и колега на проф. Долапчиев. Те бяха казани пред възпоменателната научна сесия по повод 20-тата годишнина от смъртта на проф. Долапчиев: „Докато цитираните публикации на Карман са работи на инженер, статиите на Долапчиев в новооткритите територии – от първата до последната – са работи на математик. Оттук и името му в тази област... Той бе авторитет в тази област и поради това охотно канен и внимателно изслушван на всички онези конгреси, конференции, сесии, семинари и пр., на които е присъствал и докладвал.“

* * *

Заключителна забележка: В текста са използвани следните оригинални фигури от съответните трудове на Бл. Долапчиев: фиг. 3, 6, 7, 8, 9, 10.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 98

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FACULTE DE MATHEMATIQUES ET INFORMATIQUE
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ВЪРХУ ИЗСЛЕДВАНИЯТА НА ЧЛ.-КОР. БЛ. ДОЛАПЧИЕВ ПО АНАЛИТИЧНА МЕХАНИКА [1]

ЛЮБОМИР ЛИЛОВ

Main results of corresponding member of the Bulgarian Academy of Sciences Professor Dolapchiev in the field of Analytical Mechanics concerning motion equations are presented. They are based on the variational principles of mechanics and can be applied to both holonomic and nonholonomic systems.

Ще започна с удивителния факт, че до 1965 г. проф. Долапчиев няма нито една работа по аналитична механика, въпреки че званията „доцент“ и „професор“, които е получил, са точно по аналитична механика, а от 1951 г. е и ръководител на Катедрата по аналитична механика в Софийския университет. Както свидетелства проф. Чобанов, повод да започне изследвания в областта на нехолономните системи е един елементарен курс по механика от Нилзен (Nilsen, J. Vorlesungen über elementare Mechanik. Berlin, Springer, 1935), в който курс проф. Долапчиев се натъква на непозната форма на уравнения на движение, които той в следващите си публикации нарича уравнения на Нилзен. Науката трябва да е благодарна на това, общо взето, случайно събитие, защото този начален момент отключва една кипяща изследователска дейност, довела до публикуването на 23 работи, получили световно признание и донесли на създателя си заслужена слава. През 1965 г. проф. Долапчиев е точно на 60 години и с тези изследвания той изживява втората си математическа младост. Резултатите му в аналитичната механика са едни от най-добрите, ако не и най-добрите, които той е постигнал като учен. Те продължават и обобщават

[1] Доклад, изнесен на честването на 100-годишнината от рождението му.

изследванията на акад. Ценов, друг български учен, намерил световно признание с изведените от него уравнения за нехолономни системи, известни днес като уравнения на Ценов, така че върху тази област на механиката е сложен силен български отпечатък и може да се говори за българска научна школа.

Много е трудно, ако не и невъзможно да се излагат резултати, предполагащи познаването на специфичен абстрактен математически апарат пред нехомогенна аудитория. Ще се опитам по възможно най-елементарен начин да представя същността на постиженията на проф. Долапчиев в областта на аналитичната механика.

Нека да имаме система от N материални точки с маси m_ν, радиус-вектори $\mathbf{r}_\nu(x_\nu, y_\nu, z_\nu)$, на които действат сили $\mathbf{F}_\nu$ $(\nu = 1, 2, \ldots, N)$. Ако системата е свободна, то съгласно втория закон на Нютон движението на точките се описва с уравненията

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu \left(t, \mathbf{r}_\mu, \dot{\mathbf{r}}_\mu\right), \quad \nu = 1, ..., N. \tag{1}$$

Имаме $3N$ диференциални уравнения за толкова скаларни неизвестни x_ν, y_ν, z_ν. Нека сега на системата са наложени d крайни или геометрични връзки:

$$f_\alpha \left(t, \mathbf{r}_\nu\right) = 0 \; (\alpha = 1, ..., d). \tag{2}$$

Такава система се нарича холономна. Да предположим, че тя е разрешена и d от променливите са представени като функция на останалите $n = 3N - d$ или по-общо: радиус-векторите на точките са представени като функции на времето и n параметри, наречени обобщени координати:

$$\mathbf{r}_\nu = \mathbf{r}_\nu \left(t, q_1, q_2, ..., q_n\right),$$

по такъв начин, че уравненията на връзките (2) се удовлетворяват тъждествено.

$$f_\alpha \left(t, \mathbf{r}_\nu \left(t, q_1, q_2, ..., q_n\right)\right) \equiv 0 \quad (\alpha = 1, ..., d).$$

Геометричните връзки налагат ограничения не само върху положенията на точките, но и върху техните скорости и ускорения. Действително след еднократно и двукратно диференциране на уравненията на връзките намираме

$$\sum_{\nu=1}^{N} \frac{\partial f_\alpha}{\partial \mathbf{r}_\nu} \bullet \dot{\mathbf{r}}_\nu + \frac{\partial f_\alpha}{\partial t} = 0, \tag{3}$$

$$\frac{\partial f_\alpha}{\partial \mathbf{r}_\nu} = \frac{\partial f_\alpha}{\partial x_\nu}\mathbf{i} + \frac{\partial f_\alpha}{\partial y_\nu}\mathbf{j} + \frac{\partial f_\alpha}{\partial z_\nu}\mathbf{k}, \quad \dot{\mathbf{r}}_\nu = \dot{x}_\nu \mathbf{i} + \dot{y}_\nu \mathbf{j} + \dot{z}_\nu \mathbf{k},$$

$$\sum_{\nu=1}^{N} \frac{\partial f_\alpha}{\partial \mathbf{r}_\nu} \bullet \ddot{\mathbf{r}}_\nu + \sum_{\nu=1}^{N} \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \mathbf{r}_\nu}\right) \bullet \dot{\mathbf{r}}_\nu + \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial t}\right) = 0. \tag{4}$$

Ако заместим определените от (1) ускорения $\ddot{\mathbf{r}}_\nu = \dfrac{1}{m_\nu}\mathbf{F}_\nu\left(t, \mathbf{r}_\mu, \dot{\mathbf{r}}_\mu\right)$ в двукратно диференцираните уравнения на връзките (4), то те няма да се удовлетворят. В този случай материално осъществените връзки действат на точките с допълнителни сили, наречени реакции на връзките или само реакции, така че сега движението на точките се подчинява на уравненията

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu + \mathbf{R}_\nu, \ \ \nu = 1, ..., N \tag{5}$$

и определените от (5) ускорения удовлетворяват уравненията на връзките (4). Реакциите обаче предварително не са известни и те въвеждат допълнителни $3N$ неизвестни $(R_{\nu x}, R_{\nu y}, R_{\nu z})$ Задачата на механиката става неопределена – необходими са още $n = 3N - d$. съотношения, които да допълнят съотношенията (2) и (5). Тези съотношения се получават, ако се приеме постулатът за идеалност на връзките: Сумата от елементарните работи на реакциите на връзките за произволно виртуално преместване на системата $\delta\mathbf{r}_\nu$ е равна на нула.

Постулат за идеалност на връзките:

$$\delta' A = \sum_{\nu=1}^{N} \mathbf{R}_\nu \bullet \delta\mathbf{r}_\nu = 0. \tag{6}$$

Какво е виртуално преместване? Това е съвършено фундаментален въпрос, изясняването на който е отнело десетилетия, ако не и столетия, и който периодично се дискутира дори и в наши дни. По времето на Лагранж нещата са изглеждали ясни и виртуалните премествания са били определяни като разлики между две възможни (допускани от връзките) елементарни премествания за едно и също положение на системата в един и същ момент от време:

$$\delta\mathbf{r}_\nu = \left(\dot{\mathbf{r}}'_\nu - \dot{\mathbf{r}}_\nu\right) dt,$$

където скоростите $\dot{\mathbf{r}}'$ и $\dot{\mathbf{r}}$ удовлетворяват продиференцираните уравнения на връзките (3), т.е.

$$\sum_{\nu=1}^{N} \frac{\partial f_\alpha}{\partial \mathbf{r}_\nu} \bullet \delta\mathbf{r}_\nu = 0.$$

Ако заместим реакциите от (5) в (6), намираме следната форма на постулата за идеалност на връзките, известна като общо уравнение на динамиката или принцип на Даламбер-Лагранж:

$$\sum_{v} \left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right) \bullet \delta\mathbf{r}_v = 0. \tag{7}$$

От общото уравнение на динамиката Лагранж директно извежда през 1788 г. знаменитититите си уравнения за холономни системи

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i, \qquad (i = 1, \ldots, n)\,.$$

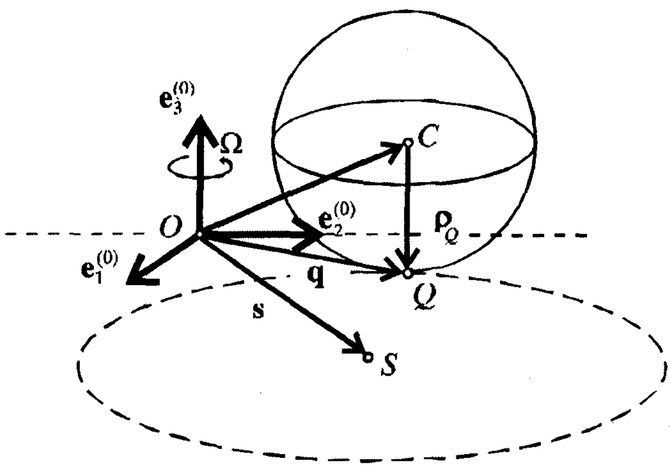
където T е кинетичната енергия на системата

$$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \dot{\mathbf{r}}_\nu^2.$$

Развитието на техниката довежда обаче до необходимостта в аналитичната механика да се разглеждат и нехолономни системи, т.е. несвободни системи, на които са наложени диференциални неинтегруеми връзки, съдържащи скоростите:

$$\varphi_\beta\left(t, \mathbf{r}_\nu, \dot{\mathbf{r}}_\nu\right) = 0 \ \ (\beta = 1, ..., g).$$

Като пример да разгледаме движението на сфера по равнина, като предполагаме абсолютно грапав контакт, т.е. сферата не може да се хлъзга и да буксува, а се търкаля по равнината.



Фиг. 1

Условието, че сферата не може да се хлъзга, означава, че скоростта на контактната точка Q е нула, което води до две неинтегруеми съотношения в тази задача.

Уравненията на Лагранж са невалидни за нехолономни системи. В десетилетията и столетията след Лагранж се появяват различни форми на уравнения за нехолономни системи, та чак до наши дни. Да спомена уравненията на Routh (1877), Voss (1885), Maggi (1896), Чаплыгин (1897), Volterra (1898), Appelle

(1899) - Gibbs (1879), Boltzmann (1902) - Hammel (1904). Особено елегантни са уравненията на Appelle (получени 20 години преди него от Gibbs и останали незабелязани):

$$\frac{\partial S}{\partial \ddot{q}_i} = Q_i,$$

където

$$S = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \ddot{\mathbf{r}}_\nu^2$$

е енергията на ускоренията. Този израз обикновено е много сложен и се пресмята доста по-трудно от кинетичната енергия. Търсейки баланс между доапеловите уравнения за нехолономни системи, които имат сложна форма, но боравят с по-простия израз на кинетичната енергия и уравненията на Апел, акад. Ценов, който е ученик на Апел, достига до следната форма уравнения на движение, известни днес като уравнения на Ценов (1952):

$$\frac{1}{2}\left(\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - 3\frac{\partial T}{\partial q_i}\right) = Q_i, \qquad (i = 1, \ldots, n).$$

Това са уравнения за холономни системи, но изведени с оглед на приложението им за нехолономни системи. В цитирания по-горе учебник на Нилзен проф. Долапчиев се натъква на следната форма уравнения на движение:

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i} = Q_i, \qquad (i = 1, \ldots, n).$$

Еднаквата структура на двете форми дава импулс на проф. Долапчиев да започне интензивни изследвания и през 1965 г. той достига до най-общата форма уравнения на движение

$$\frac{1}{n}\left(\frac{\partial \overset{(n)}{T}}{\partial \overset{(n)}{q}_i} - (n+1)\frac{\partial T}{\partial q_i}\right) = Q_i, \qquad (i = 1, \ldots, n),$$

които той нарича обобщени уравнения на Лагранж. Тук $\overset{(n)}{T} = d^nT/dt^n$. При $n = 1$ и $n = 2$ се получават съответно уравненията на Нилзен и Ценов. По друг начин тези уравнения са изведени през 1962 г. от Mangeron, D., Deleanu, S. в работата им "Sur une class d'equation de la mechanique analytique au sens de I. Tzenoff. Comptes rendues d. Ac. Bulg. Des sciences, Bd. 15, 1962". Това, което проф. Долапчиев прави повече от Mangeron и Deleanu, е, че той придава на тези уравнения и Апелов вид

$$\frac{\partial R_n}{\partial \overset{(n)}{q}_i} = Q_i \qquad (i = 1, ..., n),$$

където R_n е подходящо определена функция. Тази форма разкрива нови възможности, когато тези уравнения се приложат за нехолономни системи. Забележително постижение на проф. Долапчиев е начинът по който той свързва обобщените уравнения на Лагранж с вариационните принципи на механиката. До началото на XX век освен принципа на Даламбер широко е бил известен и принципът на Гаус за най-малката принуда: „Във всеки момент истинското движение на една механична система, подчинена на идеални връзки, се отличава от всички останали кинематически възможни (т.е. при същите връзки) движения, които системата би извършила от същата конфигурация и със същите скорости, но при други ускорения, по това, че за истинското ѝ движение функцията

$$Z = \frac{1}{2}\sum_{v}\frac{1}{m_v}\left(m_v\ddot{\mathbf{r}}_v - \mathbf{F}_v\right)^2,$$

наречена „принуда (Zwang)" има минимум." Като изразим вариационното твърдение в принципа на Гаус, ще получим

$$\sum_{v}\left(m_v\ddot{\mathbf{r}}_v - \mathbf{F}_v\right)\cdot\delta\ddot{\mathbf{r}}_v = 0, \tag{8}$$

като положенията и скоростите на точките не се варират $\delta\mathbf{r}_v = \delta\dot{\mathbf{r}}_v = 0$, а се варират ускоренията $\ddot{\mathbf{r}}_v$ по такъв начин, че да не се нарушават наложените връзки.

Както отбелязва проф. Долапчиев, едва през 1909 г. Журден забелязва, че всъщност между двата принципа (7) и (8) съществува празнина и може да се формулира следният вариационен принцип

$$\sum_{v}\left(m_v\ddot{\mathbf{r}}_v - \mathbf{F}_v\right)\cdot\delta\dot{\mathbf{r}}_v = 0,$$

в който конфигурацията на системата е фиксирана $\delta\mathbf{r}_v$, а се варират само скоростите.

По-късно Лайтингер забелязва, че принципът на Журден следва директно от принципа на Даламбер чрез пълно диференциране на (7) по времето, като се държи сметка за съотношението

$$\frac{d}{dt}\delta\mathbf{r}_v = \delta\frac{d\mathbf{r}_v}{dt}$$

и се положи след това $\delta\mathbf{r}_v = 0$. По същия начин от принципа на Журден се получава принципът на Гаус. Най-общата форма, до която се достига по този начин, е дадена от Нордхайм:

$$\sum_{v}\left(m_v\ddot{\mathbf{r}}_v - \mathbf{F}_v\right)\cdot\delta\overset{(n)}{\mathbf{r}_v} = 0 \tag{9}$$

при $\delta \mathbf{r}_\nu = \delta \dot{\mathbf{r}}_\nu = \ldots = \delta \overset{(n-1)}{\mathbf{r}_\nu} = 0$, където с $\overset{(i)}{\mathbf{r}_v} = d^i \mathbf{r}_v / dt^i$ е означена i-тата производна по времето на $\mathbf{r}_\nu$. Принципът (9) е наречен от проф. Долапчиев „обобщен принцип на Даламбер“. От обобщения принцип на Даламбер проф. Долапчиев директно получава своите обобщени уравнения на Лагранж по подобен начин, както Лагранж е получил своите уравнения от принципа на Даламбер. По този начин се получава едно красиво съответствие между различните форми уравнения на движение и съответния вариационен принцип:

- при $\delta \mathbf{r}_\nu \neq 0$ – принцип на Даламбер:

$$\sum_v \left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right) \cdot \delta \mathbf{r}_v = 0;$$

– уравнения на Лагранж:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i, \qquad (i = 1, \ldots, n).$$

- при $\delta \mathbf{r}_\nu = 0$, $\delta \dot{\mathbf{r}}_\nu \neq 0$ – принцип на Журден:

$$\sum_v \left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right) \cdot \delta \dot{\mathbf{r}}_v = 0;$$

– уравнения на Нилзен:

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i} = Q_i, \qquad (i = 1, \ldots, n).$$

- при $\delta \mathbf{r}_\nu = \delta \dot{\mathbf{r}}_\nu = 0$, $\delta \ddot{\mathbf{r}}_\nu \neq 0$ – принцип на Гаус:

$$\sum_v \left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right) \cdot \delta \ddot{\mathbf{r}}_v = 0;$$

– уравнения на Ценов:

$$\frac{1}{2}\left(\frac{\partial \ddot{T}}{\partial \dot{q}_i} - 3\frac{\partial T}{\partial q_i}\right) = Q_i, \qquad (i = 1, \ldots, n).$$

- при $\delta \mathbf{r}_\nu = \delta \dot{\mathbf{r}}_\nu = \ldots \delta \overset{(n-1)}{\mathbf{r}_v} = 0$, $\delta \overset{(n)}{\mathbf{r}_v} \neq 0$; – обобщен принцип на Даламбер:

$$\sum_v \left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right) \cdot \delta \overset{(n)}{\mathbf{r}_v} = 0.$$

– обобщени уравнения на Лагранж:

$$\frac{1}{n}\left(\frac{\partial \overset{(n)}{T}}{\partial \overset{(n)}{q_i}} - (n+1)\frac{\partial T}{\partial q_i}\right) = Q_i, \qquad (i = 1, \ldots, n).$$

Всички уравнения на движение са идентични помежду си и вариационните принципи са равностойни, когато се отнасят за холономни системи, при които е възможно да бъдат въведени обобщени координати. Не винаги обаче това е възможно. За такива холономни системи и за всички нехолономни системи използването на един или друг вариационен принцип може да доведе до съществени предимства. Проф. Долапчиев специално е разгледал този проблем и го е решил по следния начин. Той представя обобщения принцип на Даламбер като условие за стационарност на функцията Z_n.

$$\begin{gathered}\delta Z_n = 0, \qquad \delta = \delta \overset{(n)}{\mathbf{r}_v}, \quad \delta \mathbf{r}_v = \delta \dot{\mathbf{r}}_v = \cdots = \delta \overset{(n-1)}{\mathbf{r}_v} = 0, \\ Z_n = \sum_v \left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right) \cdot \overset{(n)}{\mathbf{r}_v},\end{gathered} \tag{10}$$

ако вѐличината $\left(m_v \ddot{\mathbf{r}}_v - \mathbf{F}_v\right)$ не зависи от $\overset{(n)}{\mathbf{r}_v}$. Варирането в последната формула се води по $\overset{(n)}{\mathbf{r}_v}$. Тази функция трябва да се разглежда като функция само на величините $\overset{(n)}{\mathbf{r}_v}$ и стационарността се разбира при изпълнение на наложените на системата връзки:

$$f_s\left(t,\ \mathbf{r}_v,\ \dot{\mathbf{r}}_v, \ldots,\ \overset{(m_s-1)}{\mathbf{r}_v}\right) = 0, \qquad m_s \geq 1 \quad (s = 1, \ldots, l). \tag{11}$$

Нека $m = \max\limits_{1 \leq s \leq l} m_s$. Уравненията на връзките (11) ще запишем в еднообразна форма, като в случая $m_s < m$ диференцираме съответното уравнение $(m - m_s)$ пъти пълно по времето t. Така получаваме системата

$$\varphi_s\left(t,\ \mathbf{r}_v,\ \dot{\mathbf{r}}_v, \ldots,\ \overset{(m-1)}{\mathbf{r}_v}\right) = 0 \qquad (s = 1, \ldots, l). \tag{12}$$

Ако не всички уравнения на система (12) съдържат линейно $\overset{(m-1)}{\mathbf{r}_v}$, след още едно диференциране, както отбелязва проф. Долапчиев, получаваме връзки, които са вече линейни по отношение на ускоренията от най-висок ред:

$$\sum_v \frac{\partial f_s}{\partial \overset{(m-1)}{\mathbf{r}_v}} \cdot \delta \overset{(m)}{\mathbf{r}_v} + \cdots = 0.$$

По този начин, ако n е достатъчно голямо ($n \geq (m-1)$ – в случая на линейност на уравнения (12) по отношение на $\overset{(m-1)}{\mathbf{r}_v}$, или $n \geq m$ – в случая на нелинейност), можем винаги да смятаме, че стационарността на Z_n в (10) по отношение на $\overset{(n)}{\mathbf{r}}_v$ е при линейни за $\overset{(n)}{\mathbf{r}}_v$ ограничения. Оттук критерият, който формулира проф. Долапчиев за използването на един или друг вариационен принцип, е следният: избира се онова n, за което уравненията на връзките са линейни по отношение на вариациите.

Краткият обзор на изследванията на проф. Долапчиев по аналитична механика, който направих, показва значимата следа, която той е оставил в този дял на механиката. Не бива да забравяме, че аналитичната механика е най-старият дял на механиката, където е извънредно трудно да се направи нещо ново. Толкова повече следва да се гордеем с изследванията на проф. Долапчиев и акад. Ценов, дали достоен български принос в световната наука.

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A JUMP INVERSION THEOREM FOR THE INFINITE ENUMERATION JUMP

HRISTO GANCHEV

In this paper we study partial regular enumerations for arbitrary recursive ordinal. We use the technique to obtain a jump inversion and omitting theorem for the infinite enumeration jump for the case of partial degrees.

Keywords: enumeration reducibility, enumeration jump, enumeration degrees, forcing
2000 MSC: 03D30

1. INTRODUCTION

In [2] Soskov introduces the notion of regular enumerations. Using them he proves the following jump inversion theorem:

Theorem (Soskov). *Let $k > n \geq 0$ and $B_0, \dots, B_k$ be arbitrary sets of natural numbers. Let $A \subseteq N$ and Q be a total set such that $\mathcal{P}(B_0, \dots, B_k) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \dots, B_n)$. Then there exists a total set F having the following properties:*

(i) *For all $i \leq k$, $B_i \in \Sigma^F_{i+1}$;*

(ii) *For all i $1 \leq i \leq k$, $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \dots, B_{i-1})'$;*

(iii) *$F^{(k)} \equiv_e Q$;*

(iv) *$A \not\leq_e F^{(n)}$.*

Here $\mathcal{P}(B_0, \dots)$ is the polynomial set obtained from $B_0, B_1, \dots$ as defined in Section 2.

In [1] Soskov and Baleva generalize the notion of regular enumeration and obtain the following result for the infinite case:

Theorem (Soskov, Baleva) *Let $\{B_\alpha\}_{\alpha\leq\zeta}$ be a sequence of sets of natural numbers. Let $\{A_\gamma\}_{\gamma<\zeta}$ also be a sequence of sets of natural numbers, such that for all $\gamma<\zeta$ is true that $A_\gamma \not\leq_e \mathcal{P}_\gamma$. Finally, let Q be a total set such that $\mathcal{P}_\zeta \leq_e Q$ and $\bigoplus_{\gamma<\zeta} A_\gamma^+ \leq_e Q$. Then there is a total set F such that:*

(1) *For all $\gamma\leq\zeta$ it is true that $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*

(2) *For all $\gamma\leq\zeta$, if $\gamma=\beta+1$ then $F^{(\gamma)} \equiv_e F\oplus\mathcal{P}'_\beta$ uniformly in γ;*

(3) *For all limit $\gamma\leq\zeta$ it is true that $F^{(\gamma)} \equiv_e F\oplus\mathcal{P}_{<\gamma}$ uniformly in γ;*

(4) *$F^{(\zeta)} \equiv_e Q$;*

(5) *For all $\gamma<\zeta$ it is true that $A_\gamma \not\leq_e F^{(\gamma)}$.*

In this paper we will prove that this result also holds if we want the target set F to be partial, i.e., the degree $\mathbf{d}_e(F)$ to be partial. Namely, we will prove the following theorem:

Theorem 1.1. *Let $\{B_\alpha\}_{\alpha\leq\zeta}$ be a sequence of sets of natural numbers. Let also $\{A_\gamma\}_{\gamma<\zeta}$ be a sequence of sets of natural numbers, such that for all $\gamma<\zeta$ it is true that $A_\gamma \not\leq_e \mathcal{P}_\gamma$. Finally let Q be a total set such that $\mathcal{P}_\zeta \leq_e Q$ and $\bigoplus_{\gamma<\zeta} A_\gamma^+ \leq_e Q$. Then there exists a set F such that $\mathbf{d}_e(F)$ is partial and:*

(1) *For all $\gamma\leq\zeta$ it is true that $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*

(2) *For all $\gamma\leq\zeta$, if $\gamma=\beta+1$ then $F^{(\gamma)} \equiv_e F^+\oplus\mathcal{P}'_\beta$ uniformly in γ;*

(3) *For all limit ordinals $\gamma\leq\zeta$ it is true that $F^{(\gamma)} \equiv_e F^+\oplus\mathcal{P}_{<\gamma}$ uniformly in γ;*

(4) *$F^{(\zeta)} \equiv_e Q$;*

(5) *For all $\gamma<\zeta$ it is true that $A_\gamma \not\leq_e F^{(\gamma)}$;*

(6) *F is quasiminimal over B_0, i.e. for all total sets X if $X\leq_e F$ then $X\leq_e B_0$.*

2. PRELIMINARIES

Let $W_0,\dots,W_i,\dots$ be the Gödel enumeration of the r.e. sets. We define the enumeration operator Γ_i for arbitrary set of natural numbers by $\Gamma_i(A) = \{x \mid (\exists\langle x,u\rangle\in W_i)(D_u\subseteq A)\}$, where D_u is the finite set with canonical code u. We define the relation $\leq_e$ over the sets of natural numbers by

$$A\leq_e B \iff \exists i(A=\Gamma_i(B)).$$

The relation $\leq_e$ is reflexive and transitive and defines a equivalence relation $\equiv_e$. We call the equivalence classes of $\equiv_e$ enumeration degrees.

The composition of two enumeration operators is also an enumeration operator. Beside this the index of the resulting operator is obtained uniformly from the indexes of the other ones. This means that there exists a recursive function $\mathfrak{c}$ such that $\Gamma_i(\Gamma_j(A)) = \Gamma_{\mathfrak{c}(i,j)}(A)$ for arbitrary set A.

We define the "join" operator $\oplus$ by $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. We set $A^+ = A \oplus \overline{A}$. We say that a set A of natural numbers is total iff $A \equiv_e A^+$. We say that the enumeration degree $\mathbf{a}$ is total iff there is a total set $A \in \mathbf{A}$. Otherwise we say that the enumeration degree is partial.

We define the enumeration jump to be $A' = L_A^+$, where $L_A = \{\langle x, i\rangle \mid x \in \Gamma_i(A)\}$. Using ordinal notation we can define the infinite enumeration jump. More precisely:

Let η be a recursive ordinal and let us fix an ordinal notation $e \in \mathcal{O}$ for η. For every ordinal $\alpha < \eta$ we will use the corresponding notation which is $<_{\mathcal{O}}$ then e (for an introduction on ordinal notations see [3]). Then, not distinguishing the ordinal from its notation, we define the α jump for $\alpha < \eta$ by means of transfinite induction:

(1) $A^{(0)} = A$
(2) If $\alpha = \beta + 1$ then $A^{(\alpha)} = (A^{(\beta)})'$
(3) If $\alpha = \lim(\alpha(p))$ then $A^{(\alpha)} = \{\langle p, x\rangle \mid x \in A^{(\alpha(p))}\}$.

Naturally the definition depends on the choice of the ordinal notation of α. Despite this, we can prove that if α_1 and α_2 are two different notations of α, then $A^{(\alpha_1)} \equiv_e A^{(\alpha_2)}$ (see [1], [3]), as in the case of the turing infinite jump.

We define the "polynomials" $\mathcal{P}_\alpha$ of the sets $B_0, \dots, B_\alpha, \dots$ with

Definition 2.1. Let ζ be a recursive ordinal and let $\{B_\alpha\}_{\alpha \le \zeta}$ be a sequence of sets of natural numbers. Then we define using transfinite induction the sets $\mathcal{P}_\alpha$ in the following way:

(1) $\mathcal{P}_0 = B_0$
(2) if $\alpha = \beta + 1$ then $\mathcal{P}_\alpha = \mathcal{P}'_\beta \oplus B_\alpha$;
(3) if $\alpha = \lim(\alpha(p))$ then $\mathcal{P}_\alpha = \mathcal{P}_{<\alpha} \oplus B_\alpha$, where

$$\mathcal{P}_{<\alpha} = \{\langle p, x\rangle \mid x \in \mathcal{P}_{\alpha(p)}\}$$

We also introduce the following notation:
For an arbitrary sequence of sets $\{C_\alpha\}_{\alpha<\zeta}$ we define the set $\bigoplus_{\alpha<\zeta} C_\alpha$ to be

$$\bigoplus_{\alpha<\zeta} C_\alpha = \{\langle \alpha, x\rangle \mid x \in C_\alpha\}.$$

We will consider partial functions $f : \mathbf{N} \multimap \mathbf{N}$. We will say that $f \le_e A$ iff $\langle f\rangle \le_e A$, where $\langle f\rangle$ is the graphic of f. We will use "partial" finite parts τ for which $\tau : [0, 2q+1] \longrightarrow \mathbf{N} \cup \{\perp\}$. We define the graphic of τ to be $\langle\tau\rangle = \{\langle x, y\rangle \mid x \le 2q+1 \;\&\; \tau(x) = y \ne \perp\}$ and we say that $\tau \subseteq f$ iff $\langle\tau\rangle \subseteq \langle f\rangle$. We define $\mathrm{lh}(\tau) = 2q+2$

We will assume that an effective and reversible coding of all finite sequences is fixed. Thus we have an effective and reversible coding for all finite parts. As usual from now on we will make no difference between a finite part and its code. Even more: we say that $\tau \leq \rho$ iff the inequality holds for the codes of the finite parts ρ and τ. By $\tau \subseteq \rho$ we will mean the usual extension property.

Finally we will say that the statement $\exists i P(i, x_1, \ldots, x_n, A_1, \ldots, A_k)$, where $i, x_1, \ldots, x_n \in \mathbf{N}$ and $A_1, \ldots, A_n \subseteq \mathbf{N}$, is uniformly true in $x_1, \ldots, x_n$ for all $A_1, \ldots A_k$ iff there exists a recursive function $h(x_1, \ldots, x_n)$ such that for every $x_1, \ldots, x_n \in \mathbf{N}$ and every $A_1, \ldots, A_k \subseteq \mathbf{N}$ the statement

$$P(h(x_1, \ldots, x_n), x_1, \ldots, x_n, A_1, \ldots, A_k)$$

is true.

Of course the construction of h is quite difficult and uninformative. Hence, when we have to prove that some statement is uniformly true, usually we will show a construction in which all the choices we have to make will be effective.

3. REGULAR ENUMERATIONS

The proof of the theorem in most of its parts repeats the proof of Soskov, Baleva theorem. A complete proof of the last one can be found in [1].

Let us first fix a recursive ordinal ζ and a sequence of sets $\{B_\alpha\}_{\alpha \leq \zeta}$.

The following definitions of ordinal approximation and predecessor as the proofs of their basic properties are due to Soskov and Baleva.

Definition 3.1. Let α be a recursive ordinal. We will say that $\overline{\alpha}$ is an approximation of α, iff $\overline{\alpha}$ is finite sequence of ordinals $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$, where $\alpha_0 = 0$, $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha$ and $n \geq -1$.

Definition 3.2. Let α be a recursive ordinal and let $\beta < \alpha$. Let also $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$ is an approximation of α. We define recursively the notion of β-predecessor of $\overline{\alpha}$:

a) if $\beta = \alpha_i$ for some $0 \leq i \leq n$ then set $\overline{\beta} = \langle \alpha_0, \alpha_1, \ldots, \alpha_i \rangle$;
b) if $\alpha_i < \beta < \alpha_{i+1}$ for some $0 \leq i < n$ then set $\overline{\beta}$ to be the β-predecessor of $\langle \alpha_0, \alpha_1, \ldots, \alpha_{i+1} \rangle$;
c) if $\alpha_n < \beta < \alpha$ then

1) if $\alpha = \delta + 1$ and $\beta = \delta$ set $\overline{\beta} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \beta \rangle$;
2) if $\alpha = \delta+1$ and $\beta < \delta$ then set $\overline{\beta}$ to be the β-predecessor of $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \delta \rangle$;
3) if $\alpha = \lim \alpha(p)$, $p_0 = \mu p[\alpha(p) > \alpha_n]$ and $p_1 = \mu p[\alpha(p) > \beta]$ set $\overline{\beta}$ to be the β-predecessor of $\langle \alpha_0, \alpha_1, \ldots \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p_1) \rangle$.

The following lemmas give the basic properties of the ordinal approximation and predecessor. The full proofs can be found in [1].

Lemma 3.1. *For every ordinal approximation $\overline{\alpha}$ and every $\beta < \alpha$ there is a unique β-predecessor $\overline{\beta}$ of α.*

Lemma 3.2. *Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$ be an approximation of α. Then:*

(1) *If $\beta \le \alpha_i$ for some $0 \le i \le n$ then $\overline{\beta} \preceq \overline{\alpha} \Leftrightarrow \overline{\beta} \preceq \overline{\alpha}_i$*

(2) *If for some $0 \le i \le n$, $\alpha_i \le \beta < \alpha$ and $\langle \beta_0, \beta_1, \ldots, \beta_k \rangle$ is the β-predecessor of $\overline{\alpha}$ then $i < k$ and $\alpha_l = \beta_l$ for all $l = 0, \ldots, i$*

(3) *Let $\alpha = \delta + 1$, $\alpha_n < \delta$ and $\beta \le \delta$. Then $\overline{\beta} \preceq \overline{\alpha} \Leftrightarrow \overline{\beta} \preceq \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \delta \rangle$*

(4) *Let $\alpha = \lim \alpha(p)$ be a limit ordinal and let $p_0 = \mu p[\alpha_n < \alpha(p)]$. Let also $p_1 \ge p_0$ be such that $\beta \le \alpha(p_1)$. Then*

$$\overline{\beta} \preceq \overline{\alpha} \Leftrightarrow \overline{\beta} \preceq \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p_1) \rangle$$

Lemma 3.3. *Let $\gamma < \beta < \alpha$ be ordinals, $\overline{\gamma} \preceq \overline{\beta}$ and $\overline{\beta} \preceq \overline{\alpha}$. Then $\overline{\gamma} \preceq \overline{\alpha}$.*

Let us fix an approximation $\overline{\alpha}$ of α. We define the notions of $\overline{\alpha}$-regular finite part, $\overline{\alpha}$-rank and $\overline{\alpha}$-forcing by means of transfinite recursion over α.

(i) Let first $\alpha = 0$. Then $\overline{\alpha} = \langle 0 \rangle$. 0-regular are those finite parts satisfying the condition:

If $z \in 2\mathbf{N} + 1$, $z \in \text{dom}\,(\tau)$ and $\tau(z) \neq \perp$, then $\tau(z) \in B_0$.

If dom $(\tau) = [0, 2q+1]$ we set the 0-rank $|\tau|_0$ of τ to be $q + 1$.

We will use the notation $\mathcal{R}_0$ for the set of all 0-regular finite parts.

For arbitrary finite part ρ we define:

$$\begin{aligned} \rho \Vdash_0 F_i(x) &\iff \exists v(\langle x, v \rangle \in W_i \;\&\; D_v \subseteq \langle \tau \rangle), \\ \rho \Vdash_0 \neg F_i(x) &\iff (\forall \tau \in \mathcal{R}_0)(\tau \supseteq \rho \Longrightarrow \tau \nVdash_0 F_i(x)). \end{aligned}$$

Now suppose that for all $\beta < \alpha$ the $\overline{\beta}$-regularity, $\overline{\beta}$-rank and $\overline{\beta}$-forcing are defined. We will also assume that for all $\beta < \alpha$ the function $\overline{\beta}$-rank denoted by $\lambda\tau.|\tau|_{\overline{\beta}}$ has the property:

If τ and ρ are two $\overline{\beta}$-regular finite parts such that $\tau \subseteq \rho$, then $|\tau|_{\overline{\beta}} \le |\rho|_{\overline{\beta}}$. In particular $|\tau|_{\overline{\beta}} = |\rho|_{\overline{\beta}} \iff \tau = \rho$.

(ii) Let now $\alpha = \beta + 1$. Let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Denote the set of all $\overline{\beta}$-regular finite parts by $\mathcal{R}_{\overline{\beta}}$. Let also

$$\begin{aligned} X^{\overline{\beta}}_{\langle i,j \rangle} &= \{\rho \in \mathcal{R}_{\overline{\beta}} \mid \rho \Vdash_{\overline{\beta}} F_i(j)\}, \\ S^{\overline{\beta}}_j &= \mathcal{R}_{\overline{\beta}} \cap \Gamma_j(\mathcal{P}_\beta), \end{aligned}$$

where Γ_j is the j-th enumeration operator.

If ρ is an arbitrary finite part and X is a set of $\overline{\beta}$-regular finite parts we define the function $\mu_{\overline{\beta}}(\rho, X)$ by:

$$\mu_{\overline{\beta}}(\rho, X) = \begin{cases} \mu\tau[\tau \supseteq \rho \;\&\; \tau \in X], & \text{if there is such } \tau \quad \text{(a)} \\ \mu\tau[\tau \supseteq \rho \;\&\; \tau \in \mathcal{R}_{\overline{\beta}}], & \text{if (a) is not satisfiable} \quad \text{(b)} \\ \neg!, & \text{if (a) and (b) are not satisfiable} \quad \text{(c)} \end{cases}$$

Definition 3.3. Let τ be a finite part and let $m \geq 0$. We say that ρ is $\overline{\beta}$-regular m-omitting extension of τ, iff ρ is $\overline{\beta}$-regular extension of τ, defined in $[0, q-1]$ and there are natural numbers $q_0 < q_1 < \cdots < q_m < q_{m+1} = q$ such that

a) $\rho \restriction q_0 = \tau$

b) for all $p \leq m$, it is true that $\rho \restriction q_{p+1} = \mu_{\overline{\beta}}\left(\rho \restriction (q_p + 1), X^{\overline{\beta}}_{\langle p, q_p \rangle}\right)$.

It is clear that if ρ is $\overline{\beta}$-regular m-omitting extension of τ, then $q_0, q_1, \ldots, q_{m+1}$ are unique. Even more: if ρ_1 and ρ_2 are two $\overline{\beta}$-regular m-omitting extensions of τ and $\rho_1 \subseteq \rho_2$ then $\rho_1 = \rho_2$. In other case the function $\mu_{\overline{\beta}}$ is not single valued.

Now we are ready to define the notion of $\overline{\alpha}$-regular finite part:

Let τ be a finite part defined in $[0, q-1]$ and let $r \geq 0$. We say that τ is an $\overline{\alpha}$-regular finite part with $\overline{\alpha}$-rank $r + 1$ iff there are natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \cdots < n_r < l_r < b_r < n_{r+1} = q,$$

such that for all $0 \leq j \leq r$ the following assertions hold:

(1) $\tau \restriction n_0$ *is a $\overline{\beta}$-regular finite part of $\overline{\beta}$-rank 1;*

(2) $\tau \restriction l_j = \mu_{\overline{\beta}}\left(\tau \restriction (n_j + 1), S^{\overline{\beta}}_j\right)$;

(3) $\tau \restriction b_j$ *is $\overline{\beta}$-regular j-omitting extension of $\tau \restriction l_j$;*

(4) $\tau(b_j) \in B_\alpha$;

(5) $\tau \restriction n_{j+1}$ *is a $\overline{\beta}$-regular extension of $\tau \restriction (b_j + 1)$ of rank $|\tau \restriction b_j|_{\overline{\beta}} + 1$.*

Note that directly from the definition it follows that if τ is an $\overline{\alpha}$-regular finite part, then τ is also a $\overline{\beta}$-regular finite part.

The definition of $\overline{\alpha}$-forcing for an arbitrary finite part ρ is:

$$\begin{aligned} \rho \Vdash_{\overline{\alpha}} F_i(x) \iff \exists v(\langle v, x \rangle \in W_i \ \& \ (\forall u \in D_v)\Big((u = \langle i_u, x_u, 0 \rangle \ \& \ \rho \Vdash_{\overline{\beta}} F_{i_u}(x_u)) \\ \vee\ (u = \langle i_u, x_u, 1 \rangle \ \& \ \rho \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u))\Big) \end{aligned}$$

$$\rho \Vdash_{\overline{\beta}} \neg F_i(x) \iff (\forall \tau \in \mathcal{R}_{\overline{\alpha}})(\rho \subseteq \tau \Longrightarrow \tau \nVdash_{\overline{\alpha}} F_i(x))$$

(iii) Finally let $\alpha = \underline{\lim}\, \alpha(p)$. Let $\overline{\alpha} = \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha$ and let $p_0 = \mu p[\alpha(p) > \alpha_n]$. Let also for all p, $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. Note that for $p \geq p_0$ according to Lemma 3.2

$$\overline{\alpha(p)} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p) \rangle.$$

We say that the finite part τ defined for $[0, q-1]$ is $\overline{\alpha}$-regular of $\overline{\alpha}$-rank $r + 1$ if there are natural numbers

$$0 < n_0 < b_0 < n_1 < b_1 < \cdots < n_r < b_r < n_{r+1} = q,$$

such that $0 \leq j \leq r$, it is true that:

(1) $\tau \restriction n_0$ *is a* $\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$*-regular finite part of rank* 1;

(2) $\tau \restriction b_j$ *is a* $\overline{\alpha(p_0 + 2j)}$*-regular finite part of rank* 1;

(3) $\tau(b_j) \in B_\alpha$;

(4) $\tau \restriction n_{j+1}$ *is a* $\overline{\alpha(p_0 + 2j + 1)}$*-regular finite part of rank* 1.

Note that in this case, τ is a $\overline{\alpha(p_0 + 2r + 1)}$-regular finite part of respectivly rank 1.

For every finite part ρ and every $i, x \in \mathbf{N}$ we define:

$$\rho \Vdash_{\overline{\alpha}} F_i(x) \Longleftrightarrow \exists v \left(\langle v, x \rangle \in W_i \,\&\, (\forall u \in D_v)(u = \langle p_u, i_u, x_u \rangle \,\&\, \rho \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u)) \right),$$
$$\rho \Vdash_{\overline{\alpha}} \neg F_i(x) \Longleftrightarrow (\forall \tau \in \mathcal{R}_{\overline{\alpha}})(\rho \subseteq \tau \Rightarrow \tau \not\Vdash_{\overline{\alpha}} F_i(x)).$$

This concludes the definition. The next Lemma gives the correctness of the definition and the validity of the assumption for the $\overline{\beta}$-rank.

Lemma 3.4. *Let $\alpha \leq \zeta$ and let τ be $\overline{\alpha}$-regular finite part. Then the following statements are true:*

(a) Let $\alpha = \beta + 1$. Let also $n'_0, l'_0, b'_0, \dots n'_r, l'_r, b'_r, n'_{r+1}$ and $n''_0, l''_0, b''_0, \dots n''_p, l''_p, b''_p, n''_{p+1}$ be two sequences of natural numbers satisfying (1)-(5) from **(ii)**. *Then $r = p$, $n'_{r+1} = n''_{r+1}$ and for all $0 \leq j \leq r$ we have $n'_j = n''_j$, $l'_j = l''_j$ and $b'_j = b''_j$.*

(b) Let $\alpha = \lim \alpha(p)$ and let $n'_0, b'_0, \dots n'_r, b'_r, n'_{r+1}$ and $n''_0, b''_0, \dots n''_p, b''_p, n''_{p+1}$ are two sequences of natural numbers satisfying (1)-(4) from **(iii)**. *Then $r = p$, $n'_{r+1} = n''_{r+1}$ and for all $0 \leq j \leq r$ we have $n'_j = n''_j$ and $b'_j = b''_j$.*

(c) Let ρ and τ be $\overline{\alpha}$-regular finite parts and let $\tau \subseteq \rho$. Then $|\tau|_{\overline{\alpha}} \leq |\rho|_{\overline{\alpha}}$. In particular $|\tau|_{\overline{\alpha}} = |\rho|_{\overline{\alpha}} \iff \tau = \rho$.

Proof. (a) Let $\alpha = \beta + 1$ and let $n'_0, l'_0, b'_0, \dots, n'_r, l'_r, b'_r, n'_{r+1}$ and $n''_0, l''_0, b''_0, \dots, n''_p, l''_p, b''_p, n''_{p+1}$ be two sequences of natural numbers satisfying (1)-(5) from **(ii)**. Without loss of generality we may assume that $\tau \restriction n'_0 \subseteq \tau \restriction n''_0$. Beside this, we have $|\tau \restriction n'_0|_{\overline{\beta}} = |\tau \restriction n''_0|_{\overline{\beta}} = 1$. Then considering the properties of $\overline{\beta}$-rank we obtain $\tau \restriction n'_0 = \tau \restriction n''_0$. Therefore $n'_0 = n''_0$. Let now the equality $n'_j = n''_j$ hold. Then $\tau \restriction l'_j = \mu_{\overline{\beta}}\left(\tau \restriction n'_j, S^{\overline{\beta}}_j\right) = \mu_{\overline{\beta}}\left(\tau \restriction n''_j, S^{\overline{\beta}}_j\right) = \tau \restriction l''_j$. Therefore $l'_j = l''_j$. Now considering the property of the j-omitting $\overline{\beta}$-regular extensions (mentioned after the definition) we obtain $\tau \restriction b'_j = \tau \restriction b''_j$ and therefore $b'_j = b''_j$. Now again without loss of generality we may consider $\tau \restriction n'_{j+1} \subseteq \tau \restriction n''_{j+1}$. But $|\tau \restriction n'_{j+1}|_{\overline{\beta}} = |\tau \restriction b'_j|_{\overline{\beta}} + 1 = |\tau \restriction b''_j|_{\overline{\beta}} + 1 = |\tau \restriction n''_{j+1}|_{\overline{\beta}}$. Therefore from the property of the $\overline{\beta}$-rank we obtain $n'_{j+1} = n''_{j+1}$. Now the statement $r = p$ is obvious.

(b) The proof is analogous to the previous one.

(c) Let τ and ρ be two $\overline{\alpha}$-regular finite parts and let $\tau \subseteq \rho$. From the proof of (a) we obtain that the sequence corresponding to τ and satisfying the definition of the $\overline{\alpha}$-regular finite parts is an initial part of the sequence corresponding to ρ.

Therefore $|\tau|_{\overline{\alpha}} \leq |\rho|_{\overline{\alpha}}$. If $\tau \subsetneq \rho$ then we have $|\tau|_{\overline{\alpha}} < |\rho|_{\overline{\alpha}}$, since in the contrary case we would obtain that the sequence of ρ is not monotone. □

From the definition of $\overline{\alpha}$-regular finite part and Lemma 3.4 we obtain

Corollary 3.1. *Let $\alpha = \beta + 1$, $\overline{\alpha}$ be an approximation of α and let $\overline{\beta}$ be β-predecessor of $\overline{\alpha}$. Then every $\overline{\alpha}$-regular finite part τ is $\overline{\beta}$-regular and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$.*

Lemma 3.5. *Let $1 \leq \alpha \leq \zeta$ and let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$. Then every $\overline{\alpha}$-regular finite part is $\langle \alpha_0, \ldots, \alpha_n \rangle$-regular and the $\langle \alpha_0, \ldots, \alpha_n \rangle$-rank of τ is strictly greater than $|\tau|_{\overline{\alpha}}$.*

Proof. We will use transfinite induction over α. First let $\alpha = 1$. Then $\overline{\alpha} = \langle 0, 1 \rangle$ and now the statement follows from Corollary 3.1.

Let now $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Then again (from Corollary 3.1) we obtain that τ is $\overline{\beta}$-regular finite part and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$. From Lemma 3.2 we know that $\overline{\beta}$ is of the form $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \beta_{n+1}, \ldots, \beta_{n+i} \rangle$, where $i \geq 0$. Then applying i times the induction hypothesis we obtain that τ is $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$-regular and the $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$-rank of τ is greater or equal to $|\tau|_{\overline{\beta}}$ and therefore strictly greater than $|\tau|_{\overline{\alpha}}$.

Finally let $\alpha = \lim \alpha(p)$. Let also $|\tau|_{\overline{\alpha}} = r + 1$ and let $p_0 = \mu p[\alpha(p_0) > \alpha_n]$. From the definition of $\overline{\alpha}$-regular finite part we obtain that τ is a $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_0 + 2r + 1) \rangle$-regular finite part of rank 1. From the induction hypothesis τ is a $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_0 + 2r) \rangle$-regular finite part of rank at least 2 and since τ is a $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0) \rangle$-regular finite part of rank at least $2r+2$, then τ is $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$-regular of rank at least $2r+3$ and therefore strictly greater than $r + 1$. □

Lemma 3.6. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. let also $\overline{\delta} \preceq \overline{\alpha}$. Then there is a natural number $k_{\overline{\alpha},\overline{\delta}}$, such that every $\overline{\alpha}$-regular finite part of rank greater or equal to $k_{\overline{\alpha},\overline{\delta}}$ is $\overline{\delta}$-regular.*

Proof. We will use transfinite induction over α. When $\alpha = 0$ the statement is trivial.

Now let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Let $\overline{\delta} \prec \overline{\alpha}$ (which is the interesting case). Then $\overline{\delta} \preceq \overline{\beta}$. According to the induction hypothesis there is a $k = k_{\overline{\beta},\overline{\delta}}$, such that every $\overline{\beta}$-regular finite part of rank greater or equal to k is $\overline{\delta}$-regular. Let us set $k_{\overline{\alpha},\overline{\delta}} = k$. Then according to Corollary 3.1 we obtain that k has the desired property.

Finally let $\alpha = \lim \alpha(p)$, $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$ and $\overline{\delta} \prec \overline{\alpha}$. Let also $p_0 = \mu p[\alpha(p) > \alpha_n]$, let $p_1 \geq p_0$ be such that $\alpha(p_1) > \delta$ and let us denote the $\alpha(p)$-predecessor of $\overline{\alpha}$ with $\overline{\alpha(p)}$. Applying Lemma 3.2 we obtain $\overline{\delta} \preceq \overline{\alpha}(p_1)$. Then according to the induction hypothesis every $\overline{\alpha}(p_1)$-regular finite part with rank greater or equal to $k_{\overline{\alpha}(p_1),\overline{\delta}}$ is $\overline{\delta}$-regular. It follows from the proof of the previous

Lemma that there is a natural number r, such that every $\overline{\alpha}$-regular finite part of rank at least $r+1$ is $\overline{\alpha}(p_1)$-regular of rank greater or equal to $k_{\overline{\alpha}(p_1),\overline{\delta}}$. Let us set $k_{\overline{\alpha},\overline{\beta}} = r+1$ □

Corollary 3.2. *Let $\alpha \leq \zeta$, $\overline{\alpha}$ be an approximation of α and $\overline{\beta} \preceq \overline{\alpha}$. Let also τ be $\overline{\alpha}$-regular finite part of rank greater or equal to $k_{\overline{\alpha},\overline{\beta}} + s$. Then $|\tau|_{\overline{\beta}} > s$.*

Proof. From the definition of the $\overline{\alpha}$ regular finite parts we obtain that there are natural numbers $q_0 < q_1 < \cdots < q_s$ such that $\tau \restriction q_s = \tau$ and for all j the finite parts $\tau_j = \tau \restriction q_j$ are $\overline{\alpha}$-regular with $\overline{\alpha}$-rank at least $k_{\overline{\alpha},\overline{\beta}}$ and therefore $\overline{\beta}$-regular. But $\tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_s$ and therefore $|\tau_j|_{\overline{\beta}} < |\tau_{j+1}|_{\overline{\beta}}$. Finally $|\tau_0|_{\overline{\beta}} \geq 1$, which completes the proof. □

Lemma 3.7. *Let $\alpha = \lim \alpha(p)$. Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$ and $p_0 = \mu p[\alpha(p) > \alpha_n]$. Let also $p_1 \geq p_0$ and τ be a $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p_1) \rangle$-regular finite part of rank 1. Then for every $\overline{\beta} \prec \overline{\alpha}$, if τ is $\overline{\beta}$-regular then $\beta \leq \alpha(p_1)$.*

Proof. In order to obtain a contradiction assume that τ is a $\overline{\beta}$-regular finite part for some β such that $\overline{\beta} \prec \overline{\alpha}$ and $\alpha(p_1) < \beta < \alpha$. Then $\overline{\beta}$ is the β-predecessor of

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p_1+k) \rangle,$$

where $k \geq 1$. According to Lemma 3.2 $\overline{\beta}$ is of the form

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1), \ldots, \beta \rangle.$$

As the $\overline{\beta}$-rank of τ is at least 1 then from Lemma 3.5 we obtain that the $\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1) \rangle$-rank of τ is greater than 1 which is a contradiction. □

Let $\overline{\alpha}$ be an ordinal approximation and let τ be a finite part. We introduce the following notation:

$$Reg(\tau, \overline{\alpha}) = \{ \overline{\beta} \mid \overline{\beta} \preceq \overline{\alpha} \ \&\ \tau \text{ is } \overline{\beta}\text{-regular} \}$$

Then the following is true:

Lemma 3.8. *Let $\alpha \leq \zeta$, let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$ be an approximation of α and let τ be an $\overline{\alpha}$-regular finite part. Then:*

a) if $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$ then

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in Reg(\tau, \overline{\delta});$$

b) let $\alpha = \lim \alpha(p)$. Let also $p_0 = \mu p[\alpha(p) > \alpha_n]$ and for every $p \geq p_0$ let $\overline{\alpha(p)}$ be $\alpha(p)$-predecessor of $\overline{\alpha}$. Let also $p_1 \geq p_0$ and let τ be $\overline{\alpha(p_1)}$-regular of rank 1. Then

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in Reg(\tau, \overline{\alpha(p_1)}).$$

Proof. The statement *a)* is obvious and the statement *b)* follows directly from the previous Lemma. □

Definition 3.4. We say that the sequence $A_0, \dots, A_n, \dots$ of sets of natural numbers is e-reducible to P iff there is a recursive function h such that for every n $A_n = \Gamma_{h(n)}(P)$. We say that the sequence is T-reducible to P iff there is a function χ recursive in P, such that for every n $\lambda x.\chi(n,x) = \chi_{A_n}$, where χ_{A_n} is the characteristic function of A_n.

From the definition of the enumeration jump, the e-reducibility and the T-reducibility of sequences to set we obtain the following Lemma.

Lemma 3.9. *Let P be a set such that the sequence $\{A_n\}$ is e-reducible to P. Then*

(1) *The sequence $\{A_n\}$ is uniformly T-reducible to P';*

(2) *If $R \leq_e P$ then the sequences $\{A_n \cap R\}$ and $\{C_n\}$ for which $C_n = \{\, x \mid \exists y(\langle y, x\rangle \in R \;\&\; y \in A_n\}$ are uniformly e-reducible to P.*

The full proof can be found in [2].

We introduce the following notations:

$$\begin{aligned} Z^{\overline{\alpha}}_{\langle i,j\rangle} &= \{\tau \in \mathcal{R}_{\overline{\alpha}} \mid \tau \Vdash_{\overline{\alpha}} \neg F_i(j)\} \\ O^{\overline{\alpha}}_{\tau,j} &= \{\rho \mid \rho \text{ is } \overline{\alpha}\text{-regular } j\text{-omitting extension of } \tau\} \end{aligned}$$

Proposition 3.1. *For every ordinal approximation $\overline{\alpha}$, where $\alpha \leq \zeta$ the following are true:*

(1) *$\mathcal{R}_{\overline{\alpha}} \leq_e \mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.*

(2) *The function $\lambda\tau.|\tau|_{\overline{\alpha}}$ is partially recursive in $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$;*

(3) *The sequences $\{S^{\overline{\alpha}}_j\}$ and $\{X^{\overline{\alpha}}_j\}$ are e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$;*

(4) *The sequence $\{Z^{\overline{\alpha}}_j\}$ is T-reducible to $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$;*

(5) *the functions $\lambda\tau, j.\mu_{\overline{\alpha}}\left(\tau, X^{\overline{\alpha}}_j\right)$ and $\lambda\tau, j.\mu_{\overline{\alpha}}\left(\tau, S^{\overline{\alpha}}_j\right)$ are partially recursive in $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$;*

(6) *The sequence $\{O^{\overline{\alpha}}_{\tau,j}\}$ is e-reducible to $\mathcal{P}'_\alpha$ uniformly $\overline{\alpha}$.*

Before proving the proposition let us note some properties of the sets $\mathcal{P}_\alpha$.

Lemma 3.10. (a) *If $\beta \leq \alpha \leq \zeta$ then $\mathcal{P}_\beta \leq_e \mathcal{P}_\alpha$ uniformly in α and β.*

(b) *If $\beta \leq \alpha \leq \zeta$ then $B_\beta \leq_e \mathcal{P}_\alpha$ uniformly in α and β;*

(c) *The sets $\mathcal{P}_{<\alpha}$ are total.*

Proof. (a) We must find a recursive function g, such that if $\beta \leq \alpha \leq \zeta$ then $\mathcal{P}_\beta = \Gamma_{g(\alpha,\beta)}(\mathcal{P}_\alpha)$. We will define g by recursion over the ordinals $\alpha \leq \zeta$. If $\alpha = 0$ then $g(0,0) = i_0$, where i_0 is a fixed index for the enumeration operator identity. If $\alpha = \beta$ then again $g(\alpha,\beta) = i_0$. Now let $\beta < \alpha$.

First consider $\alpha = \delta + 1$. Then $\mathcal{P}_\beta \leq_e \mathcal{P}_\delta$ and therefore $\mathcal{P}_\beta = \Gamma_{g(\delta,\beta)}(\mathcal{P}_\delta)$. But $\mathcal{P}_\delta = \Gamma_{j_0}(\Gamma_{p_0}(\mathcal{P}_\alpha))$, where j_0 is a fixed index for which $A = \Gamma_{j_0}(A')$ and p_0 is such that $A = \Gamma_{p_0}(A \oplus C)$ (j_0 and p_0 exist and do not depend on A and C). Then

$$g(\alpha, \beta) = \mathfrak{c}(g(\delta, \beta), \mathfrak{c}(j_0, p_0)).$$

For the definition of $\mathfrak{c}$ see Section 2.

Finally let $\alpha = \lim \alpha(p)$. Then there is a recursive function pr not depending on α, such that $\mathcal{P}_{\alpha(i)} = \Gamma_{pr(i)}(\mathcal{P}_{<\alpha})$. The function $m(\alpha, \beta) = \mu p[\alpha(p) \geq \beta]$, defined for the limit ordinals $\alpha \leq \zeta$ and all ordinals $\beta < \alpha$, is partially recursive. Then $\mathcal{P}_\beta \leq_e \mathcal{P}_{m(\alpha,\beta)}$ and $\mathcal{P}_{m(\alpha,\beta)} = \Gamma_{pr(m(\alpha,\beta))}(\mathcal{P}_{<\alpha})$. We set

$$g(\alpha, \beta) = \mathfrak{c}\,(g(m(\alpha, \beta), \beta), \mathfrak{c}\,(pr(m(\alpha, \beta)), p_0)).$$

(b) Follows directly from (a).

(c) Let $\alpha = \lim \alpha(p)$. We must show that $\mathbf{N} \backslash \mathcal{P}_{<\alpha} \leq_e \mathcal{P}_{<\alpha}$. Recall that $\mathcal{P}_{<\alpha} = \{\langle p, x\rangle \mid x \in \mathcal{P}_{\alpha(p)}\}$. Therefore $x \in \mathbf{N} \backslash \mathcal{P}_{<\alpha} \iff x \notin \mathcal{P}_{<\alpha} \iff x = \langle p, y\rangle \;\&\; y \notin \mathcal{P}_{\alpha(p)}$. Now according to the definition of the enumeration jump we obtain that for arbitrary set C and every z

$$z \notin C \iff 2\langle z, i_0\rangle + 1 \in C',$$

where i_0 is a fixed index for the enumeration operator identity. Now from the proof of (a) we obtain that the sequence $\mathcal{P}'_{\alpha(p)}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\alpha(p)$ and therefore the condition $x \in \mathbf{N} \backslash \mathcal{P}_{<\alpha}$ is e-reducible to $\mathcal{P}_{<\alpha}$. $\square$

Proof of Lemma 3.1. Transfinite induction over α. In the case $\alpha = 0$ the statements are clear. Now let the statements be true for every $\delta < \alpha$. First we will prove (1).

(1) First consider $\alpha = \beta + 1$ and let τ be an arbitrary finite part. Then we set the number n_0 to be $n_0 = \mu q[\tau \upharpoonright q \in \mathcal{R}_{\overline{\beta}}]$. Finding n_0 or proving that such number does not exist is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, since according to the induction hypothesis $\mathcal{R}_{\overline{\beta}} \leq_e \mathcal{P}_\beta$ uniformly in $\overline{\beta}$. If there is no such n_0 then $\tau \notin \mathcal{R}_{\overline{\beta}}$. Let n_j be defined for some $j \geq 0$. Then, if $\mu_{\overline{\beta}}\left(\tau \upharpoonright n_j, S_j^{\overline{\beta}}\right)$ is defined and $\mu_{\overline{\beta}}\left(\tau \upharpoonright n_j, S_j^{\overline{\beta}}\right) \subseteq \tau$, we set $l_j = \mathrm{lh}\left(\mu_{\overline{\beta}}\left(\tau \upharpoonright n_j, S_j^{\overline{\beta}}\right)\right)$. Since the function $\mu_{\overline{\beta}}$ is partially recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, defining l_j is r.e. in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. If we have defined l_j then we set

$$b_j = \mu q[\, q > l_j \;\&\; \tau \upharpoonright q \in O^{\overline{\beta}}_{\langle \tau \upharpoonright l_j, j\rangle}]$$

We know from the induction hypothesis that the sets $O^{\overline{\beta}}_{\langle \rho, j\rangle}$ are e-reducible to $\mathcal{P}'_\beta$ (which is a total set) uniformly in $\overline{\beta}$ and $\langle \rho, j\rangle$, and therefore setting b_j is again r.e. in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. Finally if there is a q, such that $\tau \upharpoonright q \in \mathcal{R}_{\overline{\beta}}$, we set

$$n_{j+1} = \mu q[\, q > b_j + 1 \;\&\; \tau \upharpoonright q \in \mathcal{R}_{\overline{\beta}}]$$

Knowing b_j, defining n_{j+1} is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, and therefore is r.e. in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. Then $\tau \in \mathcal{R}_{\overline{\beta}}$ iff there is n_{r+1}, which is obtained following the construction above, such that $\tau \upharpoonright n_{r+1} = \tau$ and for every $0 \le j \le r$ it is true that $\tau(b_j) \in B_\alpha$. The first condition is r.e. in the total set $\mathcal{P}'_\beta$. The second one is e-reducible to B_α. The two of them are uniform in $\overline{\alpha}$. Therefore $\mathcal{R}_{\overline{\alpha}} \le_e \mathcal{P}'_\beta \oplus B_\alpha$.

Now consider $\alpha = \lim \alpha(p)$. Let τ be an arbitrary finite part. According to Lemma 3.10 we obtain that the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Since the sets $\mathcal{R}_{\overline{\alpha(p)}}$ are e-reducible to $\mathcal{P}_{\alpha(p)}$ uniformly in $\overline{\alpha(p)}$, we obtain that the sequence $\{\mathcal{R}_{\overline{\alpha(p)}}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Analogously to the case $\alpha = \beta + 1$, we can find r.e. in $\mathcal{P}_{<\alpha}$ and uniformly in $\overline{\alpha}$ a sequence of numbers $n_0, b_0, n_1, b_1, \ldots$ satisfying the conditions of the definition of the $\overline{\alpha}$-regularity of τ. If for some of the numbers n_{r+1} is true that $n_{r+1} = \mathrm{lh}(\tau)$ and for every $0 \le j \le r$ $\tau(b_j) \in B_\alpha$ then $\tau \in \mathcal{P}_\alpha$. These conditions are e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.

(2) Follows directly from the proof of (1).

(3) The sequence $\{S^{\overline{\alpha}}_j\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$ as $S^{\overline{\alpha}}_j = \mathcal{R}_{\overline{\alpha}} \cap \Gamma_j(\mathcal{P}_\alpha)$ (Lemma 3.9). In order to prove the statement for $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ let us first assume that $\alpha = \beta + 1$. According to the definition $X^{\overline{\alpha}}_{\langle i,j\rangle} = \{\tau \in \mathcal{R}_{\overline{\alpha}} \mid \tau \Vdash_{\overline{\alpha}} F_i(j)\}$. Also

$$\tau \Vdash_{\overline{\alpha}} F_i(j) \iff \exists v\, (\langle j, v\rangle \in W_i \;\&$$

$$(\forall u \in D_v)((u = \langle 0, i_u, x_u\rangle \;\&\; \tau \Vdash_{\overline{\beta}} F_{i_u}(x_u)) \vee (u = \langle 1, i_u, x_u\rangle \;\&\; \tau \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u))$$

According to the induction hypothesis the conditions $\tau \Vdash_{\overline{\beta}} F_{i_u}(x_u))$ and $\tau \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u))$ are recursive in $\mathcal{P}'_\beta$ uniformly in i_u, x_u and $\overline{\beta}$ (the sequences $\{X^{\overline{\beta}}_k\}$ and $\{Z^{\overline{\beta}}_k\}$ are T-reducible to $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$). Therefore the condition $\tau \Vdash_{\overline{\alpha}} F_i(j)$ is e-reducible to$\mathcal{P}'_\beta$ uniformly in i, j and $\overline{\beta}$. Therefore the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.

Now let $\alpha = \lim \alpha(p)$. Then

$$\tau \Vdash_{\overline{\alpha}} F_i(j) \iff \exists v(\langle j, v\rangle \in W_i \;\&\; (\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \;\&\; \tau \Vdash_{\overline{\alpha}(p)} F_{i_u}(x_u)))$$

But the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in α. The sets $X^{\overline{\alpha}(p)}_{\langle i,j\rangle}$ are e-reducible to $\mathcal{P}_{\alpha(p)}$ uniformly in i, j and $\overline{\alpha}(p)$. Therefore the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. As $\mathcal{P}_{<\alpha}$ is a total set the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is r.e. in $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Then the condition $\tau \Vdash_{\overline{\alpha}(p_u)} F_{i_u}(x_u)$, if $\tau \in X^{\overline{\alpha}(p_u)}_{\langle i_u, x_u\rangle}$ is r.e. $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Finally we obtain that the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.

(4) Since the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$ then the condition, for given τ it is true that $(\exists \rho \in X^{\overline{\alpha}}_i)(\rho \supseteq \tau)$, is r.e. in $\mathcal{P}_\alpha$ uniformly

in i and $\overline{\alpha}$. Then the question, if for given τ is true that $(\forall\rho \supseteq \tau)(\rho \notin X_i^{\overline{\alpha}})$, i.e., if $\tau \in Z_i^{\overline{\alpha}}$, is r.e. in $\mathcal{P}'_\alpha$ uniformly in i and $\overline{\alpha}$. Therefore the sequence $\{Z_i^{\overline{\alpha}}\}$ is T-reducible to $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$.

(5) Follows directly from the definition of the function $\mu_{\overline{\alpha}}$ and the proof of (4).

(6) The reasoning is analogous to the proof of (1) and uses the fact that the function $\lambda\tau, i.\mu_{\overline{\alpha}}(\tau, X_i^{\overline{\alpha}})$ is partially recursive in $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$. □

Definition 3.5. Let τ be $\overline{\alpha}$-regular finite part with rank $r+1$. We define $B_{\overline{\alpha}}^{\tau}$ by:

a) if $\alpha = 0$, then $B_{\overline{\alpha}}^{\tau} = \{x \mid x \in \mathrm{dom}\ (\tau)\ \&\ x \in 2\mathbf{N}+1\}$

b) if $\alpha = \beta + 1$ and $n_0, l_0, b_0, \ldots, n_r, l_r, b_r, n_{r+1}$ are the numbers from the definition of the regular parts, then $B_{\overline{\alpha}}^{\tau} = \{b_0, b_1, \ldots, b_r\}$

c) if $\alpha = \lim\alpha(p)$ and $n_0, b_0, \ldots, n_r, b_r, n_{r+1}$ are the numbers from the definition of the regular parts, then $B_{\overline{\alpha}}^{\tau} = \{b_0, b_1, \ldots, b_r\}$.

Definition 3.6. Let $\overline{\zeta}$ be an approximation of ζ. We say that the partial function f from $\mathbf{N}$ in $\mathbf{N}$ is a regular enumeration respecting $\overline{\zeta}$ iff:

(1) for every finite $\rho \subseteq f$ there is a $\overline{\zeta}$-regular finite part $\tau \supseteq \rho$ such that $\tau \subseteq f$;

(2) if $\overline{\alpha} \preceq \overline{\zeta}$ and $z \in B_\alpha$ then there is an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $z \in \tau(B_{\overline{\alpha}}^{\tau})$.

It is clear from the definition, that if f is a regular enumeration, then f has $\overline{\zeta}$-regular subparts with arbitrary large rank. Then if $\overline{\alpha} \preceq \overline{\zeta}$ and $\rho \subseteq f$ there is an $\overline{\alpha}$-regular finite part $\tau \subseteq f$ such that $\rho \subseteq \tau$. In particular there are $\overline{\alpha}$-regular finite subparts of f of arbitrary rank.

If f is regular and $\overline{\alpha} \preceq \overline{\zeta}$ then with $B_{\overline{\alpha}}^{f}$ we will denote the set

$$B_{\overline{\alpha}}^{f} = \{b \mid (\exists\tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}}\ \&\ b \in B_{\overline{\alpha}}^{\tau})\}.$$

It is clear that $f(B_{\overline{\alpha}}^{f}) = B_\alpha$.

Proposition 3.2. *Let f be a regular enumeration. Then:*

(1) $B_0 \leq_e f$;

(2) *if $\alpha = \beta + 1 \leq \zeta$, then $B_\alpha \leq_e f^+ \oplus \mathcal{P}'_\beta$ uniformly in α;*

(3) *if $\alpha \leq \zeta$ is a limit ordinal, then $B_\alpha \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$ uniformly in α;*

(4) *$\mathcal{P}_\alpha \leq f^{(\alpha)}$ uniformly in α.*

Proof. Let f be a regular enumeration. It is clear that $B_0^f = 2\mathbf{N}+1$. It follows from the regularity that $B_0 = f(B_0^f)$. Therefore $B_0 \leq_e f$.

We will prove (2) and (3) using transfinite induction over α.

Let first $\alpha = \beta + 1$. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$, and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Since f is a regular enumeration, then for every finite part $\rho \subseteq f$ there is an $\overline{\alpha}$-regular finite part $\tau \subseteq f$, such that $\rho \subseteq \tau$. Therefore there is a sequence of natural numbers

$$0 < n_0 < l_0 < b_0 < \cdots < n_r < l_r < b_r < \ldots,$$

satisfying the conditions from the definition of the $\overline{\alpha}$-regular finite parts, and also satisfying that $\tau_r = f \restriction n_{r+1}$ is an $\overline{\alpha}$-regular finite part with $|\tau_r|_{\overline{\alpha}} = r+1$ for all $r \geq 0$. Therefore $B^f_{\overline{\alpha}} = \{b_0, b_1, \dots\}$. We will prove that there is a recursive in $f^+ \oplus \mathcal{P}'_\beta$, uniform in $\overline{\beta}$ procedure, which draws out the numbers $n_0, l_0, b_0, \dots$.

We know from the definition, that $\tau_0 = f \restriction n_0$ is an $\overline{\alpha}$-regular finite part with rank $|\tau_0|_{\overline{\alpha}} = 1$. According to Proposition 3.1 the set $\mathcal{R}_{\overline{\beta}}$ is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. Using the oracle f^+ we may obtain successively all the finite parts $f \restriction q$ for $q = 0, 1, \cdots$. Lemma 3.4 guarantees that τ_0 is the first from the so obtained finite parts which is in $\mathcal{R}_\beta$. Thus we obtain $n_0 = \mathrm{lh}(\tau_0)$.

Now let $r \geq -1$ and let the numbers $n_0, l_0, b_0, \dots, n_r, l_r, b_r, n_{r+1}$ have been obtained. As $S^{\overline{\beta}}_j$ is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, using the oracle $\mathcal{P}'_\beta$ we may obtain $f \restriction l_{r+1} = \mu_{\overline{\beta}}\left(f \restriction (n_{r+1}+1), S^{\overline{\beta}}_j\right)$. Thus we get $l_{r+1} = \mathrm{lh}(f \restriction l_{r+1})$. We know that $f \restriction b_{r+1}$ is a $\overline{\beta}$-regular, $r+1$-omitting extension of $f \restriction l_{r+1}$. Therefore there are numbers $l_{r+1} = q_0 < q_1 < \cdots < q_{r+1} < q_{r+2} = b_{r+1}$ such that for every $p \leq r+1$, it is true that:

$$f \restriction q_{p+1} = \mu_{\overline{\beta}}\left(f \restriction (q_p+1), X^{\overline{\beta}}_{\langle p, q_p\rangle}\right).$$

Therefore, since the sets $X^{\overline{\beta}}_j$ are recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, using successively the oracles f^+ and $\mathcal{P}'_\beta$ we may generate the finite parts $f \restriction (q_p+1)$ for $p = 0, 1, \dots r+2$. At the end of this procedure we obtain the number b_{r+1}. In order to obtain n_{r+2} we generate using the oracle f^+ the finite parts $f \restriction (b_{r+1}+1+q)$ for $q = 0, 1, \dots$. Then $n_{r+2} = \mathrm{lh}(f \restriction n_{r+2})$, where $f \restriction n_{r+2}$ is the first of the generated parts which is in $\mathcal{R}_{\overline{\beta}}$.

Thus we obtain that the set $B^f_\alpha = \{b_0, b_1, \dots\}$ is recursive in $f^+ \oplus \mathcal{P}'_\beta$ and therefore $B_\alpha = f(B^f_\alpha) \leq_e f^+ \oplus \mathcal{P}'_\beta$.

Now let $\alpha = \lim \alpha(p)$. It is clear, that the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is uniformly e-reducible to $\mathcal{P}_{<\alpha}$. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$ and let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. Since f is a regular enumeration, we can assume that f is the union of $\overline{\alpha}$-regular finite parts. Therefore there are numbers

$$0 < n_0 < b_0 < n_1 < b_1 < \cdots < n_r < b_r < \dots$$

satisfying the conditions of the definition. Since for every p the sets $\mathcal{R}_{\overline{\alpha(p)}}$ are uniformly e-reducible to $\mathcal{P}'_{\alpha(p)}$, they are also uniformly e-reducible to $\mathcal{P}_{<\alpha}$. Hence applying the procedure from above we can get the numbers $n_0, b_0, \dots, n_r, b_r, \dots$ recursively in $f^+ \oplus \mathcal{P}_{<\alpha}$. Therefore $B_\alpha = f(B^f_\alpha) \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$.

Thus in both cases the sets B^f_α are r.e. in $f^+ \oplus \mathcal{P}'_\beta$ and $f^+ \oplus \mathcal{P}_{<\alpha}$, and besides this the procedures are uniform over β and α. Therefore the reducibilities in points (2) and (3) of the theorem are uniform over α.

We will prove statement (4) with transfinite induction over α.

In the case $\alpha = 0$ the statement is (1). Now let $\alpha = \beta + 1$. Then $\mathcal{P}_\alpha = \mathcal{P}'_\beta \oplus B_\alpha$. According to the induction hypothesis $\mathcal{P}_\beta \leq_e f^{(\beta)}$ uniformly in β and therefore

$\mathcal{P}'_\beta \leq_e f^{(\alpha)}$ uniformly in α. Beside this $B_\alpha \leq_e f^+ \oplus \mathcal{P}'_\beta$ uniformly in $\overline{\alpha}$ and therefore $B_\alpha \leq_e f^{(\alpha)}$ uniformly in α. Therefore $\mathcal{P}_\alpha \leq_e f^{(\alpha)}$ uniformly in α.

Finally let $\alpha = \lim \alpha(p)$. Then $\mathcal{P}_\alpha = \mathcal{P}_{<\alpha} \oplus B_\alpha$. According to the induction hypothesis $\mathcal{P}_{\alpha(p)} \leq_e f^{(\alpha(p))}$ uniformly in $\overline{\alpha(p)}$. Therefore $\mathcal{P}_{\alpha(p)} \leq_e f^{(\alpha)}$ uniformly in $\alpha(p)$ and therefore $\mathcal{P}_{<\alpha} \leq_e f^{(\alpha)}$ uniformly in α. Beside this $B_\alpha \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$ and therefore $\mathcal{P}_\alpha \leq_e f^{(\alpha)}$ uniformly in α. ☐

Corollary 3.3. *Let f be a regular enumeration. Then $B_\alpha \leq_e f^{(\alpha)}$.*

Proof. From (5) of the proposition $\mathcal{P}_\alpha \leq f^\alpha$. But $B_\alpha \leq \mathcal{P}_\alpha$ which proves the corollary. ☐

Definition 3.7. Let f be a partial function from $\mathbf{N}$ to $\mathbf{N}$, let α be a recursive ordinal and let $i, x \in \mathbf{N}$. We define the relation $\models_\alpha$ by:

a) $\alpha = 0$
$f \models_0 F_i(x) \Leftrightarrow \exists v(\langle v, x\rangle \in W_i \;\&\; D_v \subseteq \langle f\rangle)$;

b) $\alpha = \beta + 1$
$$f \models_\alpha F_i(x) \Leftrightarrow \exists v(\langle v, x\rangle \in W_i \;\&\; (\forall u \in D_v)((u = \langle i_u, x_u, 0\rangle \;\&\; f \models_\beta F_{i_u}(x_u)) \vee (u = \langle i_u, x_u, 1\rangle \& f \models_\beta \neg F_{i_u}(x_u))));$$

c) $\alpha = \lim \alpha(p)$
$$f \models_\alpha F_i(x) \Leftrightarrow \exists v(\langle v, x\rangle \in W_i \& (\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \& f \models_{\alpha(p_u)} F_{i_u}(x_u))).$$

d) for all other cases
$f \models_\alpha \neg F_i(x) \Leftrightarrow f \not\models_\alpha F_i(x)$.

The following Lemma is true:

Lemma 3.11. *There is a partial recursive function h such that for every recursive ordinal α and every enumeration operator Γ_i, it is true that*

$$x \in \Gamma_i(f^{(\alpha)}) \Longleftrightarrow f \models_\alpha F_{h(\alpha,i)}(x)$$

Before proving the Lemma let us note that for arbitrary set C if $\alpha = \beta + 1$ then

$$C^{(\alpha)} \equiv_e \{u \mid (u = \langle 0, i_u, x_u\rangle \;\&x_u \in \Gamma_{i_u}(C^{(\beta)})) \vee (u = \langle 1, i_u, x_u\rangle \;\&x_u \notin \Gamma_{i_u}(C^{(\beta)}))\},$$

and if $\alpha = \lim \alpha(p)$ then

$$C^{(\alpha)} \equiv_e \{u \mid u = \langle p_u, i_u, x_u\rangle \;\&\; x_u \in \Gamma_{i_u}(C^{(\alpha(p_u))})\}$$

uniformly in α.

Proof of Lemma 3.11. We will show that there is a sequence of recursive functions $\{\lambda j.h_\alpha(j)\}_{\alpha\leq\zeta}$ uniform in α such that for every $\alpha \leq \zeta$ and every i the statement

$$x \in \Gamma_i(f^{(\alpha)}) \Longleftrightarrow f \models_\alpha F_{h_\alpha(i)}(x)$$

holds. We will use transfinite induction over $\alpha \leq \zeta$. First let $\alpha = 0$. We set $h_0(i) = i$. It is clear from the definition of $\models_0$ that h_0 has the desired property. Now let $\alpha = \beta + 1$. Then

$$x \in \Gamma_i(f^{(\alpha)})$$
$$\Updownarrow$$
$$\exists v(\langle x, v\rangle \in W_i \ \&\ D_v \subseteq f^{(\alpha)})$$
$$\Updownarrow$$
$$\exists v(\langle x, v\rangle \in W_i \ \&\ (\forall u \in D_v)((u = \langle 0, i_u, x_u\rangle \ \&\ x_u \in \Gamma_{i_u}(f^{(\beta)}))\vee$$
$$(u = \langle 1, i_u, x_u\rangle \ \&\ x_u \not\in \Gamma_{i_u}(f^{(\beta)}))).$$

Then from h_β we obtain

$$x \in \Gamma_i(f^{(\alpha)})$$
$$\Updownarrow$$
$$\exists v(\langle x, v\rangle \in W_i \ \&\ (\forall u \in D_v)((u = \langle 0, i_u, x_u\rangle \ \& f \models_\beta F_{h_\beta(i_u)}(x_u))\vee$$
$$(u = \langle 1, i_u, x_u\rangle \ \& f \models_\beta F_{h_\beta(i_u)}(x_u)))).$$

Consider the set W such that $\langle x, v\rangle \in W$ iff there exists v' such that $\langle x, v'\rangle \in W_i$ and

$$\forall\langle t, i, x\rangle(\langle t, h_\beta(i), x\rangle \in D_v \iff \langle t, i, x\rangle \in D_{v'})$$

Since the function h_β is recursive uniformly in β, then we can obtain recursively and uniformly in β the finite sets D_v from the finite sets $D_{v'}$. Therefore the set W is r.e. with Gödel index i_0. Thus we obtain $x \in \Gamma_{f^{(\alpha)}} \iff f \models_{i_0} (x)$. Beside this, W is obtained uniformly from the index i of the r.e. set W_i and the function h_β. Then i_0 is also obtained uniformly from i and h_β. We set $h_\alpha(i) = i_0$.

Finally let $\alpha = \lim \alpha(p)$. Then $x \in \Gamma_i(f^{(\alpha)}) \iff \exists v(\langle x, v\rangle \in W_i \ \&\ (\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \ \& x_u \in \Gamma_{i_u}(f^{(\alpha(p_u))})))$. Then, according to the induction hypothesis $x \in \Gamma_i(f^{(\alpha)}) \iff \exists v(\langle x, v\rangle \in W_i \ \&\ (\forall u \in D_v)(u = \langle p_u, x_u, i_u\rangle \ \& f \models_{\alpha(p_u)} F_{h_{\alpha(p_u)}(i_u)}(x_u)$. Let us consider the set W, for which $\langle x, v\rangle \in W$ iff there is a v' such that $\langle x, v'\rangle \in W_i$ and

$$\forall\langle p, i, x\rangle(\langle p, h_{\alpha(p)}(i), x\rangle \in D_v \iff \langle p, i, x\rangle \in D_{v'}).$$

Then, exactly as above (as the sequence of recursive functions $\{h_{\alpha(p)}\}$ is uniform in $\alpha(p)$), the finite sets D_v are obtained recursively from the finite sets $D_{v'}$, uniformly in $\{\alpha(p)\}$ and therefore uniformly in α. Then the set W is r.e. with index j_0, which is obtained uniformly from the index i and α. It is clear that $x \in \Gamma_i(f^{(\alpha)}) \iff f \models_\alpha F_{j_0}(x)$. We set $h_\alpha(i)$ to be $h_\alpha(i) = j_0$.

In both cases $h_\alpha(i)$ is uniformly obtained in i and α. □

Corollary 3.4. *Let f be a partial function from $\mathbf{N}$ to $\mathbf{N}$ and let α be a recursive ordinal. Then $A \leq_e f^{(\alpha)}$ iff there is an i such that for every x the condition $x \in A \iff f \models_\alpha F_i(x)$ is satisfied.*

Let us note that for every $\overline{\alpha} \preceq \overline{\beta}$ the relation $\Vdash_{\overline{\alpha}}$ is monotone, i.e., if $\tau \subseteq \rho$ are $\overline{\alpha}$-regular finite parts and $\tau \Vdash_{\overline{\alpha}} F_i(x)$, then $\rho \Vdash_{\overline{\alpha}} F_i(x)$, and also if $\tau \Vdash_{\overline{\alpha}} \neg F_i(x)$, then $\rho \Vdash_{\overline{\alpha}} \neg F_i(x)$.

Lemma 3.12. *Let f be a regular enumeration. Then:*
(1) for every $\overline{\alpha} \preceq \overline{\zeta}$, $f \models_\alpha F_i(x) \Leftrightarrow (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}}\ \&\ \tau \Vdash_{\overline{\alpha}} F_i(x))$;
(2) for every $\overline{\alpha} \prec \overline{\zeta}$, $f \models_\alpha \neg F_i(x) \Leftrightarrow (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}}\ \&\ \tau \Vdash_{\overline{\alpha}} \neg F_i(x))$.

Proof. We will use transfinite induction over α. First let $\alpha = 0$. Then the validity of (1) follows from the compactness of the enumeration operators Γ_i. Now let us prove (2). Let $f \models_0 \neg F_i(x)$. In order to obtain a contradiction assume that, for every $\overline{0}$-regular $\tau \subseteq f$, is true that $\tau \nVdash_{\overline{0}} \neg F_i(x)$, i.e., for every $\overline{0}$-regular $\tau \subseteq f$ there is $\rho \in \mathcal{R}_{\overline{0}}$ such that $\rho \supseteq \tau$ and $\rho \Vdash_{\overline{0}} F_i(x)$. Consider the set $S = \{\rho \in \mathcal{R}_{\overline{0}} \mid \rho \Vdash_{\overline{0}} F_i(x)\}$. It is clear that $S \leq_e \mathcal{P}_0$ and therefore there is an index j, for which $S = S_j^{\overline{0}}$. Let $\mu \subseteq f$ a $\overline{1}$-regular finite part such that $|\mu|_{\overline{1}} > j$. Such one exists, because f is regular and $1 \leq \zeta$. According to the definition of the $\overline{1}$-regular finite parts there is a $\overline{0}$-regular finite part $\rho_0 \subseteq \mu$ such that $\rho_0 \in S_j^{\overline{0}} = S$. Then $\rho_0 \subseteq f$ and from (1) $f \models_0 F_i(x)$, which is a contradiction.

Now suppose that (1) and (2) are true for every $\delta < \alpha$. We will show that the assertions are also true for α.

a) $\alpha = \beta + 1$. First we show (1). Let $f \models_\alpha F_i(x)$. Then there is v such that $\langle v, x\rangle \in W_i$ and $(\forall u \in D_v)((u = \langle i_u, x_u, 0\rangle \& f \models_\beta F_{i_u}(x_u)) \vee (u = \langle i_u, x_u, 1\rangle \& f \models_\beta \neg F_{i_u}(x_u)))$. According to the induction hypothesis we obtain $\tau_0, \tau_1 \subseteq f$ such that $(\forall u \in D_v)((u = \langle i_u, x_u, 0\rangle \& \tau_0 \Vdash_{\overline{\beta}} F_{i_u}(x_u)) \vee (u = \langle i_u, x_u, 1\rangle \& \tau_1 \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u)))$. Since one of the finite parts is extending the other and the forcing relation is monotone, we may assume $\tau_0 = \tau_1 = \tau$. Then from the definition of the $\overline{\alpha}$-forcing we obtain that $\tau \Vdash_{\overline{\alpha}} F_i(x)$.

The reverse is analogous.

Let us now prove (2). The reasoning is analogous to that of the case $\alpha = 0$. Let $f \models_\alpha \neg F_i(x)$. In order to obtain a contradiction assume that for every $\overline{\alpha}$-regular $\tau \subseteq f$ is true that $\tau \nVdash_{\overline{\alpha}} \neg F_i(x)$, i.e., for every $\overline{\alpha}$-regular $\tau \subseteq f$ there is $\rho \in \mathcal{R}_{\overline{\alpha}}$ such that $\rho \supseteq \tau$ and $\rho \Vdash_{\overline{\alpha}} F_i(x)$. Consider the set $S = \{\rho \in \mathcal{R}_{\overline{\alpha}} \mid \rho \Vdash_{\overline{\alpha}} F_i(x)\}$. It is clear that $S \leq_e \mathcal{P}_\alpha$ and therefore there is an index j for which $S = S_j^{\overline{\alpha}}$. Let $\mu \subseteq f$ be such an $\overline{\alpha + 1}$-regular finite part that $|\mu|_{\overline{\alpha+1}} > j$. Such finite part exists as f is regular and $\alpha + 1 \leq \zeta$. According to the definition of the $\overline{\alpha + 1}$-regular finite parts, there is an $\overline{\alpha}$-regular finite part $\rho_0 \subseteq \mu$ such that $\rho_0 \in S_j^{\overline{\alpha}} = S$. Then $\rho_0 \subseteq f$, $\rho_0 \Vdash_{\overline{\alpha} F_i(x)}$ and form (1) we obtain $f \models_\alpha F_i(x)$, which is a contradiction.

The opposite follows directly from (1).

b) $\alpha = \lim \alpha(p)$. First we prove (1). Let $f \models_\alpha F_i(x)$. Then there is a v such that $\langle v, x\rangle \in W_i$ and $(\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \& f \models_{\alpha(p_u)} F_{i_u}(x_u))$. Then according to the induction hypothesis, for every $u \in D_v$, $u = \langle p_u, i_u, x_u\rangle$ there is $\tau_u \subseteq f$ such that $\tau_u \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u)$. Since D_v is finite, then there is $\tau \subseteq f$ such that $\tau_u \subseteq \tau$ for all $u \in D_v$. As the forcing is monotone $\tau \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u)$ for every $u \in D_v$. Then according to the definition of the α-forcing $\tau \Vdash_{\overline{\alpha}} F_i(x)$.

Now suppose that there is $\tau \subseteq f$ such that $\tau \Vdash_{\overline{\alpha}} F_i(x)$. Then there is v such that $\langle v, x\rangle \in W_i$ and $(\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \& \tau \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u))$. Without loss of generality we may assume that τ is $\overline{\alpha(p_u)}$-regular for every $u \in D_v$. Then according to the induction hypothesis $f \models_{\alpha(p_u)} F_{i_u}(x_u)$ for every $u \in D_v$. Therefore $f \models_\alpha F_i(x)$.

The proof of (2) repeats the proof for the case $\alpha = \beta + 1$. □

Proposition 3.3. *Let f be a regular enumeration. Then f is quasiminimal over B_0, i.e., $B_0 <_e f$ and for every total set X is true that:*

$$X \leq_e f \Longrightarrow X \leq_e B_0.$$

Proof. First let us prove that $B_0 <_e f$. We know from proposition 3.2 that $B_0 \leq_e f$. It remains to show that $f \not\leq_e B_0$. In order to obtain a contradiction assume that $f \leq_e B_0$. Then the set $R = \{\tau \in \mathcal{R}_0 \mid \exists x \exists y (f(x) = y \& f(x) \neq \tau(y))\}$ is e-reducible to B_0. Then there is an index i_0 for which $R = S^0_{i_0}$. As f is regular there is a $\overline{1}$-regular finite part $\tau \subseteq f$ such that $|\tau|_{\overline{1}} > i_0$. According to the definition of the $\overline{1}$-regular finite parts, there is a number l_{i_0} such that $\tau_0 = \tau \upharpoonright l_{i_0}$ either is in $S^0_{i_0}$ or no 0-regular extension of τ_0 is in $S^0_{i_0}$. Since $\tau_0 \subseteq f$ it is clear that the first case is impossible. On the other hand, we may extend τ_0 and obtain the finite part τ_1 in such a way, that $\tau_0 \subseteq \tau_1$ and $\tau_1 \in R$. Therefore the second case is also impossible. Therefore, $f \not\leq_e B_0$.

Let us now prove the second part of the quasiminimality condition.

Let A be a total set such that $A \leq_e f$. Since A is total, then there is a total function ψ such that $\langle\psi\rangle \equiv_e A$. Since $\psi \leq_e f$, then there is an i such that $\langle\psi\rangle = \Gamma_i(\langle f\rangle)$. Now consider the set of 0-regular finite parts

$$S = \{\tau \in \mathcal{R}_0 \mid \exists x \exists y_1 \exists y_2 (y_1 \neq y_2 \ \& \ \tau \Vdash_0 F_i(\langle x, y_1\rangle) \ \& \ \tau \Vdash_0 F_i(\langle x, y_2\rangle))\}$$

The condition selecting the finite parts is r.e. and therefore $S \leq_e B_0$. Then there is a j such that $S = S^{\overline{0}}_j$. Let $\rho \subseteq f$ be a finite part such that $|\rho|_1 \geq j+1$. Such a ρ exists, because f is a regular enumeration. Let $n_0, l_0, b_0, \ldots, n_j, l_j, b_j, \ldots$ be the numbers satisfying the definition of the 1-regular finite parts for ρ. Then $\rho \upharpoonright l_j = \mu_{\overline{0}}\left(\rho \upharpoonright (n_j + 1), S^{\overline{0}}_j\right)$. According to the definition of μ either $\rho \upharpoonright l_j \in S^{\overline{0}}_j$ or none of its 0-regular extensions is in $S^{\overline{0}}_j$. Let us assume that the first holds. Then $\rho \upharpoonright l_j \Vdash_0 \langle x, y_1\rangle$ and $\rho \upharpoonright l_j \Vdash_0 \langle x, y_2\rangle$ for some x and $y_1 \neq y_2$. Then $f \models_0 \langle x, y_1\rangle$ and $f \models_0 \langle x, y_1\rangle$ and therefore $\psi(x) = y_1 \neq y_2 = \psi(x)$ which is not possible. Therefore none of the 0-regular extensions of ρ is in $S^{\overline{0}}_j$.

Now consider the set

$$S' = \left\{ \tau \in \mathcal{R}_{\overline{0}} \,\middle|\, \begin{array}{l} (\tau \supseteq \rho \lceil l_j) \;\&\; (\exists \delta_1, \delta_2 \in \mathcal{R}_{\overline{0}})(lh(\rho) \geq lh(\delta_{1/2}) \;\& \\ (\forall z \geq l_j)(\delta_{1/2}(z) \neq \perp \Rightarrow \rho(z) = \perp) \;\& \\ \exists x \exists y_1 \exists y_2 (y_1 \neq y_2 \& \delta_1 \Vdash_{\overline{0}} F_i(\langle x, y_1 \rangle) \;\&\; \delta_2 \Vdash_{\overline{0}} F_i(\langle x, y_2 \rangle)) \end{array} \right\}$$

As above, $S' = S^{\overline{0}}_{j'}$ for some j' and there is a finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S^{\overline{0}}_{j'}$ or no 0-regular extension of τ_0 is in $S^{\overline{0}}_{j}$. Let us assume that the first one holds and let $\delta_1, \delta_2, x, y_1, y_2$ satisfy the condition. As ψ is a total function, $\psi(x) = y$ for some y. Without loss of generality we may assume $y \neq y_1$. Then there is a 0-regular finite part $\tau_1 \subseteq f$ such that $\tau_1 \supseteq \tau_0$ and $\tau_1 \Vdash_0 F_i(\langle x, y \rangle)$. Therefore $\mathrm{lh}(\tau_1) \geq \mathrm{lh}(\delta_1)$ and $\delta_1(z) \neq \perp \Rightarrow \tau_1(z) = \perp$. The last one guarantees the existence of a finite part τ_1' such that $\langle \tau_1' \rangle = \langle \tau_1 \rangle \cup \langle \delta_1 \rangle$. Then $\tau_1' \supseteq \rho \lceil l_j$ and $\tau_1' \Vdash_{\overline{0}} F_i(\langle x, y \rangle)$, and $\tau_1' \Vdash_0 F_i(\langle x, y_1 \rangle)$. Therefore $\tau_1' \in S$ which contradicts the property of $\rho \lceil l_j$. Thus none of the 0-regular extensions of τ_0 is in $S^{\overline{0}}_{j'}$.

Finally consider the set

$$R = \{\tau \in \mathcal{R}_{\overline{0}} \mid \tau \supseteq \tau_0\}.$$

It is clear that $R \leq_e B_0$. All 0-regular finite subparts of f are in R and therefore $\langle \psi \rangle \subseteq \{\langle x, y \rangle \mid (\exists \tau \in R)(\tau \Vdash_0 F_i(\langle x, y \rangle))\}$. For every two finite parts $\rho_1, \rho_2 \in R$ if $\rho_1 \Vdash_{\overline{0}} F_i(\langle x, y_1 \rangle)$ and $\rho_2 \Vdash_{\overline{0}} F_i(\langle x, y_2 \rangle)$, then $y_1 = y_2$. In the contrary case the $\overline{0}$-regular extension τ_1 of τ_0 having the property $\mathrm{lh}(\tau_1) = \max\{\mathrm{lh}(\rho_1), \mathrm{lh}(\rho_2)\}$ and $(\forall z \geq \mathrm{lh}(\tau_0))(\tau_2(z) = \perp)$ is in S'. But this contradicts the property of τ_0 which was proved above. Then $\{\langle x, y \rangle \mid (\exists \tau \in R)(\tau \Vdash_0 F_i(\langle x, y \rangle))\} \subseteq \langle \psi \rangle$ and therefore these two sets coincide. But $\{\langle x, y \rangle \mid (\exists \tau \in R)(\tau \Vdash_0 F_i(\langle x, y \rangle))\} \leq_e B_0$ and therefore $\langle \psi \rangle \leq_e B_0$. □

Proposition 3.4. *Let f be a regular enumeration and $\alpha \leq \zeta$. Then the following assertions hold:*

(1) *if $\alpha = \beta + 1$, then $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}'_\alpha$;*

(2) *if α is a limit ordinal then $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$.*

Proof. First let $\alpha = \beta + 1$. Recall that $f^{(\alpha)} = L^+_{f^{(\beta)}}$, where $L_{f^{(\beta)}} = \{\langle y, z \rangle \mid y \in \Gamma_z(f^{(\beta)})\}$. There is a z_0 not depending on β such that $L_{f^{(\beta)}} = \Gamma_{z_0}(f^{(\beta)})$. Therefore

$$f \models_\beta F_{h(\beta, z_0)}(x) \iff x \in L_{f^{(\beta)}}.$$

Now applying Lemma 3.12, we obtain

$$x \in L_{f^{(\alpha)}} \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\beta}} \;\&\; \tau \Vdash_{\overline{\beta}} F_{h(\beta, z_0)}(x)),$$

$$x \in \mathbf{N} \backslash L_{f^{(\beta)}} \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\beta}} \;\&\; \tau \Vdash_{\overline{\beta}} \neg F_{h(\beta, z_0)}(x)).$$

Therefore, according to Proposition 3.1, and as the condition $\tau \subseteq f$ is uniformly recursive in f^+, we obtain that $L_{f^{(\beta)}}$ and $\mathbf{N} \backslash L_{f^{(\beta)}}$ are uniformly e-reducible $f^+ \oplus \mathcal{P}'_\beta$. Therefore $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}'_\beta$.

Now let α be a limit ordinal. Then there is a z_0 not depending on α, such that $f^{(\alpha)} = \Gamma_{z_0}(f^{(\alpha)})$. Therefore

$$x \in f^{(\alpha)} \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}} \ \& \ \tau \Vdash_{\overline{\alpha}} F_{h(\alpha,z_0)}).$$

According to Proposition 3.1 we obtain $f^{(\alpha)} \leq f^+ \oplus \mathcal{P}_\alpha$. According to Proposition 3.2, $\mathcal{P}_\alpha \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$. Therefore $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$. □

From Proposition 3.2 and 3.4 we obtain the following

Corollary 3.5. *Let f be a regular enumeration and let $\alpha \leq \zeta$. Then:*
(1) if $\alpha = \beta + 1$, then $f^{(\alpha)} \equiv_e f^+ \oplus \mathcal{P}'_\beta$;
(2) if α is a limit ordinal, then $f^{(\alpha)} \equiv_e f^+ \oplus \mathcal{P}_{<\alpha}$.

The following two definitions will be helpful in proving the existence of regular enumerations.

Let us fix a total function σ, such that for every $\alpha \leq \zeta$ $\sigma(\alpha) \in B_\alpha$.

Definition 3.8. Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. We say that τ is $\overline{\alpha}$-complete for σ if

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \Rightarrow \sigma(\beta) \in \tau(B^{\tau}_{\overline{\beta}}).$$

Now let us fix a sequence of sets of natural numbers $\{A_\gamma\}_{\gamma<\zeta}$ such that $(\forall \gamma < \zeta)(A_\gamma \not\leq_e \mathcal{P}_\gamma)$.

Definition 3.9. let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. We say that the finite part τ is $\overline{\alpha}$-omitting in respect to $\{A_\gamma\}$ iff for every $\overline{\beta} \in Reg(\tau, \overline{\alpha})$ the following is true:

If $\beta = \delta + 1$, $\overline{\delta}$ is the δ predecessor of $\overline{\beta}$ and $|\tau|_{\overline{\beta}} = r + 1$, then for every $p \leq r$ there exist a $q_p \in$ dom (τ) and a $\overline{\delta}$-regular finite part $\rho_{p+1} \subseteq \tau$ such that

a) $\rho_{p+1} \Vdash_{\overline{\delta}} F_p(q_p) \ \& \ \tau(q_p) \not\in A_\delta$;

b) $\rho_{p+1} \Vdash_{\overline{\delta}} \neg F_p(q_p) \ \& \ \tau(q_p) \in A_\delta$.

Note, that, as for all x the assertion $x \in A_\delta \vee x \not\in A_\delta$ holds, then the conditions *a)* and *b)* are equivalent to

a′) $\tau(q_p) \not\in A_\delta \Longrightarrow \rho_{p+1} \Vdash_{\overline{\delta}} F_p(q_p)$;

b′) $\tau(q_p) \in A_\delta \Longrightarrow \rho_{p+1} \Vdash_{\overline{\delta}} \neg F_p(q_p)$.

If $\overline{\delta} = \langle \delta_0, \delta_1, \ldots, \delta \rangle$ is an approximation of δ and $\delta < \alpha$, then we will note the approximation $\langle \delta_0, \delta_1, \ldots, \delta, \alpha \rangle$ of α with $\langle \overline{\delta}, \alpha \rangle$.

Now we are ready to prove that the regular enumerations exist.

Proposition 3.5. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. Then the following assertions hold:*

(1) For every $\overline{\alpha}$-regular finite part τ and every $y \in \mathbf{N}$ there is a $\overline{\alpha}$-regular extension ρ of τ such that $|\rho|_{\overline{\alpha}} = |\tau|_{\overline{\alpha}} + 1$, $\rho(\mathrm{lh}(\tau)) = y$, ρ is $\overline{\alpha}$-omittimg and $\overline{\alpha}$-complete.

(2) For every $\overline{\delta} \prec \overline{\alpha}$, for every $\overline{\delta}$-regular τ of rank 1 and every $y \in \mathbf{N}$ there is a $\overline{\delta}, \alpha$-regular extension ρ of τ of rank 1 such that $\rho(\mathrm{lh}(\tau)) = y$, ρ is $\overline{\delta}, \alpha$-omitting and $\overline{\delta}, \alpha$-complete.

Proof. We will prove simultaneously (1) and (2) with transfinite induction over α.

a) $\alpha = 0$. In this case (2) is trivial. Now let us consider (1). Let τ be 0-regular finite part and let $y \in \mathbf{N}$. Set ρ to be

$$\rho(x) = \begin{cases} \tau(x), & x < lh(\tau) \\ y, & x = lh(\tau) \\ \sigma(0), & x = lh(\tau) + 1 \\ \neg!, & x > lh(\tau) + 1 \end{cases}$$

Then ρ is a 0-regular finite part satisfying all the desired properties.

b) Let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. First we prove (1).

Let τ be $\overline{\alpha}$-regular finite part and let $y \in \mathbf{N}$. Let also dom $(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r + 1$. Note, that according to the induction hypothesis for (1), it is true that for every $\overline{\beta}$-regular finite part θ, every set $Z \subseteq \mathcal{R}_{\overline{\beta}}$ and every $y \in \mathbf{N}$ the function $\mu_{\overline{\beta}}(\theta * y, Z)$ has a value. Let us denote n_{r+1} with q. As τ is $\overline{\beta}$-regular, then $\rho' = \mu_{\overline{\beta}}(\tau * y, S^{\overline{\beta}}_{r+1})$ is defined. Then let $l_{r+1} = lh(\rho')$. We will construct a special $\overline{\beta}$-regular $r+1$-omitting extension of ρ'. We will define with induction over $p \leq r+2$ the $\overline{\beta}$-regular finite parts ρ_p and the numbers q_p. Set $q_0 = l_{r+1}$ and $\rho_0 = \rho'$. Assume that for some $p < r+2$ the number q_p and the finite part ρ_p are defined. Consider the set

$$C = \{x \mid (\exists \rho \supseteq \rho_p)(\rho \in \mathcal{R}_{\overline{\beta}} \;\&\; \rho(q_p) = x \;\&\; \rho \Vdash_{\overline{\beta}} F_p(q_p))\}.$$

Note that

$$x \notin C \iff (\forall \rho \in \mathcal{R}_{\overline{\beta}})(\rho \supseteq (\rho_p * x) \Longrightarrow \rho \nVdash_{\overline{\beta}} F_p(q_p)).$$

From the definition of C and Proposition 3.1 we obtain $C \leq_e \mathcal{P}_\beta$ and therefore $C \neq A_\beta$. Let x_0 be the least number such that

$$x_0 \in A_\beta \,\&\, x_0 \notin C \;\vee\; x_0 \notin A_\beta \,\&\, x_0 \in C.$$

Then set $\rho_{p+1} = \mu_{\overline{\beta}}\left(\rho_p * x_0, X^{\overline{\beta}}_{\langle p, q_p \rangle}\right)$ and $q_{p+1} = \mathrm{lh}(\rho_{p+1})$.

Now we obtain that $\rho'' = \rho_{r+2}$ is a $\overline{\beta}$-regular $r+1$-omitting extension ρ_0. Set $b_{r+1} = \mathrm{lh}(\rho'')$. Finally set ρ to be a $\overline{\beta}$-regular extension of ρ'', such that

$|\rho|_{\overline{\beta}} = |\rho''|_{\overline{\beta}} + 1$, $\rho(b_{r+1}) = \sigma(\alpha)$, ρ is a $\overline{\beta}$-omitting and $\overline{\beta}$-complete. Then ρ satisfies (1) from the theorem. Indeed, from the construction of ρ we obtain that ρ is an $\overline{\alpha}$-regular extension of $\tau * y$ and $|\rho|_{\overline{\alpha}} = |\tau|_{\overline{\alpha}} + 1$. In order to show that ρ is $\overline{\alpha}$-complete in respect to σ recall that according to Lemma 3.8

$$\overline{\delta} \in Reg(\rho, \overline{\alpha}) \iff \overline{\delta} = \overline{\alpha} \vee \overline{\delta} \in Reg(\rho, \overline{\beta}).$$

Now fix a $\overline{\delta} \in Reg(\rho, \overline{\alpha})$. If $\overline{\delta} = \overline{\alpha}$ (i.e., $\delta = \alpha$) then $\sigma(\alpha) = \rho(b_{r+1})$. If $\delta \in Reg(\rho, \overline{\beta})$, then, since ρ is $\overline{\beta}$-complete finite part, there is a $b_\delta \in dom(\rho)$, such that $\sigma(\delta) = \rho(b_\delta)$. Therefore ρ is $\overline{\alpha}$-complete.

Now let us prove that ρ is $\overline{\alpha}$-omitting. Fix $\overline{\delta + 1} \in Reg(\rho, \overline{\alpha})$. Then again according to Lemma 3.8 either $\overline{\delta} = \overline{\beta}$ or $\overline{\delta + 1} \in Reg(\rho, \overline{\beta})$ holds. First let $\delta = \beta$. Then as $|\rho|_{\overline{\alpha}} = r + 2$, fix a $p \leq r + 1$. Consider the finite part ρ_{p+1} and the number q_p from the construction. If $\rho_{p+1}(q_p) \in A_\beta$, it follows from the construction that $\rho_{p+1}(q_p)$ is not in the corresponding set C. Now according to the note made after the definition of C, we have $\rho_{p+1} \Vdash_{\overline{\beta}} \neg F_p(q_p)$. Therefore the condition (a') from the definition of the $\overline{\alpha}$-omitting holds. On the other hand, if $\rho_{p+1}(q_p) \notin A_\beta$ holds, then ρ_{p+1} is the least $\overline{\beta}$-regular extension of $\rho_p * (\rho_{p+1}(q_p))$ such that $\rho_{p+1} \Vdash_{\overline{\beta}} F_p(q_p)$ and there for the condition (b') from the definition of the $\overline{\alpha}$-omitting is satisfied.

If $\overline{\delta + 1} \in Reg(\rho, \overline{\beta})$, then we obtain the omitting conditions from the fact that ρ is a $\overline{\beta}$-omitting finite part.

Now let us prove (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1.

1) $\delta = \beta$. Then $\overline{\delta} = \overline{\beta}$ and beside this $\overline{\beta}$ is the β-predecessor of $\overline{\delta}, \alpha$. Let $n_0 = \text{lh}(\tau)$ and $\rho_0 = \mu_{\overline{\beta}}\left(\tau * y, S_0^{\overline{\beta}}\right)$. Let also ρ_1 be a 0-omitting, $\overline{\beta}$-regular extension of ρ_0, built as above, let $b_1 = \text{lh}(\rho_1)$, and let ρ be a $\overline{\beta}$-complete, $\overline{\beta}$-omitting extension of ρ_1, such that $\rho_1(b_1) = \sigma(\alpha)$ and $|\rho|_{\overline{\beta}} = |\rho_1|_{\overline{\beta}} + 1$. It is clear that ρ is a $\langle\overline{\delta}, \alpha\rangle$-regular finite part with rank 1, which is α-complete and α-omitting.

2) $\delta < \beta$. Then according to Lemma 3.2 the β-predecessor of $\langle\overline{\delta}, \alpha\rangle$ is $\langle\overline{\delta}, \beta\rangle$ and $\overline{\delta} \prec \overline{\beta}$ holds. Using the induction hypothesis extend τ to a $\langle\overline{\delta}, \beta\rangle$-regular finite part ρ_1 of rank 1, such that $\rho_1(\text{lh}(\tau)) = y$. Then we extend ρ_1 to a $\langle\overline{\delta}, \alpha\rangle$-complete and $\langle\overline{\delta}, \alpha\rangle$-omitting finite part ρ of rank 1 as in the prove of (1).

c) Let $\alpha = \lim \alpha(p)$. Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha\rangle$ and let $p_0 = \mu p[\alpha_n < \alpha(p)]$. As in the previous case, let us first prove (1).

Let τ be an α-regular finite part with rank $r + 1$ and let $y \in \mathbf{N}$. It is clear that τ is an $\overline{\alpha(p_0 + 2r + 1)}$-regular finite part with rank 1. According to the induction hypothesis for (2) there is an $\langle\alpha(p_0 + 2r + 1), \alpha(p_0 + 2r + 2)\rangle$-regular extension ρ_0 of τ of rank 1 such that $\rho_0(\text{lh}(\tau)) = y$. Set $b_{r+1} = \text{lh}(\rho_0)$. Again, according to the induction hypothesis for (2), we construct a $\langle\alpha(p_0 + 2r + 1), \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle$-regular extension ρ of ρ_0 of rank 1, such that $\rho(b_{r+1}) = \sigma(\alpha)$ and ρ is $\langle\alpha(p_0 + 2r + 1), \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle$-complete and $\langle\alpha(p_0 + 2r + 1), \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle$-omitting. Note that $\langle\alpha(p_0 + 2r + 1), \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle = \overline{\alpha(p_0 + 2r + 3)}$. Therefore ρ is an $\overline{\alpha}$-regular finite part of rank $r + 2$. It remains to show that ρ is $\overline{\alpha}$-complete and $\overline{\alpha}$-omitting. Let $\overline{\beta} \in Reg(\rho, \overline{\alpha})$. Then

$\overline{\beta} = \overline{\alpha}$ or $\overline{\beta} \in Reg(\tau, \overline{\alpha(p_0 + 2r + 3)})$. In both cases it follows from the construction that $\sigma(\beta) \in \rho(B^{\rho}_{\overline{\beta}})$.

In order to show, that ρ is $\overline{\alpha}$-omitting, let us assume that $\beta = \delta + 1$. Then $\beta \neq \alpha$ and therefore $\overline{\beta} \in Reg(\tau, \overline{\alpha(p_0 + 2r + 3)})$. As ρ is $\overline{\alpha(p_0 + 2r + 3)}$-omitting then it satisfies the omitting conditions in respect to β.

Finally let us show (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part with rank 1. Let $y \in \mathbf{N}$ and let also $p_\delta = \mu p[\delta < \alpha(p)]$. According to the induction hypothesis for (2), there is a $\langle\overline{\delta}, \alpha(p_\delta)\rangle$-regular extension ρ_1 of τ such that $\rho_1(\mathrm{lh}(\tau)) = y$ and ρ_1 has $\langle\overline{\delta}, \alpha(p_\delta)\rangle$-rank 1. Then again according to the induction hypothesis for (2) we obtain a $\langle\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)\rangle$-regular extension ρ of ρ_1, which has rank 1 and for which $\rho(b_0) = \sigma(\alpha)$ holds and which also is $\langle\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)\rangle$-complete and $\langle\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)\rangle$-omitting. Then ρ is $\langle\overline{\delta}, \alpha\rangle$-regular extension of τ with rank 1 which is $\langle\overline{\delta}, \alpha\rangle$-complete and $\langle\overline{\delta}, \alpha\rangle$-omitting. □

Note that from the proof we have that the construction is recursive in the set

$$(\bigoplus_{\gamma<\zeta} A^+_\gamma) \oplus \sigma \oplus \mathcal{P}_\alpha.$$

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. Let us fix an arbitrary approximation $\overline{\zeta}$ of ζ. We will construct recursively in Q a sequence of finite regular parts $\{\tau_s\}$ such that $\tau_s \subseteq \tau_{s+1}$ and that the partial function $f = \bigcup_s \tau_s$ is a regular enumeration. Using the previous propositions and some additional reasoning we will see that the set $F \underset{+}{=} \langle f\rangle$ has the desired properties.

As Q is total and $\mathcal{P}_\zeta \leq_e Q$ then according to Lemma 3.2 there are a recursive in Q function $\sigma(\gamma, i)$, such that for every $\gamma \leq \zeta$ the function $\lambda i.\sigma(\gamma, i)$ is enumerating B_γ. Let us fix σ. When constructing the sequence $\{\tau_s\}$, we will ensure that every finite part τ_s is $\overline{\zeta}$-regular of $\overline{\zeta}$-rank equal to $s + 1$, and τ_{s+1} is $\overline{\zeta}$-omitting in respect to $\{A_\gamma\}$ and $\overline{\zeta}$-complete in respect to $\sigma_s = \lambda\gamma.\sigma(\gamma, (s)_1)$ where $s = \langle(s)_0, (s)_1\rangle$. Let us also fix a recursive in Q enumeration $y_0, y_1, \ldots, y_s, \ldots$ of Q.

We begin by setting τ_0 to be an arbitrary $\overline{\zeta}$-regular finite part with $\overline{\zeta}$-rank 1. Let τ_s be constructed. Then according to Proposition 3.5 we can obtain recursively in Q a $\overline{\zeta}$-regular extension τ_{s+1} of τ_s, such that $\tau_{s+1}(\mathrm{lh}(\tau_s)) = y_s$, $|\tau_{s+1}|_{\overline{\zeta}} = |\tau_s|_{\overline{\zeta}} + 1$ and τ_{s+1} is $\overline{\zeta}$-omitting and $\overline{\zeta}$-complete in respect to σ_s. Note that τ_{s+1} is strictly extending τ_s.

First let us show that f is a regular enumeration.

Note that f is a partial function from $\mathbf{N}$ in $\mathbf{N}$, and for every $\rho \subseteq f$ there is an index s, such that $\rho \subseteq \tau_s$. Then consider $\overline{\gamma} \preceq \overline{\zeta}$ and $z \in B_\gamma$. Let us fix an s such big that every $\overline{\zeta}$-regular finite part of $\overline{\zeta}$-rank at least s is $\overline{\gamma}$-regular (such an s exists according to Lemma 3.2). We can also choose s such that $z = \sigma(\gamma, (s)_1)$ holds. Then as τ_{s+1} is of $\overline{\zeta}$-rank $s + 2$ and is $\overline{\zeta}$-complete in respect to $\sigma_s = \lambda\gamma.\sigma(\gamma, (s)_1)$ we obtain that $z \in \tau_{s+1}(B^{\tau_{s+1}}_{\overline{\gamma}})$. Therefore f is a regular enumeration.

Now we show that $f^{(\zeta)} \equiv_e Q$.

It is clear that $f^+ \leq_e Q$. Beside this as f is regular then, according to Proposition 3.4, $f^{(\zeta)} \leq_e f^+ \oplus \mathcal{P}_\zeta \leq_e Q$. From the proof of Proposition 3.2 we obtain a recursive in $f^+ \oplus \mathcal{P}_\zeta$ procedure which gives us the sequence $q_s = lh(\tau_s)$. It is also true that

$$y \in Q \iff \exists s(y = f(q_s)),$$

and $f(q_s)$ is always defined. Thus $Q \leq_e f^{(\zeta)}$ and therefore $f^{(\zeta)} \equiv_e Q$.

It remains to prove that for every $\gamma < \zeta$, $A_\gamma \not\leq f^{(\gamma)}$ is satisfied.

To obtain a contradiction assume that for some $\gamma < \zeta$, $A_\gamma \leq f^{(\gamma)}$ holds. Then the set $f^{-1}(A_\gamma) = \{x \mid \exists y(\langle x, y\rangle \in \langle f\rangle \ \&\ y \in A_\gamma)\}$ is also e-reducible to $f^{(\gamma)}$. Then there is an index i, for which

$$x \in C \iff f \models_\gamma F_i(x).$$

Let $\overline{\gamma+1}$ be the $\gamma+1$-predecessor of $\overline{\zeta}$ and let $\overline{\gamma}$ be the γ-predecessor of $\overline{\gamma+1}$. Let s be so big that every $\overline{\zeta}$-regular finite part is $\overline{\gamma+1}$-regular of $\overline{\gamma+1}$-rank greater or equal to i (such an s exists according to Lemma 3.2). Then τ_{s+1} is $\overline{\gamma+1}$-regular and $|\tau_{s+1}|_{\overline{\gamma+1}} > i$. As τ_{s+1} is $\overline{\zeta}$-omitting finite part there is a $q \in \operatorname{dom}(\tau_{s+1})$ and a $\overline{\gamma}$-regular finite part $\rho \subseteq \tau_{s+1}$ such that:

$$\rho \Vdash_{\overline{\gamma}} F_i(q) \ \&\ \tau_{s+1}(q) \notin A_\gamma \ \vee\ \rho \Vdash_{\overline{\gamma}} \neg F_i(q) \ \&\ \tau_{s+1}(q) \in A_\gamma.$$

Therefore

$$f(q) \in A_\gamma \Longrightarrow (\exists \rho \subseteq f)(\rho \Vdash_{\overline{\gamma}} F_i(q)) \ \&\ f(q) \notin A_\gamma \Longrightarrow (\exists \rho \subseteq f)(\rho \Vdash_{\overline{\gamma}} \neg F_i(q))$$

Then according to the Truth Lemma (Lemma 3.12),

$$f \models_\gamma F_i(q) \iff q \notin C,$$

which is a contradiction. □

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A REDUCIBILITY IN THE THEORY OF ITERATIVE COMBINATORY SPACES

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The notion of iterative combinatory space introduced in the past by the present author gave the framework for an algebraic generalization of a part of Computability Theory. In the present paper a reducibility concerning iterative combinatory spaces is considered, as well as the corresponding equivalence. A statement of J. Zashev about variants of iterative combinatory spaces is shown to fail under certain interpretations of the notion of variant in the terms of the relations in question.

Keywords: combinatory space, iterative combinatory space, computable element, computable mapping

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1. SOME PRELIMINARIES

According to Definition II.1.1 in [5], a combinatory space is a 9-tuple

$$\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F),$$

where $\mathcal{F}$ is a partially ordered semigroup, I is its identity, $\mathcal{C} \subsetneq \mathcal{F}$, $\Pi : \mathcal{F}^2 \to \mathcal{F}$, $\Sigma : \mathcal{F}^3 \to \mathcal{F}$, $L, R, T, F \in \mathcal{F}$, and the following conditions are identically satisfied, when $\varphi, \psi, \theta, \chi$ range over $\mathcal{F}$, and a, b, c range over $\mathcal{C}$:

$$\begin{gathered}\forall c(\varphi c \geq \psi c) \;\Rightarrow\; \varphi \geq \psi,\\ \Pi(a,b) \in \mathcal{C},\; L\,\Pi(a,b) = a,\; R\,\Pi(a,b) = b,\\ \Pi(\varphi,\psi)c = \Pi(\varphi c, \psi c),\; \Pi(I, \psi c)\theta = \Pi(\theta, \psi c),\; \Pi(c, I)\theta = \Pi(c, \theta),\\ T \neq F,\; Tc \in \mathcal{C},\; Fc \in \mathcal{C},\end{gathered}$$

$$\Sigma(T,\varphi,\psi)=\varphi,\ \Sigma(F,\varphi,\psi)=\psi,\ \theta\,\Sigma(\chi,\varphi,\psi)=\Sigma(\chi,\theta\varphi,\theta\psi)$$
$$\Sigma(\chi,\varphi,\psi)c=\Sigma(\chi c,\varphi c,\psi c),\ \Sigma(I,\varphi c,\psi c)\theta=\Sigma(\theta,\varphi c,\psi c),$$
$$\varphi\geq\psi,\ \theta\geq\chi\ \Rightarrow\ \Sigma(I,\varphi,\theta)\geq\Sigma(I,\psi,\chi)$$

(the same notion is named "semicombinatory space" in [3,4]). The definition implies that multiplication, Π and Σ are monotonically increasing operations. If for some given σ,χ in $\mathcal{F}$ the equation $\theta=\Sigma(\chi,\theta\sigma,I)$ has a least solution θ, and this solution has certain additional nice properties, then the solution in question is called *the iteration of σ controlled by χ*, and it is denoted by $[\sigma,\chi]$. [1] In the present paper it will be also called *the $\mathcal{S}$-iteration of σ controlled by χ*, and the notation $[\sigma,\chi]^{\mathcal{S}}$ will be also used for it.

The triple $(\mathcal{F},I,\mathcal{C})$ will be further called *the kernel* of the combinatory space $(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$. We shall often consider pairs of combinatory spaces having one and the same kernel. The following statement concerning such pairs can be obtained as an immediate corollary of the definition of iteration.

Lemma 1.1. *Let $\mathcal{S}_i=(\mathcal{F},I,\mathcal{C},\Pi_i,L_i,R_i,\Sigma_i,T_i,F_i)$, $i=0,1$, be combinatory spaces, let χ_0,χ_1 be elements of $\mathcal{F}$ such that $\Sigma_0(\chi_0,\varphi,\psi)=\Sigma_1(\chi_1,\varphi,\psi)$ for all φ,ψ in $\mathcal{F}$, and let σ,ι be elements of $\mathcal{F}$ such that ι is the $\mathcal{S}_0$-iteration of σ controlled by χ_0. Then ι is also the $\mathcal{S}_1$-iteration of σ controlled by χ_1.*

A combinatory space $\mathcal{S}=(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$ is called *iterative* if the iteration of σ controlled by χ exists for any σ and χ in $\mathcal{F}$. A notion of computability in iterative combinatory spaces was studied, and some versions of the First Recursion Theorem and of the Normal Form Theorem are among the results about it (intuitively, the elements of $\mathcal{F}$ play the role of functions in that theory, ordinary computability in the set of the natural numbers and abstract first order computability in the sense of Moschovakis [1] being particular instances). The considered computability is a relative one, namely for any subset $\mathcal{B}$ of $\mathcal{F}$ some elements of $\mathcal{F}$ and some operations in $\mathcal{F}$ are called $\mathcal{S}$-computable in $\mathcal{B}$ (however, mainly the particular case of an empty $\mathcal{B}$ will matter for the present paper).

Numerous examples of iterative combinatory spaces are given in the books [2,5]. A class of such examples (actually the simplest ones) is indicated in Example II.1.2 of [5], the iterativeness of the corresponding combinatory spaces being established in Section II.4 of [5]. The construction of these examples looks as follows. We take an infinite set M, an injective mapping J of M^2 into M, partial mappings L and R of M into M such that $L(J(s,t))=s$, $R(J(s,t))=t$ for all s,t in M, as well as two total mappings T and F of M into M and a partial predicate H on M such that $H(T(u))$ is true and $H(F(u))$ is false for any u in M (any 7-tuple (M,J,L,R,T,F,H) with such components is called a *computational structure*). Then we consider the 9-tuple $(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$, where $\mathcal{F}$ consists of all partial mappings of M into M, the multiplication in $\mathcal{F}$ is defined by $\varphi\psi=\lambda u.\,\varphi(\psi(u))$, the inequality $\varphi\geq\psi$ means that φ is an extension of ψ, I is

[1] The precise definition of iteration can be found in Section II.3 of [5], and, up to an exchange of the second and the third arguments of Σ, also in [3,4].

the identity mapping of M onto itself, $\mathcal{C}$ consists of all total constant mappings of M into M, $\Pi(\varphi,\psi) = \lambda u.\, J(\varphi(u),\psi(u))$ for any φ and ψ in $\mathcal{F}$, and we have $\Sigma(\chi,\varphi,\psi)(u) = v$ iff either $H(\chi(u))$ is true and $\varphi(u) = v$, or $H(\chi(u))$ is false and $\psi(u) = v$. It is shown that each 9-tuple constructed in such a way is an iterative combinatory space, and the equality $[\sigma,\chi](u) = v$ holds iff there are a non-negative integer m and a finite sequence $w_0, w_1, \ldots, w_m$ of elements of M such that $w_0 = u$, $w_m = v$, $H(\chi(w_j))$ is true and $w_{j+1} = \sigma(w_j)$ for $j = 0, 1, \ldots, m-1$, whereas $H(\chi(w_m))$ is false. The combinatory spaces of this kind will be called here *pf-spaces* (*combinatory spaces of partial functions*). A pf-space will be called *ordinary* if its last two components are constant functions. Without naming them so, the ordinary pf-spaces are considered already in [2] – they actually form the content of Example 1 in Section II.1.3 there, and their iterativeness is shown in Section III.3.2 of the book.

The next two examples indicate certain concrete pf-spaces corresponding to computational structures whose first component is the set $\mathbb{N}$ of the non-negative integers.

Example 1.1. Let J be the bijection from $\mathbb{N}^2$ to $\mathbb{N}$ defined by

$$J(s,t) = \frac{(s+t)(s+t+1)}{2} + s\,,$$

L, R, T, F be the functions from $\mathbb{N}$ to $\mathbb{N}$ defined by the equalities

$$L(J(s,t)) = s,\ \ R(J(s,t)) = t,\ \ T(u) = 1,\ \ F(u) = 0,$$

and H be the predicate that is false at 0 and true at all other elements of $\mathbb{N}$. Then $(\mathbb{N}, J, L, R, T, F, H)$ is a computational structure, and we may consider its corresponding pf-space.

Example 1.2. The same as the previous example, except that T is defined by means of the equality $T(u) = u+1$ (the corresponding pf-space is not an ordinary one).

Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be an iterative combinatory space. The notion of $\mathcal{S}$-computability (coinciding with $\mathcal{S}$-computability in the empty set in the terminology of [5]) is defined as follows. An element of $\mathcal{F}$ will be called *$\mathcal{S}$-computable* if this element can be obtained from the elements L, R, T, F by means of multiplication, the operation Π and $\mathcal{S}$-iteration (if $\mathcal{B}$ is a subset of $\mathcal{F}$ then an element of $\mathcal{F}$ is called *$\mathcal{S}$-computable in $\mathcal{B}$* if this element can be obtained from elements of the set $\{L, R, T, F\} \cup \mathcal{B}$ by means of the three operations in question). A mapping Γ of $\mathcal{F}^n$ into $\mathcal{F}$ will be called *$\mathcal{S}$-computable* if for arbitrary $\theta_1, \ldots, \theta_n$ in $\mathcal{F}$ there is an explicit expression for $\Gamma(\theta_1, \ldots, \theta_n)$ through $L, R, T, F, \theta_1, \ldots, \theta_n$ by means of multiplication, the operation Π and $\mathcal{S}$-iteration, the form of this expression not depending on the choice of $\theta_1, \ldots, \theta_n$ ($\mathcal{S}$-computability of Γ in a given subset $\mathcal{B}$ of $\mathcal{F}$ is defined similarly, but the expression for $\Gamma(\theta_1, \ldots, \theta_n)$ may contain now also notations for some fixed elements of $\mathcal{B}$).

Remark 1.1. By the equality $[\sigma, F] = I$, the function I is $\mathcal{S}$-computable. The mapping Σ is also $\mathcal{S}$-computable, since (as shown in Section II.5 of [5])

$$\Sigma(\chi, \varphi, \psi) = [R_\star \psi][R_\star^2 \varphi R]\Pi(\chi, L_\star),$$

where $[\sigma] = R[\sigma R, L]$, $L_\star = \Pi(T, I)$, $R_\star = \Pi(F, I)$. Hence adding I to the initial elements and Σ to the used operations in the above definitions would not enlarge the set of the $\mathcal{S}$-computable elements of $\mathcal{F}$ and the set of the $\mathcal{S}$-computable mappings of $\mathcal{F}^n$ into $\mathcal{F}$.

Example 1.1 (continuation). Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be the combinatory space indicated in Example 1.1. Then all $\mathcal{S}$-computable elements of $\mathcal{F}$ are one-argument partial recursive functions. However, the converse statement is not true. For instance the primitive recursive function θ defined by the equality $\theta(u) = |u - 1|$ is not $\mathcal{S}$-computable. To prove this, we consider the family of all pre-images of the sets $\{0\}$ and $\mathbb{N} \setminus \{0\}$ under products of finitely many L's and R's (the function I being also regarded as such a product). Let $\mathcal{T}$ be the topology in $\mathbb{N}$ having as a prebase this family. The functions J, L, R, T, F can be easily shown to be continuous with respect to $\mathcal{T}$. It follows from here by Exercise II.4.21 of [5] that all functions from $\mathcal{F}$ have open domains and are continuous with respect to $\mathcal{T}$. On the other hand, the function θ is not continuous with respect to $\mathcal{T}$ since $\theta^{-1}\{0\} = \{1\}$, and the set $\{1\}$ is not open in $\mathcal{T}$ because any open set containing 1 contains also some number distinct from 1, namely some number of the form $J(0, J(0, \ldots J(0, J(0, 2)) \ldots))$.

Example 1.2 (continuation). Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be the combinatory space from Example 1.2. Then again all $\mathcal{S}$-computable elements of $\mathcal{F}$ are one-argument partial recursive functions, but now the converse statement is also true. In view of Theorem I.3.1 of [5] it is sufficient to show the $\mathcal{S}$-computability of the function $\lambda u.\, u \dot{-} 1$, where $u \dot{-} 1$ is $u - 1$ for $u \in \mathbb{N} \setminus \{0\}$ and 0 for $u = 0$. Its $\mathcal{S}$-computability is seen from the fact that $R(J(u, u) + 1) = u - 1$ for any positive integer u, and therefore $\lambda u.\, u \dot{-} 1 = \Sigma(I, RT\Pi(I, I), F)$. [2]

2. REDUCIBILITY OF AN ITERATIVE COMBINATORY SPACE TO A GIVEN ONE

We shall again be interested in pairs of combinatory spaces having one and the same kernel (as in Lemma 1.1).

Lemma 2.1. *Let $(\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be combinatory spaces, and τ be such an element of $\mathcal{F}$ that $\Pi_0(a, b) = \tau\Pi_1(a, b)$ for all $a, b \in \mathcal{C}$. Then $\Pi_0(\varphi, \psi) = \tau\Pi_1(\varphi, \psi)$ for all $\varphi, \psi \in \mathcal{F}$.*

---

[2] Actually a slight generalization of Theorem I.3.1 of [5] holds that allows an arbitrary function from $\mathcal{F}$ coinciding with the function $\lambda u.\, u \dot{-} 1$ on the positive integers to be used instead of it. $RT\Pi(I, I)$ is such a function already (the functions $RT^2\Pi(\theta, T)$ with $\theta \in \{L, R, T, F\}$ are also such ones).

Proof. Let φ, ψ be arbitrary elements of $\mathcal{F}$. Since $\Pi_i(a,b) = \Pi_i(a,I)b$, $i = 0,1$, we see that $\Pi_0(a,I)b = \tau\Pi_1(a,I)b$ for all $a,b \in \mathcal{C}$, hence $\Pi_0(a,I) = \tau\Pi_1(a,I)$ for all $a \in \mathcal{C}$. Therefore

$$\Pi_0(I,\psi c)a = \Pi_0(a,\psi c) = \Pi_0(a,I)\psi c = \tau\Pi_1(a,I)\psi c = \tau\Pi_1(a,\psi c) = \tau\Pi_1(I,\psi c)a$$

for all $a, c \in \mathcal{C}$, hence $\Pi_0(I,\psi c) = \tau\Pi_1(I,\psi c)$ for all $c \in \mathcal{C}$. It follows from here that

$$\Pi_0(\varphi,\psi)c = \Pi_0(\varphi c,\psi c) = \Pi_0(I,\psi c)\varphi c = \tau\Pi_1(I,\psi c)\varphi c = \tau\Pi_1(\varphi c,\psi c) = \tau\Pi_1(\varphi,\psi)c$$

for all $c \in \mathcal{C}$, and this proves the equality $\Pi_0(\varphi,\psi) = \tau\Pi_1(\varphi,\psi)$. □

Whenever $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0,1$, are combinatory spaces with one and the same kernel, we set

$$P_{\mathcal{S}_0}^{\mathcal{S}_1} = \Pi_1(L_0, R_0),\ Q_{\mathcal{S}_0}^{\mathcal{S}_1} = \Sigma_1(L_0, T_0R_0, F_0R_0),\ \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1} = \Sigma_1(I, T_0, F_0).$$

Lemma 2.2. *Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0,1$, be combinatory spaces. Then*

$$\Pi_1(\varphi,\psi) = P_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\varphi,\psi),\quad \Sigma_1(\chi,\varphi,\psi) = \Sigma_0(Q_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\chi,I),\varphi,\psi)$$

for all φ,ψ,χ in $\mathcal{F}$. If T_0 and F_0 belong to $\mathcal{C}$ then $Q_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\chi,I) = \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}\chi$, thus $\Sigma_1(\chi,\varphi,\psi) = \Sigma_0(\dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}\chi,\varphi,\psi)$ in that case.

Proof. The first of the equalities follows by Lemma 2.1 from the fact that

$$\Pi_1(a,b) = P_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(a,b)$$

for all $a, b \in \mathcal{C}$. For the general case in the rest of the proof we first observe that

$$Q_{\mathcal{S}_0}^{\mathcal{S}_1}\Pi_0(\chi,I) = \Sigma_1(\chi,T_0,F_0)$$

for all $\chi \in \mathcal{F}$ (we get this equality by applying Proposition II.1.8 of [5] to the "mixed" combinatory space $(\mathcal{F}, I, \mathcal{C}, \Pi_0, L_0, R_0, \Sigma_1, T_1, F_1)$). In the case when T_0 and F_0 belong to $\mathcal{C}$, we also have the equality

$$\dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}\chi = \Sigma_1(\chi,T_0,F_0),$$

because then, by Proposition II.1.2 of [5], we have $T_0 = T_0c$, $F_0 = F_0c$ for any $c \in \mathcal{C}$. On the other hand, $\Sigma_0(\Sigma_1(\chi,T_0,F_0),\varphi,\psi) = \Sigma_1(\chi,\varphi,\psi)$, since

$$\begin{gathered}\Sigma_0(\Sigma_1(\chi,T_0,F_0),\varphi,\psi)c = \Sigma_0(\Sigma_1(\chi c,T_0c,F_0c),\varphi c,\psi c) = \\ \Sigma_0(I,\varphi c,\psi c)\Sigma_1(\chi c,T_0c,F_0c) = \Sigma_1(\chi c,\Sigma_0(I,\varphi c,\psi c)T_0c,\Sigma_0(I,\varphi c,\psi c)F_0c) = \\ \Sigma_1(\chi c,\Sigma_0(T_0c,\varphi c,\psi c),\Sigma_0(F_0c,\varphi c,\psi c)) = \Sigma_1(\chi,\Sigma_0(T_0,\varphi,\psi),\Sigma_0(F_0,\varphi,\psi))c = \\ \Sigma_1(\chi,\varphi,\psi)c.\end{gathered}$$

for all $c \in \mathcal{C}$. $\square$

Corollary 2.1. *Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be two combinatory spaces with one and the same kernel $(\mathcal{F}, I, \mathcal{C})$, and let $\mathcal{S}_0$ be iterative. Then $\mathcal{S}_1$ is also iterative, and for any $\sigma, \chi \in \mathcal{F}$ the equality*

$$[\sigma, \chi]^{\mathcal{S}_1} = [\sigma, Q^{\mathcal{S}_1}_{\mathcal{S}_0}\Pi_0(\chi, I)]^{\mathcal{S}_0}$$

holds. Thus if T_0 and F_0 belong to $\mathcal{C}$, then $[\sigma, \chi]^{\mathcal{S}_1} = [\sigma, \dot{Q}^{\mathcal{S}_1}_{\mathcal{S}_0}\chi]^{\mathcal{S}_0}$.

Proof. By Lemma 1.1 and the above lemma. $\square$

If $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ and $\mathcal{S}' = (\mathcal{F}, I, \mathcal{C}, \Pi', L', R', \Sigma', T', F')$ are two iterative combinatory spaces with the same kernel, then $\mathcal{S}'$ will be called *reducible to* S if the elements L', R', T', F' and the mappings Π', Σ' are $\mathcal{S}$-computable. Clearly the space $\mathcal{S}$ is reducible to itself (thanks to the $\mathcal{S}$-computability of Σ). Making use of Corollary 2.1, we see that the iteration operation in any iterative combinatory space reducible to $\mathcal{S}$ is a $\mathcal{S}$-computable mapping of $\mathcal{F}^2$ into $\mathcal{F}$, and therefore the introduced reducibility of iterative combinatory spaces is transitive. The iterative combinatory space $\mathcal{S}'$ will be called *equipowerful with* $\mathcal{S}$ if $\mathcal{S}'$ is reducible to $\mathcal{S}$ and $\mathcal{S}$ is reducible to $\mathcal{S}'$.

The space $\mathcal{S}'$ will be said to be *quasi-reducible* to the space $\mathcal{S}$ if all $\mathcal{S}'$-computable elements of $\mathcal{F}$ are $\mathcal{S}$-computable. Of course, if $\mathcal{S}'$ is reducible to $\mathcal{S}$ then $\mathcal{S}'$ is quasi-reducible to $\mathcal{S}$ (thanks to the $\mathcal{S}$-computability of the $\mathcal{S}'$-iteration). We do not know whether the converse implication holds, however the equipowerfulness of $\mathcal{S}$ and $\mathcal{S}'$ turns out to be equivalent to their mutual quasi-reducibility (i.e. to the equality of the set of the $\mathcal{S}$-computable elements of $\mathcal{F}$ and the set of the $\mathcal{S}'$-computable ones).

Theorem 2.1. *Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces. Then the next three conditions are equivalent:*

(i) *$\mathcal{S}_0$ is equipowerful with $\mathcal{S}_1$;*

(ii) *the set of the $\mathcal{S}_0$-computable elements of $\mathcal{F}$ coincides with the set of the $\mathcal{S}_1$-computable ones;*

(iii) *the elements $P^{\mathcal{S}_1}_{\mathcal{S}_0}$, $Q^{\mathcal{S}_1}_{\mathcal{S}_0}$, L_1, R_1, T_1, F_1 of $\mathcal{F}$ are $\mathcal{S}_0$-computable, and its elements $P^{\mathcal{S}_0}_{\mathcal{S}_1}$, $Q^{\mathcal{S}_0}_{\mathcal{S}_1}$, L_0, R_0, T_0, F_0 are $\mathcal{S}_1$-computable.*

In the case when T_0, F_0, T_1, F_1 belong to $\mathcal{C}$, the condition (iii) *can be replaced by*

(iii$^{\cdot}$) *the elements $P^{\mathcal{S}_1}_{\mathcal{S}_0}$, $\dot{Q}^{\mathcal{S}_1}_{\mathcal{S}_0}$, L_1, R_1, T_1, F_1 of $\mathcal{F}$ are $\mathcal{S}_0$-computable, and its elements $P^{\mathcal{S}_0}_{\mathcal{S}_1}$, $\dot{Q}^{\mathcal{S}_0}_{\mathcal{S}_1}$, L_0, R_0, T_0, F_0 are $\mathcal{S}_1$-computable.*

Proof. The implication (i)$\Rightarrow$(ii) is clear from what was said in the paragraph before the theorem. The implications (ii)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii$^{\cdot}$) follow from the fact that multiplication, Π_i and Σ_i preserve $\mathcal{S}_i$-computability for $i = 0, 1$. The validity of the implication (iii)$\Rightarrow$(i) in the general case and of the implication (iii$^{\cdot}$)$\Rightarrow$(i) in the case when T, F, T', F' belong to $\mathcal{C}$ are seen from Lemma 2.2. $\square$

Now several examples concerning the notions introduced follow (Corollary 2.1 is used in some of them for showing the iterativeness of the constructed new combinatory spaces).

Example 2.1. The pf-space considered in Example 1.1 is reducible to the one considered in Example 1.2, but these two spaces are not equipowerful.

Example 2.2. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be an iterative combinatory space that is symmetric in the sense of [5], i.e. the equality $\Pi(\varphi c, I)\theta = \Pi(\varphi c, \theta)$ holds for all $\varphi, \theta \in \mathcal{F}$ and all $c \in \mathcal{C}$ (in particular, $\mathcal{S}$ can be any pf-space). Let $\mathcal{S}_1 = (\mathcal{F}, I, \mathcal{C}, \Pi_1, R, L, \Sigma, T, F)$, where Π_1 is the mapping of $\mathcal{F}^2$ into $\mathcal{F}$ obtained from Π by exchanging its arguments, i.e. $\Pi_1(\varphi, \psi) = \Pi(\psi, \varphi)$ for all $\varphi, \psi \in \mathcal{F}$. Then $\mathcal{S}_1$ is an iterative combinatory space that is equipowerful with $\mathcal{S}$ (as indicated in Exercise II.1.2 of [5], the combinatory space $\mathcal{S}_1$ is also symmetric).

Example 2.2 (continuation). The assumption in Example 2.2 about the symmetry of $\mathcal{S}$ cannot be omitted without making other changes in the example. However, the definition of Π_1 is equivalent to another one that makes the symmetry assumption superfluous. In fact, an application of Lemma 2.2 in the situation from the example shows that $\Pi_1(\varphi, \psi) = \Pi_1(L, R)\Pi(\varphi, \psi)$, hence the equality

$$\Pi_1(\varphi, \psi) = \Pi(R, L)\Pi(\varphi, \psi) \tag{2.1}$$

holds for all $\varphi, \psi \in \mathcal{F}$. Now it is clear that we would get the same combinatory space $\mathcal{S}_1 = (\mathcal{F}, I, \mathcal{C}, \Pi_1, R, L, \Sigma, T, F)$ in the considered situation if we would define Π_1 by means of the equality (2.1). However, such a definition of $\mathcal{S}_1$ has the advantage that $\mathcal{S}_1$ turns out to be always an iterative combinatory space equipowerful with $\mathcal{S}$ (no symmetry of $\mathcal{S}$ is already needed). Checking everything in this statement is straightforward except for the fact that $\mathcal{S}$ is reducible to $\mathcal{S}_1$. The reducibility of $\mathcal{S}$ to $\mathcal{S}_1$ can be shown by proving the equality

$$\Pi(\varphi, \psi) = \Pi_1(L, R)\Pi_1(\varphi, \psi), \tag{2.2}$$

and this equality follows by Lemma 2.1 from the fact that, as it is easy to be verified, $\Pi(a, b) = \Pi_1(L, R)\Pi_1(a, b)$ for all $a, b \in \mathcal{C}$. [3]

Example 2.3. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be any iterative combinatory space, and let

$$\begin{aligned}
\mathcal{S}_0 &= (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma_0, F, T),\\
\mathcal{S}_1 &= (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma_1, \Pi(T, I), \Pi(F, I)),\\
\mathcal{S}_2 &= (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma_2, \Pi(I, T), \Pi(I, F)),
\end{aligned}$$

where Σ_0,Σ_1 and Σ_2 are defined by means of the equalities

[3] A proof of the equality (2.2) by using Lemma 2.2 is also possible, namely

$$\Pi_1(L, R)\Pi_1(\varphi, \psi) = \Pi(R, L)\Pi(L, R)\Pi(R, L)\Pi(\varphi, \psi) =$$
$$\Pi(R, L)\Pi(R, L)\Pi(\varphi, \psi) = \Pi(R, L)\Pi_1(\varphi, \psi) = \Pi(\varphi, \psi).$$

$$\Sigma_0(\chi,\varphi,\psi)=\Sigma(\chi,\psi,\varphi),\ \ \Sigma_1(\chi,\varphi,\psi)=\Sigma(L\chi,\varphi,\psi),\ \ \Sigma_2(\chi,\varphi,\psi)=\Sigma(R\chi,\varphi,\psi)$$

(cf. Exercise II.1.1 in [5]). Then $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are iterative combinatory spaces that are equipowerful with $\mathcal{S}$ (the equalities

$$\Sigma(\chi,\psi,\varphi)=\Sigma_1(\Pi(\chi,I),\varphi,\psi)=\Sigma_2(\Pi(I,\chi),\varphi,\psi),$$
$$T=L\Pi(T,I)=R\Pi(I,T),\ F=L\Pi(F,I)=R\Pi(I,F)$$

are used in the proof of the reducibility of $\mathcal{S}_1$ and $\mathcal{S}_2$ to $\mathcal{S}$).

Example 2.4. Let M be an infinite set, m_0 and m_1 be two distinct elements of M, and J be a bijection from M^2 to M such that $J(m_0,m_0)=m_0$, $J(m_0,m_1)=m_1$ (we may for instance set $M=\mathbb{N}$, $m_0=0$, $m_1=1$, and take J as in Example 1.1). Let L and R be the mappings of M into M defined by means of the equalities $L(J(s,t))=s$, $R(J(s,t))=t$. Then clearly $L(m_0)=R(m_0)=L(m_1)=m_0$, $R(m_1)=m_1$. We define a new mapping J' of M^2 into M by means of the equality

$$J'(s,t)=J(L(s),J(R(s),t)).$$

It is easily seen that the equality $J'(s,t)=u$ is equivalent to the pair of equalities $s=J(L(u),L(R(u)))$, $t=R(R(u))$. Therefore J' is also a bijection from M^2 to M, and after setting

$$L'(u)=J(L(u),L(R(u))),\ R'(u)=R(R(u))$$

we have $L'(J'(s,t))=s$, $R'(J'(s,t))=t$ for all $s,t\in M$. Moreover, we have also the equalities $J'(m_0,m_0)=m_0$, $J'(m_0,m_1)=m_1$. Now let us consider the computational structures (M,J,L,R,T,F,H) and (M,J',L',R',T,F,H), where $T(u)=m_1$, $F(u)=m_0$ for all $u\in M$, and H is a partial predicate on M such that $H(m_1)$ is true, $H(m_0)$ is false. Let $\mathcal{S}=(\mathcal{F},I,\mathcal{C},\Pi,L,R,\Sigma,T,F)$ and $\mathcal{S}'=(\mathcal{F},I,\mathcal{C},\Pi',L',R',\Sigma,T,F)$ be the pf-spaces corresponding to these two computational structures. Since $\Pi'(\varphi,\psi)=\Pi(L\varphi,\Pi(R\varphi,\psi))$ for all $\varphi,\psi\in\mathcal{F}$, and the equalities $L'=\Pi(L,LR)$, $R'=R^2$ hold, the pf-space $\mathcal{S}'$ is reducible to $\mathcal{S}$. However, we shall show that $\mathcal{S}'$ is not equipowerful with $\mathcal{S}$, i.e. $\mathcal{S}$ is not reducible to $\mathcal{S}'$. This will be shown by proving that the $\mathcal{S}$-computable element $\Pi(T,F)$ of $\mathcal{F}$ is not $\mathcal{S}'$-computable. For that purpose, let us denote by A the smallest subset of M containing the elements m_0 and m_1 and closed under application of J'. It is easy to show by induction that the image of A under any $\mathcal{S}'$-computable function from $\mathcal{F}$ is a subset of A. On the other hand, $\Pi(T,F)(u)=J(m_1,m_0)$ for all $u\in M$, but $J(m_1,m_0)$ does not belong to A, because $m_0\neq J(m_1,m_0)$, $m_1\neq J(m_1,m_0)$, and $J'(s,t)\neq J(m_1,m_0)$ whenever $s\neq J(m_1,m_0)$, due to the equalities $L(m_0)=L(m_1)=m_0$ and $L'(J(m_1,m_0))=J(m_1,m_0)$.

Remark 2.1. The above example shows how to construct an infinite sequence $\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2,\ldots$ of pf-spaces not differing from one another out of their fourth to sixth components and having the property that $\mathcal{S}_j$ is reducible to $\mathcal{S}_i$ without being equipowerful with it, whenever $j>i$.

Remark 2.2. Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces with one and the same kernel, and let

$$\mathcal{D} = \{P_{\mathcal{S}_0}^{\mathcal{S}_1}, Q_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\}$$

(or $T_0, F_0 \in \mathcal{C}$, $\mathcal{D} = \{P_{\mathcal{S}_0}^{\mathcal{S}_1}, \dot{Q}_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\}$). An application of Lemma 2.2 shows that the operations Π_1, Σ_1 and consequently also the iteration in $\mathcal{S}_1$ are $\mathcal{S}_0$-computable in the set $\mathcal{D}$, hence all $\mathcal{S}_1$-computable elements of $\mathcal{F}$ are $\mathcal{S}_0$-computable in $\mathcal{D}$. If $\mathcal{S}_0$ is reducible to $\mathcal{S}_1$ then also the converse is true, hence in this case the $\mathcal{S}_1$-computability of an element of $\mathcal{F}$ is equivalent to its $\mathcal{S}_0$-computability in $\mathcal{D}$.

Intuitively, an iterative combinatory space can be considered as a certain kind of programming system. The intuitive interpretation of the reducibility of the space $\mathcal{S}'$ to the space $\mathcal{S}$ is as emulability of all $\mathcal{S}'$-programs (including the ones that may use oracles) by corresponding $\mathcal{S}$-programs. The quasi-reducibility of $\mathcal{S}'$ to $\mathcal{S}$ can be interpreted similarly, but with having in view only the programs that do not use oracles. Of course, the equipowerfulness will be interpreted as emulability in both directions.

3. ON A STATEMENT OF JORDAN ZASHEV

If (M, J, L, R, T, F, H) is a computational structure whose component J is a bijection from M^2 to M, then the corresponding pf-space $(\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ has the property that $\Pi(L, R) = I$. In a remark on page 78 of [7] Jordan Zashev indicates a way for improving the exposition of the theory for iterative combinatory spaces with this property (assuming that the elements T and F belong to $\mathcal{C}$). According to him the examples of combinatory spaces given in [2,5] do not give reasons to consider the abandonment of the equality $\Pi(L, R) = I$ as essential for the scope of the theory, since, as he writes, "*all of them have more or less obvious variants in which the last equality is true*". No definition is given in [7] for the used notion of variant, and of course no proof or disproof of the quoted statement can be expected without such a definition. We shall present now a refutation of the statement in question for the case when "variant" is interpreted as an iterative combinatory space that is quasi-reducible to the given one. Of course this will also show the failure of the statement for the stronger interpretations as an iterative combinatory space reducible to the given one or as an iterative combinatory space equipowerful with it.

Let us call an iterative combinatory space $(\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ a *Z-space* if the equality $\Pi(L, R) = I$ holds. We shall indicate some ordinary pf-spaces to which no Z-space is quasi-reducible, and this will be the promised refutation, since, as we mentioned in Section 1, the ordinary pf-spaces and all pf-spaces are the subject of some examples in [2] and in [5], respectively. The following lemma will be used.

Lemma 3.1. *Let (M, J, L, R, T, F, H) be a computational structure, and let the corresponding pf-space $\mathcal{S}$ be such that some Z-space with the same kernel is* *quasi-reducible to $\mathcal{S}$. Then there is an $\mathcal{S}$-computable bijection from M to the range of J.*

Proof. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$, and let $(\mathcal{F}, I, \mathcal{C}, \Pi', L', R', \Sigma', T', F')$ be a Z-space that is quasi-reducible to $\mathcal{S}$. Then the element $\Pi(L', R')$ of $\mathcal{F}$ is $\mathcal{S}$-computable thanks to the $\mathcal{S}$-computability of L' and R'. By Lemma 2.2, the equality $\Pi'(L', R') = \Pi'(L, R)\Pi(L', R')$ holds, hence $\Pi'(L, R)\Pi(L', R') = I$. Therefore $\Pi(L', R')$ is an injective mapping of M into M. Taking into account the definition of Π, we conclude that in fact $\Pi(L', R')$ is an injective mapping of M into the range of J. To show that any element of the range of J is a value of $\Pi(L', R')$, let us consider such an element u. Then $u = J(s, t)$ for some s and t in M. Denoting by a and b the elements of $\mathcal{C}$ with values s and t, respectively, we consider the element $\Pi'(a, b)$ of $\mathcal{C}$. Let v be the value of this constant function. The equalities $L'\Pi'(a, b) = a$, $R'\Pi'(a, b) = b$ imply that $L'(v) = s$, $R'(v) = t$, hence $\Pi(L', R')(v) = u$. □

Having the above lemma at our disposal, we shall proceed by indicating some computational structures (M, J, L, R, T, F, H) such that T and F are constant mappings of M into M, and, if $\mathcal{S}$ is the corresponding pf-space, then no $\mathcal{S}$-computable bijection from M to the range of J exists.

Example 3.1. We consider a computational structure (M, J, L, R, T, F, H) of the following kind. The set M is the closure of A under formation of ordered pairs, where A is some non-empty set, and none of its elements is an ordered pair, J is the function from M^2 to M defined by the equality $J(s, t) = (s, t)$, L and R are the functions from the range of J to M defined by means of the equalities $L(J(s, t)) = s$, $R(J(s, t)) = t$, T and F are the constant functions from M to M with values (o, o) and o, respectively, where o is some distinguished element of A, H is the predicate on M that is false on A and true everywhere in $M \setminus A$. Let $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ be the pf-space corresponding to this computational structure. Under the additional assumption that A is finite and has more than one element, we shall show that no Z-space with the same kernel is quasi-reducible to $\mathcal{S}$. Let A have k elements, where $k > 1$. Suppose there is an $\mathcal{S}$-computable bijection θ_0 from M to the range of J. Then θ_0 is a computable bijection from M to $M \setminus A$. From here a contradiction will be produced as follows. We define inductively a family $\mathcal{M}$ of subsets of M by the clauses that $A \in \mathcal{M}$ and $X \times Y \in \mathcal{M}$ whenever $X, Y \in \mathcal{M}$. One proves by induction that all members of $\mathcal{M}$ are non-empty finite sets, and, whenever $Z \in \mathcal{M}$, then either $Z = A$, or the cardinality of Z is divisible by k^2. Another induction shows that each element of M belongs to exactly one member of $\mathcal{M}$. By means of a third induction we prove that whenever θ is an $\mathcal{S}$-computable element of $\mathcal{F}$, the image by θ of any member of $\mathcal{M}$ is a subset of some member of $\mathcal{M}$. In particular, the mapping θ_0 will have this property. Since θ_0 is a bijection from M to $M \setminus A$, each member of $\mathcal{M}$ different from A will be the union of its subsets that are images by θ_0 of members of $\mathcal{M}$, and these subsets will be pairwise disjoint. Let Z be the member of $\mathcal{M}$ that contains as a subset the image by θ_0 of the set A. Clearly $Z \neq A$, and therefore the cardinality of Z is divisible by k^2. On the other hand, this cardinality must be equal to the sum of k and some numbers divisible by k^2, and this is a contradiction.

Remark 3.1. We could reason in the same way as above if we would make the functions L and R total by additionally setting $L(u) = R(u) = u$ for all $u \in A$. On the other hand, as seen from [6], the situation would become essentially different if we would make them total in the way from [1], namely by setting $L(o) = R(o) = o$, $L(u) = R(u) = (o, o)$ for all $u \in A \setminus \{o\}$. Then, independently of the cardinality of A, there would be a Z-space having the same kernel as $\mathcal{S}$ and reducible to it.

4. AN EXTENSION OF THE CONSIDERED REDUCIBILITY

The application of an iterative combinatory space $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ for the characterization of some concrete computability notion usually makes use of $\mathcal{S}$-computability in certain subset $\mathcal{B}$ of $\mathcal{F}$. The intuitive interpretation of $\mathcal{S}$ as a programming system can be transferred also to pairs $(\mathcal{S}, \mathcal{B})$ by replacing $\mathcal{S}$-computability with $\mathcal{S}$-computability in $\mathcal{B}$. The case of $\mathcal{S}$-computability will then correspond to the pair $(\mathcal{S}, \emptyset)$. It is natural to extend the reducibility notions introduced in Section 2 for the case of two pairs $(\mathcal{S}, \mathcal{B})$ and $(\mathcal{S}', \mathcal{B}')$, where $\mathcal{S}$ and $\mathcal{S}'$ are iterative combinatory spaces with one and the same kernel, and $\mathcal{B}, \mathcal{B}'$ are subsets of their first component. Here are the corresponding definitions.

If $\mathcal{S} = (\mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F)$ and $\mathcal{S}' = (\mathcal{F}, I, \mathcal{C}, \Pi', L', R', \Sigma', T', F')$ are iterative combinatory spaces, and $\mathcal{B}$, $\mathcal{B}'$ are subsets of $\mathcal{F}$, then the pair $(\mathcal{S}', \mathcal{B}')$ will be called *reducible to* the pair $(\mathcal{S}, \mathcal{B})$ if the elements L', R', T', F', all elements of $\mathcal{B}'$ and the mappings Π', Σ' are $\mathcal{S}$-computable in $\mathcal{B}$. The pair $(\mathcal{S}', \mathcal{B}')$ will be said to be *quasi-reducible* to the pair $(\mathcal{S}, \mathcal{B})$ if all elements of $\mathcal{F}$ that are $\mathcal{S}'$-computable in $\mathcal{B}'$ are also $\mathcal{S}$-computable in $\mathcal{B}$. If each of the pairs $(\mathcal{S}, \mathcal{B})$ and $(\mathcal{S}', \mathcal{B}')$ is reducible to the other one then these pairs will be called *equipowerful.*

As in Section 2 the reducibility is seen to be reflexive and transitive, and it implies quasi-reducibility. Also, Theorem 2.1 remains valid after replacing the combinatory spaces with pairs of the considered kind, the proof being quite similar. Here is the result of the replacements.

Theorem 4.1. *Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces. and $\mathcal{B}_0$, $\mathcal{B}_1$ be subsets of $\mathcal{F}$. Then the next three conditions are equivalent:*

(i) *$(\mathcal{S}_0, \mathcal{B}_0)$ is equipowerful with $(\mathcal{S}_1, \mathcal{B}_1)$;*

(ii) *the set of the elements of $\mathcal{F}$ that are $\mathcal{S}_0$-computable in $\mathcal{B}_0$ coincides with the set of the ones that are $\mathcal{S}_1$-computable in $\mathcal{B}_1$;*

(iii) *the elements of the set $\{P_{\mathcal{S}_0}^{\mathcal{S}_1}, Q_{\mathcal{S}_0}^{\mathcal{S}_1}, L_1, R_1, T_1, F_1\} \cup \mathcal{B}_1$ are $\mathcal{S}_0$-computable in $\mathcal{B}_0$, and the elements of the set $\{P_{\mathcal{S}_1}^{\mathcal{S}_0}, Q_{\mathcal{S}_1}^{\mathcal{S}_0}, L_0, R_0, T_0, F_0\} \cup \mathcal{B}_0$ are $\mathcal{S}_1$-computable in $\mathcal{B}_1$.*

In the case when T_0, F_0, T_1, F_1 belong to $\mathcal{C}$, the condition (iii) *can be replaced by*

(iii˙) *the elements of the set* $\{P^{\mathcal{S}_1}_{\mathcal{S}_0}, \dot{Q}^{\mathcal{S}_1}_{\mathcal{S}_0}, L_1, R_1, T_1, F_1\} \cup \mathcal{B}_1$ *are* $\mathcal{S}_0$*-computable in* $\mathcal{B}_0$, *and the elements of the set* $\{P^{\mathcal{S}_0}_{\mathcal{S}_1}, \dot{Q}^{\mathcal{S}_0}_{\mathcal{S}_1}, L_0, R_0, T_0, F_0\} \cup \mathcal{B}_0$ *are* $\mathcal{S}_1$*-computable in* $\mathcal{B}_1$.

The statements in Remark 2.2 can be strengthened in the following way.

Remark 4.1. Let $\mathcal{S}_i = (\mathcal{F}, I, \mathcal{C}, \Pi_i, L_i, R_i, \Sigma_i, T_i, F_i)$, $i = 0, 1$, be iterative combinatory spaces with one and the same kernel, and let

$$\mathcal{D} = \{P^{\mathcal{S}_1}_{\mathcal{S}_0}, Q^{\mathcal{S}_1}_{\mathcal{S}_0}, L_1, R_1, T_1, F_1\}$$

(or $T_0, F_0 \in \mathcal{C}$, $\mathcal{D} = \{P^{\mathcal{S}_1}_{\mathcal{S}_0}, \dot{Q}^{\mathcal{S}_1}_{\mathcal{S}_0}, L_1, R_1, T_1, F_1\}$). Then the pair $(\mathcal{S}_1, \emptyset)$ is reducible to the pair $(\mathcal{S}_0, \mathcal{D})$. If $\mathcal{S}_0$ is reducible to $\mathcal{S}_1$ then $(\mathcal{S}_1, \emptyset)$ and $(\mathcal{S}_0, \mathcal{D})$ are equipowerful.

The following obvious monotonicity can also be mentioned: if $\mathcal{S}_0$ and $\mathcal{S}_1$ are iterative combinatory spaces with one and the same kernel $(\mathcal{F}, I, \mathcal{C})$, and $\mathcal{B}_0$, $\mathcal{B}_1$ are subsets of $\mathcal{F}$ such that the pair $(\mathcal{S}_0, \mathcal{B}_0)$ is reducible to the pair $(\mathcal{S}_1, \mathcal{B}_1)$, then for any subset $\mathcal{E}$ of $\mathcal{F}$ the pair $(\mathcal{S}_0, \mathcal{B}_0 \cup \mathcal{E})$ is reducible to the pair $(\mathcal{S}_1, \mathcal{B}_1 \cup \mathcal{E})$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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ON THE 2-COLORING DIAGONAL VERTEX FOLKMAN NUMBERS WITH MINIMAL POSSIBLE CLIQUE NUMBER

NIKOLAY KOLEV, NEDYALKO NENOV

For a graph G the symbol $G \xrightarrow{v} (p,p)$ means that in every 2-coloring of the vertices of G, there exists a monochromatic p-qlique. The vertex diagonal Folkman numbers

$$F_v(p,p;p+1) = \min\{|V(G)| : G \xrightarrow{v} (p,p) \text{ and } K_{p+1} \not\subset G\}$$

are considered. We prove that $F_v(p,p;p+1) \leq \frac{13}{12}p!$, $p \geq 4$.

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1. NOTATIONS

We consider only finite non-oriented graphs without loops and multiple edges. We call a p-clique of a graph G a set of p vertices, each two of which are adjacent. The largest positive integer p such that the graph G contains a p-clique is denoted by $cl(G)$.

In this paper we shall use also the following notations:
$V(G)$ – the vertex set of G;
$E(G)$ – the edge set of G;
$\overline{G}$ – the complementary graph of G;
$G[X]$, $X \subseteq V(G)$ – the subgraph of G, induced by X;
$G - X$, $X \subseteq V(G)$ – the subgraph of G, induced by $V(G)\backslash X$;

K_n – the complete graph on n vertices;
$\Gamma_G(v)$ – the neighbors of v in G;
C_n – the simple cycle on n vertices;
$\alpha(G)$ – the independence number of G, i.e. $\alpha(G) = cl(\overline{G})$;
$Aut(G)$ – the group of all automorphisms of G.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

Let $G_1, G_2, \ldots, G_k$ be graphs and $V(G_i) \cap V(G_j) = \emptyset$, $i \neq j$. We denote by $\bigcup_{i=1}^{k} G_i$ the graph G for which $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and $E(G) = \bigcup_{i=1}^{k} E(G_i)$.

The Ramsey number $R(p, q)$ is the smallest natural number n such that for an arbitrary n-vertex graph G either $cl(G) \geq p$ or $\alpha(G) \geq q$.

2. RESULTS

Definition 2.1. Let G be a graph and p, q be positive integers. A 2-coloring

$$V(G) = V_1 \cup V_2, \; V_1 \cap V_2 = \emptyset$$

of the vertices of G is said to be (p, q)-free, if V_1 contains no p-cliques and V_2 contains no q-cliques of G. The symbol $G \xrightarrow{v} (p, q)$ means that every 2-coloring of $V(G)$ is not (p, q)-free. The vertex Folkman numbers are defined by the inequality

$$F_v(p, q; s) = \min\{|V(G)| : G \xrightarrow{v} (p, q) \text{ and } cl(G) \subset s\}.$$

The numbers $F_v(p, p; s)$ are called diagonal Folkman numbers.

In this paper we consider the diagonal Folkman numbers $F_v(p, p; p + 1)$. Only two exact values of these numbers are known:

$$F_v(2, 2; 3) = 5; \tag{2.1}$$

$$F_v(3, 3; 4) = 14, \text{ [5] and [14]}. \tag{2.2}$$

The equality (2.1) is well known and easy to prove. The inequality $F_v(3, 3; 4) \leq 14$ was proved in [5], and the inequality $F_v(3, 3; 4) \geq 14$ was verified by means of computer in [11].

The following bounds are known for these numbers:

$$F_v(p, p; p + 1) \leq \lfloor 2p!\,(e - 1) \rfloor - 1, \; p \geq 2, \text{ [4]};$$

$$F_v(p, p; p + 1) \leq \lfloor p!\, e \rfloor - 2, \; p \geq 3, \text{ [6]}.$$

In [10] N. Nenov significantly improved these values proving that

$$F_v(p,p;p+1) \leq \frac{35}{24}p!, \ p \geq 4. \tag{2.3}$$

The inequality (2.3) was proved using the following

Theorem 1. $F_v(p+1,p+1;p+2) \leq (p+1)F_v(p,p;p+1), \ p \geq 2$

As this result was only stated in [10], we shall supply the proof of Theorem 1 here. In this paper we shall improve the inequality (2.3) by proving the following

Theorem 2. $F_v(p,p;p+1) \leq \dfrac{13}{12}p!, \ p \geq 4.$

Theorem 2 is proved by induction on p. As the inductive step follows trivially from Theorem 1, it remains to prove only the inductive base $p = 4$, i.e.

Theorem 3. $F_v(4,4;5) \leq 26.$

We shall note that from Theorem 1 it follows that $F_v(4,4;5) \leq 35$, [9]. In [9] it was also proved that $F_v(4,4;5) \geq 16$.

Let G and G_1 be two graphs and $V(G) \xrightarrow{\varphi} V(G_1)$ be a homomorphism of graphs (i.e. if $[a,b] \in E(G)$, then $[\varphi(a), \varphi(b)] \in E(G_1)$). If $V_1 \cup V_2$ is a (p,q)-free 2-coloring of $V(\varphi(G))$, then it is easy to see that $\varphi^{-1}(V_1) \cup \varphi^{-1}(V_2)$ is a (p,q)-free 2-coloring of $V(G)$.

That is why we have the following

Proposition 2.1[10]. *Let G and G_1 be graphs and $V(G) \xrightarrow{\varphi} V(G_1)$ be a homomorphism. Then from $G \xrightarrow{v} (p,q)$ it follows $G_1 \xrightarrow{v} (p,q)$.*

3. PROOF OF THEOREM 1

In the case when $p \leq 3$ Theorem 1 follows from (2.1), (2.2) and Theorem 3. So we can now consider $p \geq 4$. Let G be a graph such that $G \xrightarrow{v} (p,p)$, $cl(G) = p$ and

$$|V(G)| = F_v(p,p;p+1). \tag{3.1}$$

We consider the graph

$$P = G_1 \cup G_2 \cup \ldots \cup G_{p+1} \cup K_{p+1},$$

where each of the graphs G_i, $i = 1,2,\ldots,p+1$ is an isomorphic copy of G and $V(K_{p+1}) = \{a_1, \ldots, a_{p+1}\}$. The graph $\widetilde{P}$ is obtained from P by connecting the vertex a_i with every vertex from G_i, $i = 1, \ldots, p+1$. The graph L is obtained from $\widetilde{P}$ by adding a new vertex b such that

$$\Gamma_L(b) = \bigcup_{i=1}^{p+1} V(G_i).$$

We shall prove that

$$L \xrightarrow{v} (p+1, p+1). \tag{3.2}$$

Assume the opposite and let $V_1 \cup V_2$ be a $(p+1, p+1)$-free 2-coloring of L. Without loss of generality we can consider $b \in V_1$. Define the sets

$$W_i = V(G_i) \cup \{b, a_i\},\ i = 1, \ldots, p+1.$$

It is clear that $L[W_i] = \overline{K_2} + G_i$, where $V(\overline{K_2}) = (b, a_i)$. As $G \xrightarrow{v} (p, p)$ we have $a_i \in V_1$, $i = 1, 2, \ldots, p+1$. We have obtained that V_1 contains the $(p+1)$-clique $\{a_1, \ldots, a_{p+1}\}$, which is a contradiction. Thus (3.2) is proved.

From the definition of L and $cl(G) = p$ we have

$$cl(G) = p + 1. \tag{3.3}$$

From (3.1) we have

$$|V(L)| = (p+1)F_v(p, p; p+1) + p + 2. \tag{3.4}$$

In each of the graphs G_i, $i = 1, \ldots, p$ (i.e. without G_{p+1}) we choose vertices $x_i, y_i \in V(G_i)$ such that $[x_i, y_i] \notin E(G_i)$ (as G_i is not a complete graph then such vertices exist). Define the sets:

$$X_i = \Gamma_{G_i}(x_i) \cup \{a_i\} \cup \{b\} \tag{3.5}$$

and

$$Y_i = \Gamma_{G_i}(y_i) \cup \{a_i\} \cup \{b\},\ i = 1, \ldots, p.$$

From $cl(G_i) = p$ it follows that $\Gamma_{G_i}(x_i)$ and $\Gamma_{G_i}(y_i)$ do not contain p-cliques. As the vertices b and a_i are not adjacent we have

$$X_i \text{ and } Y_i \text{ do not contain } (p+1)\text{-cliques for } i = 1, \ldots, p. \tag{3.6}$$

Let us note that

$$\Gamma_L(x_i) = X_i \text{ and } \Gamma_L(y_i) = Y_i. \tag{3.7}$$

We denote by R the graph that is obtained from L by deleting the vertices x_i, y_i, $i = 1, \ldots, p$ and the edges connecting them and by adding two new vertices x and y such that

$$\Gamma_R(x) = \bigcup_{i=1}^{p} X_i,\ \Gamma_R(y) = \bigcup_{i=1}^{p} Y_i. \tag{3.8}$$

It is clear that

$$|V(R)| = |V(L)| - 2(p-1).$$

From the last equality and (3.7) we have

$$|V(R)| = (p+1)F_v(p, p; p+1) - p + 4.$$

As $p \geq 4$, we have

$$|V(R)| \leq (p+1)F_v(p,p;p+1). \tag{3.9}$$

We shall show that

$$cl(R) < p+2. \tag{3.10}$$

Assume the opposite, i.e. $cl(R) \geq p+2$ and let A be a $(p+2)$ - clique of the graph R. As $L - \{x,y\}$ is a subgraph of the graph L and $cl(L) = p+1$, it follows that $x \in A$ or $y \in A$. Without loss of generality we can assume that $x \in A$. We consider the $(p+1)$ - clique $A' = A - x$. From (3.8) it follows that

$$A' \subseteq \bigcup_{i=1}^{p} X_i,\ i = 1, ..., p. \tag{3.11}$$

As $|A'| = p+1$ from (3.11) it follows that some of the sets X_i contain two vertices from A'. Without loss of generality we can assume that X_1 contains two vertices from A'. As b and a_1 are not adjacent in R, from (3.5), $i = 1$ it follows that there is a vertex w such that

$$w \in A' \cap \Gamma_{G_1}(x_1).$$

As

$$\Gamma_R(w) \cap V(G_i - x_i - y_i) = \emptyset,\ i = 2, ..., p+1$$

and $a_2, ..., a_{p+1} \notin \Gamma_R(w)$ it follows that $A' \cap V(G_i - x_i - y_i) = \emptyset, i \geq 2$, and $a_2, ..., a_{p+1} \notin A'$.

As

$$\Gamma_{G_i}(x_i) \subseteq V(G_i - x_i - y_i),$$

we conclude that

$$A' \cap X_i = \emptyset \text{ or } A' \cap X_i = \{b\},\ i = 2, \ldots, p+1.$$

Hence from (3.11) it follows that $A' \subseteq X_1$, which contradicts (3.6). Thus (3.10) is proved.

Consider the mapping $V(L) \xrightarrow{\varphi} V(R)$, which is defined as follows:

$$v \xrightarrow{\varphi} v, \quad \text{if } v \neq x_i, v \neq y_i,\ i = 1, 2, \ldots, p;$$

$$x_i \xrightarrow{\varphi} x,\ y_i \xrightarrow{\varphi} y\ ,\ i = 1, 2, \ldots, p.$$

From (3.7) and (3.8) it follows that φ is a homomorphism from L to R. From (3.2) and proposition (2.1) we have $R \xrightarrow{v} (p+1, p+1)$. This fact and (3.10) give

$$F_v(p+1, p+1; p+2) \leq |V(R)|.$$

This inequality and (3.9) complete the proof of the theorem.

4. (4,4)-FREE 2-COLORING OF THE GRAPH OF GREENWOOD AND GLEASON

The complementary graph of the graph of Greenwood and Gleason Q is given on figure 1. This graph has the property

$$\alpha(Q) = 2,\ cl(Q) = 4 \qquad [2]. \tag{4.1}$$

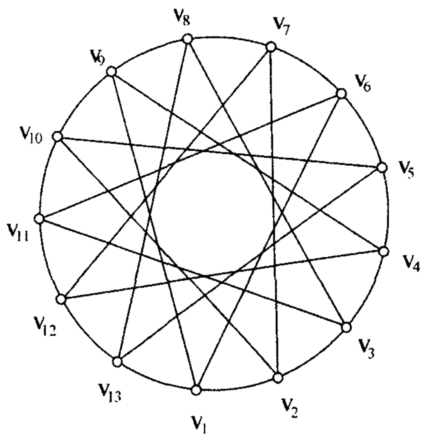

Fig. 1. Graph $Q \longrightarrow \overline{Q}$

Using this graph Greenwood and Gleason proved that $R(3,5) = 14$.
In [7] N. Nenov proved that

$$Q \xrightarrow{v} (3,4). \tag{4.2}$$

It is easy to see that 2-coloring

$$V(Q) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\} \cup \{v_7, v_8, v_{11}, v_{12}, v_{13}\} \tag{4.3}$$

is (4,4)-free and hence $Q \overset{v}{\nrightarrow} (4,4)$.
The complementary graph $\overline{Q}$ contains the 13-cycles:

$$C_{13}^{(1)} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\},$$

$$C_{13}^{(2)} = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_5, v_{10}, v_2, v_7, v_{12}, v_4, v_9\}.$$

Let us note that $E(\overline{Q}) = E(C_{13}^{(1)}) \cup E(C_{13}^{(2)})$.
These two cycles are equivalent as the mapping

$$\varphi = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} \\ v_1 & v_6 & v_{11} & v_3 & v_8 & v_{13} & v_5 & v_{10} & v_2 & v_7 & v_{12} & v_4 & v_9 \end{pmatrix}$$

$$= (v_1)(v_2, v_6, v_{13}, v_9)(v_3, v_{11}, v_{12}, v_4)(v_5, v_8, v_{10}, v_7)$$

is an automorphism of $\overline{Q}$ (and hence of Q) and $\varphi\left(C_{13}^{(1)}\right)=C_{13}^{(2)}, \varphi\left(C_{13}^{(2)}\right)=C_{13}^{(1)}$.

We shall also need the cyclic automorphism of Q:

$$\xi=\begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} \\ v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_1 \end{pmatrix}.$$

A straightforward computation shows that

$$\varphi\xi=\xi^5\varphi. \tag{4.4}$$

Let $\langle\varphi,\xi\rangle$ be the subgroup of $Aut(Q)$, generated by φ and ξ. From (4.4) it follows that $\langle\xi\rangle$ is a normal subgroup of $\langle\varphi,\xi\rangle$. Hence from (4.4) it also follows that $|\langle\varphi,\xi\rangle|=52$. As $Aut(Q)$ acts transitively on Q, we have $|Aut(Q)|=13|St(v_1)|$. It is easy to see that $|St(v_1)|=4$ and hence $|Aut(Q)|=52$. Thus we proved the following

Proposition 4.1. $Aut(Q)=\langle\varphi,\xi\rangle$.

From this and (4.4) we obtain:

Proposition 4.2. *Each element of $Aut(Q)$ is of the kind $\xi^l\varphi^k$, where $0\le k\le 3$, $0\le l\le 12$.*

Define the following sets:

$$M=\{v_1,v_2,v_3,v_4,v_5,v_6,v_9,v_{10}\},$$

$$S=\{v_1,v_2,v_3,v_4,v_6,v_7,v_9\}.$$

We shall use and prove the following propositions:

Proposition 4.3. *Let $V_1\cup V_2$ be a (4,4)-free coloring of $V(Q)$ such that $|V_1|=8$ and $|V_2|=5$. Then there exists $\psi\in Aut(Q)$ such that $V_1=\psi(M)$.*

Proposition 4.4. *Let $V_1\cup V_2$ be a (4,4)-free coloring such that $|V_1|=7$ and $|V_2|=6$. Then there exists $\psi\in Aut(Q)$ such that either $V_1\subset\psi(M)$ or $V_1=\psi(S)$.*

Define the following sets:

$$M_0=\{v_1,v_2,v_3,v_4,v_5,v_6,v_9,v_{10}\}=M,$$

$$M_1=\{v_1,v_2,v_3,v_4,v_5,v_6,v_{10},v_{11}\}=\xi^5\varphi^2(M),$$

$$M_2=\{v_1,v_6,v_{11},v_3,v_8,v_{13},v_2,v_7\}=\varphi(M),$$

$$M_3=\{v_1,v_6,v_{11},v_3,v_8,v_{13},v_7,v_{12}\}=\xi^{-1}\varphi^3(M),$$

$$S_0=\{v_1,v_2,v_3,v_4,v_6,v_7,v_9\}=S,$$

$$S_1=\{v_1,v_2,v_3,v_4,v_6,v_7,v_{12}\}=\xi\varphi(S),$$

$$S_2=\{v_1,v_2,v_3,v_4,v_9,v_{11},v_{12}\}=\xi^3\varphi^2(S),$$

$$S_3 = \{v_1, v_2, v_3, v_4, v_6, v_{11}, v_{12}\} = \xi^2\varphi^3(S).$$

Using Propositions 4.2, 4.3 and 4.4 it is easy to prove

Proposition 4.5. *Let $V_1 \cup V_2$ be a (4,4)-free coloring of $V(Q)$. Then there exists an integer $0 \le k \le 12$ such that $V_1 \subset \xi^k M_i$ for some $0 \le i \le 3$ or $V_1 \subset \xi^k S_i$ for some $0 \le i \le 3$.*

In order to prove these propositions we shall need the following lemmas:

Lemma 4.1. *If C is a simple 4-cycle and if C is an induced subgraph of $\overline{Q}$ then there exists $\psi \in Aut(Q)$, such that $C = \psi(\{v_1, v_2, v_6, v_7\})$.*

Lemma 4.2. *If D is a simple chain of length 4 and if v_1 is the starting point of D and v_5 – the endpoint of D, then*
1) $D = \{v_1, v_9, v_{10}, v_5\}$ *or*
2) $D = \{v_1, v_9, v_4, v_5\}$ *or*
3) $D = \{v_1, v_2, v_{10}, v_5\}$.

Lemma 4.3. *If D is a simple chain of length 4, and if v_1 is the starting point and v_7 – the endpoint, then*
1) $D = \{v_1, v_{13}, v_{12}, v_7\}$ *or*
2) $D = \{v_1, v_{13}, v_8, v_7\}$ *or*
3) $D = \{v_1, v_9, v_8, v_7\}$.

Lemmas 4.2 and 4.3 are trivial and their proof is a straightforward check of all possibilities.

Lemma 4.4. *If $\overline{Q}$ contains an induced subgraph isomorphic to C_{2s+1} for some positive integer s, then this subgraph contains at least 3 consequent vertices in at least one of the two cycles: $C_{13}^{(1)}$ and $C_{13}^{(2)}$ of $\overline{Q}$.*

Lemma 4.5. *$\overline{Q}$ does not contain an induced subgraph isomorphic to C_7.*

Lemma 4.6. *If C is a simple 5-cycle, which is an induced subgraph of $\overline{Q}$, then there is $\psi \in Aut(Q)$ such that $C = \psi(\{v_1, v_2, v_3, v_4, v_9\})$ or $C = \psi(\{v_1, v_2, v_3, v_8, v_9\})$.*

The detailed proofs of all the propositions and lemmas from this paragraph with the exception of lemmas 4.2 and 4.3, which are obvious, will be supplied in part 7.

5. DESCRIPTION OF THE MAIN CONSTRUCTION

We consider two isomorphic copies Q and Q' of the graph Q (see Fig. 1). Denote

$$V(Q) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\},$$

$$V(Q') = \{w_1, v_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}\},$$

We consider the graph L such that $V(L) = V(Q) \cup V(Q')$. $E(L)$ will be defined below.

We define

$$\Gamma_L(w_1) \cap V(Q) = \varphi(M) = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}.$$

$$\Gamma_L(w_i) \cap V(Q) = \xi^{i-1}\left(\Gamma_L(w_1) \cap V(Q)\right), \; 1 \leq i \leq 13.$$

$$E' = \{[w_i v_j] \mid w_i \in V(Q'),\; v_j \in \Gamma_L(w_i) \cap V(Q)\}.$$

Now we define the edge set of L:

$$E(L) = E(Q) \cup E(Q') \cup E'.$$

We extend the automorphism ξ of Q which is defined above to a mapping from L to L, namely:

$$\xi = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13})(w_1, w_2, w_3, w_4, w_5, w_6, w_7,$$
$$w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}).$$

From the construction of L it is easy to see that this extension of ξ is an isomorphism of L, which we shall also denote by ξ.

As L has 26 vertices, it will be enough to prove that $L \xrightarrow{v} (4, 4)$ and $cl(L) < 5$, in order to prove Theorem 3.

6. PROOF OF THEOREM 3

We shall first prove that $cl(L) < 5$. Assume the opposite. Let S be a 5-clique in L. As Q and Q' are isomorphic, by (4.1) we have $cl(Q) = cl(Q') = 4$. Hence $S \not\subset Q$ and $S \not\subset Q'$. Therefore we have the following 4 cases:

First case. $|S \cap Q| = 4$, $|S \cap Q'| = 1$.

Using ξ without loss of generality we can consider $w_1 \in S$.

But $\Gamma_L(w_1) \cap Q = \varphi(M)$, which is isomorphic to M, which has no 4-cliques by (4.3).

Second case. $|S \cap Q| = 1$, $|S \cap Q'| = 4$.

Now using ξ, without loss of generality we can consider that $v_1 \in S$. From the construction in Section 5 via trivial computation it follows that

$$\Gamma_L(v_1) \cap Q' = \{w_1, w_{13}, w_{12}, w_9, w_8, w_7, w_4, w_2\}$$

$$= \varphi^3 (\{w_1, w_2, w_3, w_4, w_5, w_6, w_9, w_{10}\}).$$

This subgraph is isomorphic to M, which has no 4-cliques by (4.3).

Third case. $|S \cap Q| = 3$, $|S \cap Q'| = 2$.

Using ξ, without loss of generality we can consider $w_1 \in S$. Again using ξ we reduce this case to the following subcases:

Subcase 3.1. $S \cap Q' = \{w_1, w_3\}$. Now from the construction in Section 5 we have:

$$\Gamma_L(w_1) \cap \Gamma_L(w_4) \cap Q = \{v_2, v_3, v_8, v_{13}\},$$

which has no 3-cliques.

Subcase 3.2. $S \cap Q' = \{w_1, w_4\}$. Now

$$\Gamma_L(w_1) \cap \Gamma_L(w_3) \cap Q = \{v_1, v_6, v_{11}, v_3\},$$

which has no 3-cliques.

Subcase 3.3. $S \cap Q' = \{w_1, w_5\}$. Now

$$\Gamma_L(w_1) \cap \Gamma_L(w_5) \cap Q = \{v_{11}, v_6, v_7, v_2\},$$

which has no 3-cliques.

Subcase 3.4. $S \cap Q' = \{w_1, w_7\}$. Now

$$\Gamma_L(w_1) \cap \Gamma_L(w_7) \cap Q = \{v_7, v_8, v_{13}, v_1, v_6\},$$

which is isomorphic to C_5 and has no 3-cliques.

Fourth case. $|S \cap Q'| = 3$, $|S \cap Q| = 2$.

Using ξ, without loss of generality we can assume that $v_1 \in S$. Again, using ξ, we reduce this case to the following subcases:

Subcase 4.1. $S \cap Q = \{v_1, v_3\}$. Now from the construction in Section 5 we have:

$$\Gamma_L(v_1) \cap \Gamma_L(v_3) \cap Q' = \{w_1, w_9, w_4, w_2\},$$

which has no 3-cliques.

Subcase 4.2. $S \cap Q = \{v_1, v_4\}$. Now

$$\Gamma_L(v_1) \cap \Gamma_L(v_4) \cap Q = \{w_2, w_7, w_4, w_{12}\},$$

which has no 3-cliques.

Subcase 4.3. $S \cap Q = \{v_1, v_5\}$. Now

$$\Gamma_L(v_1) \cap \Gamma_L(v_5) \cap Q' = \{w_{13}, w_{12}, w_4, w_8\},$$

which has no 3-cliques.

Subcase 4.4. $S \cap Q = \{v_1, v_7\}$. Now

$$\Gamma_L(v_1) \cap \Gamma_L(v_7) \cap Q = \{w_1, w_{13}, w_8, w_7, w_2\},$$

which is isomorphic to C_5 and therefore has no 3-cliques.

Thus we have completed the proof of the fact that $cl(L) < 5$.

It remains to prove $L \xrightarrow{v} (4,4)$ only.

Assume that $V_1 \cup V_2$ is a (4,4)-free vertex coloring of L. Then $V_1 \cap Q$ and $V_2 \cap Q$ must be a (4,4)-free vertex coloring of Q. Then, according to Proposition 4.5 and having in mind that ξ can be continued to an automorphism of L, we have the following five groups of cases (totally 32 cases).

First group of cases: when there is $0 \le n \le 12$, $n \in \mathbb{N}$, such that $V_1 \cap Q \subseteq \xi^n(M_0)$. Thus without loss of generality we can assume that $V_1 \cap Q \subseteq M$.

Case 1.1. When $V_1 \supset M = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\}$, $V_2 \supseteq \{v_7, v_8, v_{11}, v_{12}, v_{13}\}$. Now we have:

$$v_9, v_2, v_5 \in V_1, \quad \text{therefore } w_8 \in V_2,\ w_3 \in V_2;$$

$$v_2, v_4, v_6 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$v_9, v_2, v_6 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$w_9, w_5, w_3 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_8, w_5, v_7 \in V_2, \quad \text{therefore } w_2 \in V_1.$$

Now w_2, w_{12}, v_1, v_4 is a 4-clique in V_1. We have completed the proof of case 1.1. Note that $v_4 \notin V_2$ and $v_9 \notin V_2$ because of the 4-cliques v_7, v_9, v_{11}, v_{13} and v_4, v_7, v_{11}, v_{13}. Therefore we have only 6 other cases in this group of cases:

Case 1.2. Replace v_1, i.e.

$$V_1 \supset \{v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\},\ V_2 \supset \{v_1, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

We have $\{v_9, v_2, v_6\} \subset V_1 \cap \Gamma_L(w_9)$ and $\{v_1, v_8, v_{11}\} \subset V_2 \cap \Gamma_L(w_9)$. So whatever the color of w_9, either $\{w_9, v_1, v_8, v_{11}\}$ or $\{w_9, v_9, v_2, v_6\}$ is a monochromatic 4-clique.

Case 1.3. Replace v_2, i.e.

$$V_1 \supset \{v_1, v_3, v_4, v_5, v_6, v_9, v_{10}\},\ V_2 \supset \{v_2, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

We have $v_2, v_8, v_{11} \in V_2 \cap \Gamma_L(w_9)$ and $v_9, v_3, v_6 \in V_1 \cap \Gamma_L(w_9)$.

So whatever the color of w_9, either $\{w_9, v_2, v_8, v_{11}\}$ or $\{w_9, v_9, v_3, v_6\}$ is a monochromatic 4-clique.

Case 1.4. Replace v_3, i.e.

$$V_1 \supset \{v_1, v_2, v_4, v_5, v_6, v_9, v_{10}\},\ V_2 \supset \{v_3, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

The proof is similar to the one in case 1.1. We have:

$$v_9, v_2, v_5 \in V_1, \quad \text{therefore } w_8 \in V_2,\ w_3 \in V_2;$$

$$v_2, v_4, v_6 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$v_9, v_2, v_6 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_7, w_5, w_8 \in V_2, \quad \text{therefore } w_2 \in V_1;$$

$$w_3, w_5, w_9 \in V_2, \quad \text{therefore } w_{12} \in V_1.$$

Now w_2, w_{12}, v_1, v_4 is a 4-clique in V_1.

Note that the proof was precisely the same as the one of case 1.1.

Case 1.5. Replace v_5, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_9, v_{10}\},\ V_2 \supset \{v_5, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

Now $\{v_5, v_7, v_{11}\} \subset V_2 \cap \Gamma_L(w_5)$ and $\{v_2, v_4, v_6\} \subset V_1 \cap \Gamma_L(w_5)$.

Now whatever the color of w_5, either $\{w_5, v_2, v_4, v_6\}$ or $\{w_5, v_5, v_7, v_{11}\}$ is a monochromatic 4-clique.

Case 1.6. Replace v_6, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_9, v_{10}\},\ V_2 \supset \{v_6, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_3, v_{10} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_9, v_2, v_5 \in V_1, \quad \text{therefore } w_8 \in V_2;$$

$$v_6, v_8, v_{12} \in V_2, \quad \text{therefore } w_7 \in V_1;$$

$$w_9, v_8, v_{11} \in V_2, \quad \text{therefore } w_{11} \in V_1;$$

$$w_7, w_{11}, v_4 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$w_5, w_9, v_6 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_{12}, v_1, v_4 \in V_1, \quad \text{therefore } w_2 \in V_2.$$

Now $\{w_2, w_5, w_8, v_7\}$ is a 4-clique in V_2.

Case 1.7. Replace v_{10}, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\}, \ V_2 \supset \{v_7, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}.$$

We have $v_7, v_{10}, v_{13} \in V_2 \cap \Gamma_L(w_8)$ and $v_9, v_2, v_5 \in V_1 \cap \Gamma_L(w_8)$.

Now whatever the color of w_8, either $\{w_8, v_7, v_{10}, v_{13}\}$ or $\{w_8, v_9, v_2, v_5\}$ is a monochromatic 4-clique.

Second group of cases: when there is $0 \le k \le 12$, $k \in \mathbb{N}$ such that $V_1 \cap Q \subseteq \xi^k(M_1)$. Without loss of generality we can assume that $V_1 \cap Q \subset M_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\}$. We have the following cases.

Case 2.1.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_{11}, v_4, v_2 \in V_1, \quad \text{therefore } w_{10} \in V_2;$$

$$v_{11}, v_5, v_1 \in V_1, \quad \text{therefore } w_{12} \in V_2.$$

Now $\{w_{10}, w_{12}, v_9, v_{12}\}$ is a 4-clique in V_2.

Now note that $v_{11}, v_3 \notin V_2$ because of the 4-cliques $\{v_7, v_9, v_{13}, v_{11}\}$ and $\{v_3, v_7, v_9, v_{13}\}$. So only 6 other cases are possible in this group.

Case 2.2. Replace v_1, i.e.

$$V_1 \supset \{v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_1, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_8, v_{12} \in V_2, \quad \text{therefore } w_2 \in V_1;$$

$$v_7, v_9, v_{13} \in V_2, \quad \text{therefore } w_8 \in V_1;$$

$$v_{11}, v_4, v_2 \in V_1, \quad \text{therefore } w_{10} \in V_2;$$

$$w_{10}, v_9, v_{12} \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_2, w_8, w_{12} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_3, v_6, v_{10} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_8, w_6, w_9 \in V_2, \quad \text{therefore } w_3 \in V_1;$$

$$v_{10}, v_4, v_6 \in V_1, \quad \text{therefore } w_4 \in V_2;$$

$$w_4, w_6, w_{10} \in V_2, \quad \text{therefore } w_{13} \in V_1.$$

Now $\{w_3, w_{13}, v_2, v_5\}$ is a 4-clique in V_1.

Case 2.3. Replace v_2, i.e.

$$V_1 \supset \{v_1, v_3, v_4, v_5, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_2, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_2, v_9, v_{12} \in V_2, \quad \text{therefore } w_{10} \in V_1;$$

$$v_2, v_9, v_{13} \in V_2, \quad \text{therefore } w_3 \in V_1;$$

$$v_7, v_9, v_{13} \in V_2, \quad \text{therefore } w_7 \in V_1.$$

Now $\{w_3, w_7, w_{10}, v_4\}$ is a 4-clique in V_1.

Case 2.4. Replace v_4, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_5, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_4, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_{10}, v_6, v_3 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_{11}, v_2, v_5 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$v_{11}, v_5, v_1 \in V_1, \quad \text{therefore } w_{12} \in V_2;$$

$$v_7, v_9, v_{13} \in V_2, \quad \text{therefore } w_7 \in V_1;$$

$$w_{12}, v_9, v_{12} \in V_2, \quad \text{therefore } w_{10} \in V_1;$$

$$w_5, w_9, w_{12} \in V_2, \quad \text{therefore } w_3 \in V_1;$$

$$w_7, w_{10}, w_3 \in V_1, \quad \text{therefore } w_{13} \in V_2;$$

$$w_3, v_3, v_5 \in V_1, \quad \text{therefore } w_6 \in V_2.$$

Now $\{w_6, w_{13}, v_7, v_{13}\}$ is a 4-clique in V_2.

Case 2.5. Replace v_5, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_5, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_4, v_{11} \in V_1, \quad \text{therefore } w_{12} \in V_2;$$

$$v_{11}, v_4, v_2 \in V_1, \quad \text{therefore } w_{10} \in V_2.$$

Now $\{w_{10}, w_{12}, v_9, v_{12}\}$ is a 4-clique in V_2.

Case 2.6. Replacing v_6, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_{10}, v_{11}\}, \ V_2 \supset \{v_6, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

The proof is word by word the same as the proof of case 2.5.

Case 2.7. Replacing v_{10}, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_6, v_{11}\}, \ V_2 \supset \{v_7, v_8, v_9, v_{10}, v_{12}, v_{13}\}.$$

The proof again is word by word the same as the proof of case 2.5.

Third group of cases. Let there be such $0 \leq k \leq 12$, $k \in \mathbb{N}$, such that $V_1 \cap Q \subseteq \xi^k(M_2)$. As ξ is an automorphism of L, we can consider without loss of generality $V_1 \cap Q \subseteq M_2 = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}$.

Case 3.1. Let

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}, \ V_2 \supset \{v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$
$$v_6, v_3, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$
$$v_2, v_6, v_{13} \in V_1, \quad \text{therefore } w_{13} \in V_2;$$
$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2.$$

Now $\{w_2, w_6, w_9, w_{13}\}$ is a 4-clique in V_2.

Now note that $v_3, v_2 \notin V_2$ because of the 4-cliques $\{v_3, v_5, v_9, v_{12}\}$ and $\{v_2, v_5, v_9, v_{12}\}$. So we have only 6 other cases in this group.

Case 3.2. Replace v_1, i.e.

$$V_1 \supset \{v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}, \ V_2 \supset \{v_1, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

Now $v_1, v_5, v_{12} \in V_2 \cap \Gamma_L(w_{13})$ and $v_2, v_6, v_{13} \in V_1 \cap \Gamma_L(w_{13})$ so whatever the color of w_{13} either $\{w_{13}, v_1, v_5, v_{12}\}$ or $\{w_{13}, v_2, v_6, v_{13}\}$ will be a monochromatic 4-clique.

Case 3.3. Replace v_6, i.e.

$$V_1 \supset \{v_1, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}, \ V_2 \supset \{v_6, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_4, v_6, v_{10} \in V_2, \quad \text{therefore } w_4 \in V_1;$$
$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$
$$v_7, v_{11}, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$
$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$
$$w_2, w_6, w_9 \in V_1, \quad \text{therefore } w_{13} \in V_1;$$
$$w_4, w_{13}, v_1 \in V_1, \quad \text{therefore } w_7 \in V_2.$$

Now $\{w_7, w_9, v_6, v_9\}$ is a 4-clique in V_2.

Case 3.4. Replace v_{11}, i.e.

$$V_1 \supset \{v_1, v_6, v_3, v_8, v_{13}, v_2, v_7\}, \ V_2 \supset \{v_{11}, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have, similarly to case 3.1:

$$v_6, v_3, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_2, v_6, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$v_2, v_6, v_{13} \in V_1, \quad \text{therefore } w_{13} \in V_2.$$

Now $w_2, w_6, w_9, w_{13} \in V_2$ is a monochromatic 4-clique.

Case 3.5. Replace v_8, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_{13}, v_2, v_7\}, \ V_2 \supset \{v_8, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

Now $v_8, v_5, v_{12} \in V_2 \cap \Gamma_L(w_6)$ and $v_6, v_3, v_{13} \in V_1 \cap \Gamma_L(w_6)$. Whatever the color of w_6, either $\{w_6, v_8, v_5, v_{12}\}$, or $\{w_6, v_6, v_3, v_{13}\}$ will be a monochromatic 4-clique.

Case 3.6. Replace v_{13}, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_2, v_7\}, \ V_2 \supset \{v_{13}, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_1, w_9 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$v_5, v_{12}, v_9 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$v_{13}, v_{10}, v_4 \in V_2, \quad \text{therefore } w_{11}, w_3 \in V_1.$$

Subcase 3.6.1. Let $w_6 \in V_1$. Now $w_6, v_3, v_6 \in V_1$, therefore $w_4 \in V_2$.
Also $w_6, w_3, w_{12} \in V_1$, hence $w_{10} \in V_2$.
Now $\{w_{10}, w_4, v_{10}, v_4\}$ is a 4-clique in V_2.

Subcase 3.6.2. Let $w_6 \in V_2$.
We have $w_2, w_6, w_9 \in V_2$, hence $w_{13} \in V_1$.
Now $w_{13}, v_1, v_7 \in V_1$, therefore $w_7 \in V_2$.
From $w_3, w_{13}, v_2 \in V_1$ follows $w_{10} \in V_2$.
Now $\{w_7, w_{10}, v_{12}, v_9\}$ is a 4-clique in V_2.

Case 3.7. Replace v_7, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2\}, \ V_2 \supset \{v_7, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_1, v_8, v_{11} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_2, v_6, v_{13} \in V_1, \quad \text{therefore } w_{13} \in V_2;$$

$$v_6, v_3, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$w_6, w_9, w_{13} \in V_2, \quad \text{therefore } w_2 \in V_1;$$

$$v_7, v_5, v_9 \in V_2, \quad \text{therefore } w_8 \in V_1;$$

$$v_7, v_4, v_{10} \in V_2, \quad \text{therefore } w_5 \in V_1;$$

$$w_2, w_5, w_8 \in V_1, \quad \text{therefore } w_{11} \in V_2.$$

Now $\{v_{10}, w_{13}, w_{11}, w_9\}$ is a 4-clique in V_2.

Fourth group of cases.

Assume there is $k \in \mathbb{N}$, $0 \leq k \leq 12$ such that $V_1 \cap Q \subseteq \xi^k(M_3)$. As ξ is an automorphism, without loss of generality we can assume that $V_1 \cap Q \subseteq M_3 = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}$.

Case 4.1. Let

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}, \ V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_6, v_3, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12}, w_{13} \in V_1;$$

$$w_2, v_4, v_2 \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Now note that $v_{11}, v_{12} \notin V_2$ because of the 4-cliques $\{v_{12}, v_2, v_5, v_9\}$ and $\{v_{11}, v_2, v_5, v_9\}$. So we have 6 more cases in this group of cases.

Case 4.2. Replace v_1, i.e.

$$V_1 \supset \{v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}, \ V_2 \supset \{v_1, v_5, v_{10}, v_4, v_9, v_2\}.$$

We have:

$$v_{12}, v_8, v_6 \in V_1, \quad \text{therefore } w_6, w_7 \in V_2;$$

$$v_2, v_5, v_9 \in V_2, \quad \text{therefore } w_3, w_8 \in V_1;$$

$$v_1, v_4, v_{10} \in V_2, \quad \text{therefore } w_4 \in V_1;$$

$$w_4, v_3, v_6 \in V_1, \quad \text{therefore } w_1 \in V_2;$$

Subcase 4.2.1. Let $w_{11} \in V_1$. We have:

$$w_4, w_8, w_{11} \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, v_2, v_4 \in V_2, \quad \text{therefore } w_5 \in V_1;$$

$$w_2, v_2, v_9 \in V_2, \quad \text{therefore } w_9 \in V_1.$$

Now $\{w_{11}, w_5, w_9, v_{11}\}$ is a 4-clique in V_1.

Subcase 4.2.2. Let $w_{11} \in V_2$. We have:

$$w_1, w_7, w_{11} \in V_2, \quad \text{therefore } w_5 \in V_1;$$

$$w_5, v_6, v_{12} \in V_1, \quad \text{therefore } w_{12} \in V_2;$$

$$w_5, w_8, v_7 \in V_1, \quad \text{therefore } w_2 \in V_2.$$

Now $\{w_2, w_{12}, v_1, v_4\}$ is a 4-clique in V_2.
Thus case 4.2 is over.

Case 4.3. Replace v_6, i.e.

$$V_1 \supset \{v_1, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}, \; V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2, v_6\}.$$

Now $v_2, v_6, v_9 \in V_2 \cap \Gamma_L(w_9)$ and $v_1, v_8, v_{11} \in V_1 \cap \Gamma_L(w_9)$. Whatever the color of w_9, either $\{w_9, v_2, v_6, v_9\}$, or $\{w_9, v_1, v_8, v_{11}\}$ is a 4-clique.

Case 4.4. Replace v_3, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_8, v_{13}, v_7, v_{12}\}, \; V_2 \supset \{v_3, v_5, v_{10}, v_2, v_4, v_9\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_7, v_{11}, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_1, v_{12}, v_8 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12}, w_{13} \in V_1;$$

$$w_2, v_2, v_4 \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Case 4.5. Replace v_8, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_{13}, v_7, v_{12}\}, \; V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2, v_8\}.$$

Now $v_8, v_2, v_4 \in V_2 \cap \Gamma_L(w_2)$ and $v_7, v_1, v_3 \in V_1 \cap \Gamma_L(w_2)$. Whatever the color of w_2, either $\{w_2, v_8, v_2, v_4\}$, or $\{w_2, v_1, v_3, v_7\}$ is a 4-clique.

Case 4.6. Replace v_{13}, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_7, v_{12}\}, \; V_2 \supset \{v_{13}, v_5, v_{10}, v_2, v_4, v_9\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_{12}, v_3, v_6 \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_2, v_2, v_4 \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Case 4.7. Replace v_7, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_{12}\}, \ V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2, v_7\}.$$

We have:

$$v_1, v_8, v_{11} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_1, v_8, v_{12} \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$v_3, v_6, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$v_4, v_7, v_{10} \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Fifth group of cases.

Now we assume there is $k \in \mathbb{N}$, $0 \le k \le 12$ and $0 \le i \le 3$ that $Q \cap V_1 = S_i$. We have the following possibilities:

Case 5.1.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_7, v_9\}, \ V_2 \supset \{v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_3, v_7 \in V_1, \quad \text{therefore } w_1 \in V_2;$$

$$v_1, v_4, v_7 \in V_1, \quad \text{therefore } w_7 \in V_2;$$

$$v_3, v_7, v_9 \in V_1, \quad \text{therefore } w_{10} \in V_2;$$

$$v_3, v_6, v_9 \in V_1, \quad \text{therefore } w_4 \in V_2.$$

Now $\{w_1, w_4, w_7, w_{10}\}$ is a 4-clique in V_2.

Case 5.2.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_7, v_{12}\}, \ V_2 \supset \{v_5, v_8, v_9, v_{10}, v_{11}, v_{13}\}.$$

Now $v_5, v_8, v_{11} \in V_2 \cap \Gamma_L(w_6)$ and $v_3, v_6, v_{12} \in V_1 \cap \Gamma_L(w_6)$. Whatever the color of w_6, either $\{w_6, v_5, v_8, v_{11}\}$, or $\{w_6, v_3, v_6, v_{12}\}$ will be a monochromatic a 4-clique.

Case 5.3.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_9, v_{11}, v_{12}\}, \ V_2 \supset \{v_5, v_6, v_7, v_8, v_{10}, v_{13}\}.$$

Now $v_2, v_9, v_{11} \in V_1 \cap \Gamma_L(w_9)$ and $v_6, v_8, v_{10} \in V_2 \cap \Gamma_L(w_9)$. Now whatever the color of w_9, either $\{w_9, v_2, v_9, v_{11}\}$, or $\{w_9, v_6, v_8, v_{10}\}$ will be a monochromatic a 4-clique.

Case 5.4.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_{11}, v_{12}\},\ V_2 \supset \{v_5, v_7, v_8, v_9, v_{10}, v_{13}\}.$$

We have:

$$v_1, v_3, v_{12} \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$v_1, v_4, v_{11} \in V_1, \quad \text{therefore } w_4 \in V_2;$$

$$v_3, v_6, v_{12} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_2, v_6, v_{12} \in V_1, \quad \text{therefore } w_{13} \in V_2.$$

Now $\{w_{13}, w_2, w_4, w_6\}$ is a 4-clique in V_2.

The above considerations, Proposition 4.5 and the fact that ξ is an automorphism of L prove Theorem 3.

7. PROOFS OF THE PROPOSITIONS AND LEMMAS FROM SECTION 4

Proof of Lemma 1. Let C be the wanted 4-cycle. Then using φ without loss of generality we have:

$$|E(C) \cap E(C_{13}^{(1)})| \geq |E(C) \cap E(C_{13}^{(2)})|, \ i.e.$$

$$|E(C) \cap E(C_{13}^{(1)})| \geq 2.$$

Case 1. If $|E(C) \cap E(C_{13}^{(1)})| = 4$, then using ξ we may assume that $C = \{v_1, v_2, v_3, v_4\}$, but $Q(\{v_1, v_2, v_3, v_4\})$ is not a simple 4-cycle.

Case 2. If $|E(C) \cap E(C_{13}^{(1)})| = 3$. As $3 > 2$, then there are two edges in $E(C) \cap E(C_{13}^{(1)})$ with a common vertex. Then using ξ we may assume that $\{v_1, v_2, v_3\} \subseteq V(C)$.

But $\Gamma_{\overline{Q}}(v_1) \cup \Gamma_{\overline{Q}}(v_3) = \{v_2\}$ and hence this case is impossible.

Case 3. If $|E(C) \cap E(C_{13}^{(1)})| = 2$. If there are two adjacent edges in $E(C) \cap E(C_{13}^{(1)})$, then using ξ we would have $\{v_1, v_2, v_3\} \subset V(C)$, which is impossible as mentioned above. Then the two edges in $E(C) \cap E(C_{13}^{(1)})$ are not adjacent. Using ξ we can assume that $\{v_1, v_2\} \subseteq V(C)$. Now we must have at least one edge from v_1 or v_2 in $E(C) \cap H_2$.

The possibilities are $v_1v_9, v_1v_6, v_2v_{10}, v_2v_7$.

Thus we obtain two 4-cycles $\{v_1, v_2, v_{10}, v_9\}$ and $\{v_1, v_2, v_6, v_7\}$, which are equivalent: $\varphi(\{v_1, v_2, v_{10}, v_9\}) = \{v_1, v_6, v_7, v_2\}$.

Thus the lemma is proved. □

The proofs of Lemma 4.2 and 4.3 are trivial.

Proof of Lemma 4.4. As $E(\overline{Q}) = E(C_{13}^{(1)}) \cup E(C_{13}^{(2)})$ and using φ we can consider that $|E(C_{2s+1}) \cap E(C_{13}^{(1)})| \geq |E(C_{2s+1}) \cap E(C_{13}^{(2)})|$.

Therefore $|E(C_{2s+1}) \cap E(C_{13}^{(1)})| \geq s+1$ and as $2s+1$ is odd we have at least two adjacent edges in $E(C_{2s+1}) \cap E(C_{13}^{(1)})$. □

Proof of Lemma 4.5. Assume that C is an induced subgraph of $\overline{Q}$, isomorphic to C_7. Using the previous lemma and ξ we obtain $\{v_1, v_2, v_3\} \subset V(C_7)$.

Assign $V(C) = \{v_1, v_2, v_3, a, b, c, d\}$.

Then

$$\begin{aligned} d &\in \Gamma_{\overline{Q}}(v_1)/\{v_2\} = \{v_6, v_9, v_{13}\} \\ a &\in \Gamma_{\overline{Q}}(v_3)/\{v_2\} = \{v_4, v_8, v_{11}\} \end{aligned} \tag{7.1}$$

Now let us observe that

$$\begin{gathered} C \text{ does not contain 4 consequent vertices} \\ \text{in any of the cycles } C_{13}^{(1)} \text{ and } C_{13}^{(2)}. \end{gathered} \tag{7.2}$$

Indeed, if (7.2) is not correct, using φ and ξ, we can assume that $\{v_1, v_2, v_3, v_4\} \subset V(C)$. But each vertex of $\overline{Q}$ is adjacent to at least one of these 4 vertices. As C has 7 vertices, it cannot be a simple cycle. Thus (7.2) is proved.

From (7.2) we have that $d \neq v_{13}, a \neq v_4$.

Case 1. Let $a = v_8$. Now $b \neq v_9$ as v_1, v_2, v_3, v_8, v_9 is a simple 5-cycle.

Also $b \neq v_{13}$ by (7.2).

As $b \in \Gamma_{\overline{Q}}(a)/\{v_3\} = \{v_7, v_9, v_{13}\}$ it remains $b = v_7$, but $v_2, v_7 \in E(\overline{Q})$, which is a contradiction.

Case 2. Let $a = v_{11}$.

Then $b \in \Gamma_{\overline{Q}}(v_{11})/\{v_3\} = \{v_{10}, v_{12}, v_6\}$.

As $v_6, v_1, v_{10}, v_2 \in E(\overline{Q})$, it follows $b = v_{12}$

Now $c \in \Gamma_{\overline{Q}}(v_{12})/\{v_{11}\} = \{v_{13}, v_4, v_7\}$.

But $v_{13}, v_1, v_4, v_3, v_7, v_2 \in E(\overline{Q})$, which is a contradiction.

The lemma is proved. □

Proof of Lemma 4.6.

From Lemma 4.4, using ξ, we have $\{v_1, v_2, v_3\} \subset V(C)$. Assign $V(C) = \{v_1, v_2, v_3, c, d\}$.

If C contain 4 consequent vertices on one of the two cycles $C_{13}^{(1)}$, $C_{13}^{(2)}$, i.e. without loss of generality $V(C) = \{v_1, v_2, v_3, v_4, d\}$, then

$$d \in \Gamma_{\overline{Q}}(v_1) \cap \Gamma_{\overline{Q}}(v_4) = \{v_9\}.$$

Hence $C = \{v_1, v_2, v_3, v_4, v_9\}$ and we are through.

So we can consider

$$C \text{ does not contain 4 consequent vertices on any of the cycles } C_{13}^{(1)},\ C_{13}^{(2)}. \tag{7.3}$$

Then $c \neq v_4, d \neq v_{13}$.

Case 1. If $c = v_{11}$. Then

$$d \in \Gamma_{\overline{Q}}(v_{11}) \cap \Gamma_{\overline{Q}}(v_1) = \{v_6\},$$

and hence

$$C = \{v_1, v_2, v_3, v_6, v_{11}\} = \varphi^{-1}\xi^{-3}(\{v_1, v_2, v_3, v_4, v_9\}),$$

and we are through.

Case 2. If $c = v_8$. Then

$$d \in \Gamma_{\overline{Q}}(v_8) \cap \Gamma_{\overline{Q}}(v_1) = \{v_9, v_{13}\},$$

but $d \neq v_{13}$ and hence $C = \{v_1, v_2, v_3, v_8, v_9\}$. The lemma is proved. □

Propositions 4.1 and 4.2 are trivial.

Before proving proposition 4.3 we shall introduce the following notation.

Assign:

$$\varphi_j = \xi^{j-1}\varphi\xi^{j-1},\ j = 1, \ldots, 13;$$

(i.e. $\varphi = \varphi_1$ – we shall continue to use both φ and φ_1 farther).

$$\eta_j = \varphi_j^2,\ j = 1, \ldots, 13;$$

$$\eta = \varphi^2;$$

(we have $\eta = \eta_1$ in these assignments).

The "geometric" interpretation of these automorphisms is the following:

φ_j replaces $C_{13}^{(1)}$ and $C_{13}^{(2)}$, leaving the vertex v_j fixed;

η_j is a reflection around the vertex v_j.

Proof of Proposition 4.3. From the statement of the theorem, we have $|\overline{Q}[V_1]| = 8$, $\alpha(\overline{Q}[V_1]) < 4$, $cl(\overline{Q}[V_1]) = 2$. We shall use the classification of all such graphs, given on p.194 in [3].

Note that all the three configurations contain a simple 4-cycle $w_1w_2w_3w_4$ and two simple 4-chains w_1abw_3, w_2cdw_4. We already know from Lemma 4.1 that any simple 4-cycle can be obtained from $\{v_1, v_6, v_7, v_2\}$ via an automorphism $\psi \in Aut(\overline{Q})$. So without loss of generality we have $\{v_1, v_2, v_7, v_6\} \subset V_1$. Now using $v_2v_6 = \xi(v_1v_5)$ and Lemma 4.2, we have the following possible simple 4-chains v_2cdv_6:

$$1)\ v_2v_{10}v_{11}v_6 \quad 2)\ v_2v_{10}v_5v_6 \quad 3)\ v_3v_{11}v_{11}v_6. \tag{7.4}$$

Using Lemma 4.3 we have the following possibilities for v_1abv_7:

$$1)\ v_1v_{13}v_{12}v_7 \quad 2)\ v_1v_{13}v_8v_7 \quad 3)\ v_1v_9v_8v_7. \tag{7.5}$$

Combining (7.4) and (7.5), we have:

$$V_1 = \{v_1, v_{13}, v_{12}, v_7, v_2, v_{10}, v_{11}, v_6\} = \xi^9\eta_{10}(M);$$
$$V_1 = \{v_2, v_{10}, v_{11}, v_6, v_1, v_{13}, v_8, v_7\}.$$

Now $V_2 = \{v_3, v_4, v_5, v_9, v_{12}\}$ contains 4 clique $v_3v_5v_9v_{12}$.

$$V_1 = \{v_2, v_{10}, v_{11}, v_6, v_1, v_9, v_8, v_7\} = \xi^5(M);$$
$$V_1 = \{v_2, v_{10}, v_5, v_6, v_1, v_{13}, v_{12}, v_7\} = \xi^{-1}\varphi(M);$$
$$V_1 = \{v_2, v_{10}, v_5, v_6, v_1, v_{13}, v_8, v_7\} = \xi^7\varphi\eta_{10}(M);$$
$$V_1 = \{v_2, v_{10}, v_5, v_6, v_1, v_9, v_8, v_7\} = \xi^4\eta_{10}(M);$$
$$V_1 = \{v_2, v_3, v_{11}, v_6, v_1, v_{13}, v_{12}, v_7\} = \xi^{-3}(M);$$
$$V_1 = \{v_2, v_3, v_{11}, v_6, v_1, v_{13}, v_8, v_7\} = \varphi(M);$$
$$V_1 = \{v_2, v_3, v_{11}, v_6, v_1, v_9, v_8, v_7\} = \xi^8\varphi\eta_{10}(M).$$

Proposition 4.3 is proved. □

Proof of Proposition 4.4. From the statement of the theorem it follows that $|\overline{Q}[V_1]| = 7$, $\alpha(\overline{Q}[V_1]) < 4$, $cl(\overline{Q}[V_1]) = 2$. We shall use the classification of all such graphs on p.194 in [3].

We shall need the following corollary from this classification, which can be easily proved independently:

$$\begin{array}{c}\text{If } G \text{ is a graph with } |G| = 7,\ \alpha(G) < 4,\ cl(G) = 2.\\ \text{Now } G \text{ contains either } C_7 \text{ or } C_5 \text{ as an induced subgraph.}\end{array} \tag{7.6}$$

Now from (7.6) and Lemma 4.5 we see that $\overline{Q}[V_1]$ contains an induced subgraph, isomorphic to C_5. From Lemma 4.6 we have the following cases:

Case 1. Let $V_1 \supset \{v_1, v_2, v_3, v_4, v_8, v_9\}$.
Now we have as v_1v_2 is (4,4)-free:

$$v_1, v_4, v_8 \in V_1, \quad \text{therefore } v_{10}, v_{11} \in V_2;$$

$$v_2, v_4, v_8 \in V_1, \quad \text{therefore } v_6, v_{11} \in V_2.$$

Then for the seventh vertex of V_1 we have the following possibilities: v_5, v_7, v_{13}, v_{12}.

If the seventh vertex of V_1 is v_5 or v_{13}, then $V_1 \subset \{v_{13}, v_1, v_2, v_3, v_4, v_5, v_8, v_9\} = \xi^{-1}(M)$.

If the seventh vertex is either v_7 or v_{12}, then $V_1 \subset \{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{12}\} = \xi\varphi(M)$.

Case 2. Let $V_1 \supset \{v_1, v_2, v_3, v_8, v_9\}$.

Now we assume that $v_4 \in V_2$ (otherwise we fall in the conditions of the previous case).

We can consider $v_{13} \in V_2$.

Otherwise, i.e. if $v_{13} \in V_1$, then $\eta_2(V_1)$ would be a 7-vertex subgraph of Q with the wanted properties and $\eta_2(V_1) \subset \{v_1, v_2, v_3, v_4, v_8, v_9\}$ and this lead to the previous case.

Now note that $v_4v_7v_{11}v_{13}$ is a 4-clique in Q. As $cl(Q[V_2]) < 4$ and as we already proved $v_4, v_{13} \in V_2$ then $v_7 \in V_1$ or $v_{11} \in V_1$. From the clasification on p.192 in [3] we see that there must be an edge outside the 5-cycle. So we have the following possibilities for the remaining 2 vertices of V_1: $v_7v_6; v_7v_{12}; v_{11}v_6; v_{11}v_{12}; v_{11}v_{10}$.

We have

Subcase 2.1.
$V_1 = \{v_1, v_2, v_3, v_6, v_7, v_8, v_9\} \subset \{v_6, v_7, v_8, v_9, v_{11}, v_1, v_2, v_3\} = \xi^{-5}\varphi\eta_{10}(M)$.

Subcase 2.2.
$V_1 = \{v_1, v_2, v_3, v_8, v_9, v_6, v_{11}\} \subseteq \{v_6, v_7, v_8, v_9, v_{11}, v_1, v_2, v_3\} = \xi^{-5}\varphi\eta_{10}(M)$.

Subcase 2.3.
$V_1 = \{v_1, v_3, v_7, v_8, v_9, v_{12}\} \subset \{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{12}\} = \xi\varphi(M)$.

Subcase 2.4. $V_1 = \{v_1, v_2, v_3, v_8, v_9, v_{11}, v_{12}\}$. Now $V_2 = \{v_4, v_5, v_6, v_7, v_{10}, v_{13}\}$ and hence $G(V_2)$ contains the 4-clique v_4, v_7, v_{10}, v_{13}.

Subcase 2.5.
$V_1 = \{v_1, v_2, v_3, v_8, v_{10}, v_{11}\} \subseteq \{v_8, v_9, v_{10}, v_{11}, v_1, v_2, v_3, v_6, \} = \xi^{-5}\varphi(M)$.
Case 2 is over.

Case 3. Let $V_1 \supset \{v_1, v_2, v_3, v_4, v_9\}$, but $v_8 \notin V_1$, i.e. $v_8 \in V_2$.
Using η_9 we can consider as in the previous case that $v_{10} \in V_2$.
As there must be an edge outside the 5-cycle we have the following possibilities for the other vertices in V_1: $v_5v_{13}; v_5v_6; v_6v_{11}; v_6v_7; v_{11}v_{12}; v_{12}v_{13}; v_{12}v_7$.

Subcase 3.1.
$V_1 = \{v_{13}, v_1, v_2, v_3, v_4, v_5, v_9\} \subset \{v_{13}, v_1, v_2, v_3, v_4, v_5, v_8, v_9\} = \xi^{-1}(M)$.

Subcase 3.2.
$V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\} \subset \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\} = M$.

Subcase 3.3.
$V_1 = \{v_1, v_2, v_3, v_4, v_6, v_9, v_{11}\} \subset \{v_1, v_2, v_3, v_4, v_6, v_9, v_{10}, v_{11}\} = \xi^3\varphi\eta_{10}(M)$.

Subcase 3.4.
$V_1 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_9\} = S_1$.

Subcase 3.5.

$V_1 = \{v_1, v_2, v_3, v_4, v_9, v_{11}, v_{12}\} = \xi^3\varphi^2(S)$.

Subcase 3.6.
$V_1 = \{v_{12}, v_{13}, v_1, v_2, v_3, v_4, v_9\} \subset \{v_{12}, v_3, v_1, v_2, v_3, v_4, v_8, v_9\} = \xi^{-2}\eta_{10}(M)$.

Subcase 3.7.
$V_1 = \{v_1, v_2, v_3, v_4, v_7, v_9, v_{12}, v_7\} \subset \{v_1, v_2, v_3, v_4, v_7, v_8, v_{12}\} = \xi\varphi(M)$.
This proposition is proved. □

Proof of Proposition 4.5. As $R(3,4) = 9$ we have two possibilities only: $|V_1| = 8$, $|V_2| = 5$ and $|V_1| = 7$, $|V_1| = 6$
Now Proposition 4.5 follows from (4.4) and propositions 4.1, 4.3, 4.4. □
All statements from Section 4 are proved.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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AN EXAMPLE OF A 16-VERTEX FOLKMAN EDGE (3,4)-GRAPH WITHOUT 8-CLIQUES

NIKOLAY KOLEV, NEDYALKO NENOV

In [6] we computed the edge Folkman number $F(3,4;8) = 16$. There we used and announced without proof that in any blue-red coloring of the edges of the graph $K_1 + C_5 + C_5 + C_5$ there is either a blue 3-clique or red 4-clique. In this paper we give a detailed proof of this fact.

Keywords: Folkman graph, Folkman number
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1. INTRODUCTION

Only finite non-oriented graphs without multiple edges and loops are considered. We call a p-clique of the graph G a set of p vertices each two of which are adjacent. The largest positive integer p such that G contains a p-clique is denoted by $\mathrm{cl}(G)$. A set of vertices of the graph G none two of which are adjacent is called an independent set. In this paper we shall also use the following notations:

- $V(G)$ is the vertex set of the graph G;
- $E(G)$ is the edge set of the graph G;
- $N(v)$, $v \in V(G)$ is the set of all vertices of G adjacent to v;
- $G[V]$, $V \subseteq V(G)$ is the subgraph of G induced by V;
- $\chi(G)$ is the chromatic number of G;

- K_n is the complete graph on n vertices;
- C_n is the simple cycle on n vertices.

The equality $C_n = v_1 v_2 \dots v_n$ means that $V(C_n) = \{v_1, \dots, v_n\}$ and

$$E(C_n) = \{[v_i, v_{i+1}], i = 1, \dots, n-1\} \cup \{[v_1, v_n]\}$$

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$ where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

Let G and H be two graphs. We shall say that H is a subgraph of G and we shall denote $H \subseteq G$ when $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.1. A 2-coloring

$$E(G) = E_1 \cup E_2, \qquad E_1 \cap E_2 = \emptyset, \tag{1.1}$$

is called a blue-red coloring of the edges of the graph G (the edges in E_1 are blue and the edges in E_2 are red).

We define for blue-red coloring (1.1) and for an arbitrary vertex $v \in V(G)$

$$N_i(v) = \{x \in N(v) \mid [v, x] \in E_i\}, \quad i = 1, 2;$$
$$G_i(v) = G[N_i(v)].$$

Definition 1.2. Let H be a subgraph of G. We say that H is a monochromatic subgraph in the blue-red coloring (1.1) if $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$. If $E(H) \subseteq E_1$ we say that H is a blue subgraph, and if $E(H) \subseteq E_2$ we say that H is a red subgraph.

Definition 1.3. The blue-red coloring (1.1) is called (p, q)-free, if there are no blue p-cliques and no red q-cliques. The symbol $G \to (p, q)$ means that any blue-red coloring of $E(G)$ is not (p, q)-free. If $G \to (p, q)$ then G is called edge Folkman (p, q)-graph.

Let p, q and r be positive integers. The Folkman number $F(p, q; r)$ is defined by the equality

$$F(p, q; r) = \min\{|V(G)| : G \to (p, q) \text{ and } \mathrm{cl}(G) < r\}.$$

In [1] Folkman proved that

$$F(p, q; r) \text{ exists } \iff r > \max\{p, q\}.$$

That is why the numbers $F(p, q; r)$ are called Folkman numbers. Only few Folkman numbers are known. An exposition of the results on the Folkman numbers was given in [6]. In [6] we computed a new Folkman number, namely $F(3, 4; 8) = 16$. This result is based upon the fact that $K_1 + C_5 + C_5 + C_5 \to (3, 4)$, which was announced without proof in [6]. In this paper we give a detailed proof of this fact. So, the aim of this paper is to prove the following

Main Theorem. *Let* $G = K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$, *where* $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$ *are copies of the 5-cycle* C_5. *Then* $G \to (3,4)$.

2. AUXILIARY RESULTS

Lemma 2.1. *Let* $E(G) = E_1 \cup E_2$ *be a* $(3,4)$*-free red-blue coloring of the edges of the graph* G. *Then:*

(a) $G_1(v)$ *is a red subgraph*, $v \in V(G)$;

(b) $(E(G_2(v)) \cap E_1) \cup (E(G_2(v)) \cap E_2)$ *is a* $(3,3)$*-free red-blue coloring of* $E(G_2(v))$, $v \in V(G)$. *Thus* $G_2(v) \not\to (3,3)$.

Proof. The statement of (a) is obvious. Assume that (b) is not true. Then, since there is no blue 3-clique, $G_2(v)$ contains a red 3-clique. This red 3-clique together with the vertex v form a red 4-clique, which is a contradiction. □

Corollary 2.1. *Let* $E(G) = E_1 \cup E_2$ *be a* $(3,4)$*-free blue-red coloring of* $E(G)$. *Then:*

(a) $\operatorname{cl}(G_1(v)) \leq 3$, $v \in V(G)$;

(b) $\operatorname{cl}(G_2(v)) \leq 5$, $v \in V(G)$;

(c) $G_2(v) \not\supseteq K_3 + C_5$, $v \in V(G)$.

Proof. The statement of (a) follows from Lemma 2.1(a). The statements of (b) and (c) follow from Lemma 2.1(b), since $K_6 \to (3,3)$, [4] and $K_3 + C_5 \to (3,3)$, [2]. □

Lemma 2.2 ([5]). *Let* $G = C_5 + H$, *where* $V(H) = \{x, y, z\}$ *and* $E(H) = \{[x, y], [x, z]\}$. *Let* $E(G) = E_1 \cup E_2$ *be a* $(3,3)$*-free blue-red coloring of* $E(G)$. *Then* H *is monochromatic in this coloring.*

Lemma 2.3 ([3]). *Let* $G = C_5 + K_2$ *and* $E(G) = E_1 \cup E_2$ *be a* $(3,3)$*-free blue-red coloring of* $E(G)$ *such that* $E(C_5) \subseteq E_i$. *Then* $E(K_2) \in E_i$.

Lemma 2.4. *Let* $G = K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$, *where* $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$ *are copies of the 5-cycle* C_5 *and* $V(K_1) = \{a\}$. *Let* $E(G) = E_1 \cup E_2$ *be a blue-red coloring of* $E(G)$ *such that* $\operatorname{cl}(G_1(a)) \leq 3$ *and* $G_2(a) \not\to (3,3)$. *Then, up to numeration of the 5-cycles* $C_5^{(1)}$, $C_5^{(2)}$ *and* $C_5^{(3)}$, *we have:*

(a) $N_1(a) \supset V(C_5^{(1)})$ *and* $N_1(a) \cap V(C_5^{(2)})$ *is an independent set;*

(b) $N_2(a) \supset V(C_5^{(3)})$ *and* $N_2(a) \cap V(C_5^{(2)})$ *is not an independent set.*

Proof. Let $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$ and $C_5^{(3)} = w_1w_2w_3w_4w_5$. We shall use the following obvious fact

$$\chi(C_5) = 3. \tag{2.1}$$

It follows from (2.1) that

$$N_1(a) \cap V(C_5^{(i)}) \text{ or } N_2(a) \cap V(C_5^{(i)}) \text{ is not an independent set, } i = 1, 2, 3. \tag{2.2}$$

By (2.2) and Corollary 2.1(b), at least one of the sets $N_2(a) \cap V(C_5^{(i)})$, $i = 1, 2, 3$, is an independent set. Thus, at least one of the sets $N_1(a) \cap V(C_5^{(i)})$, $i = 1, 2, 3$, is not an independent set. Without loss of generality we can assume that

$$N_1(a) \cap V(C_5^{(1)}) \text{ is not an independent set.} \tag{2.3}$$

It follows from Corollary 2.1(a) and (2.3) that $N_1(a) \cap V(C_5^{(2)}) = \emptyset$ or $N_1(a) \cap V(C_5^{(3)}) = \emptyset$. Let for example $N_1(a) \cap V(C_5^{(3)}) = \emptyset$. Then

$$N_2(a) \supset V(C_5^{(3)}). \tag{2.4}$$

We have from (2.3) and Corollary 2.1(a) that $N_1(a) \cap V(C_5^{(2)})$ is an independent set. Thus, it follows from (2.1) that $N_2(a) \cap V(C_5^{(2)})$ is not an independent set. This fact, together with (2.4) and Corollary 2.1(c), gives $N_2(a) \cap V(C_5^{(1)}) = \emptyset$. Hence, $N_1(a) \supseteq V(C_5^{(1)})$. The Lemma is proved. □

Lemma 2.5. *Let $G = K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$, where $C_5^{(i)}$, $i = 1, 2, 3$, are copies of the 5-cycle C_5. Let $E(G) = E_1 \cup E_2$ be a blue-red coloring such that some of the cycles $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$ is not monochromatic. Then this coloring is not $(3,4)$-free.*

Proof. Let $V(K_1) = \{a\}$, $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$ and $C_5^{(3)} = w_1w_2w_3w_4w_5$. Assume the opposite, i.e. $E(G) = E_1 \cup E_2$ is $(3,4)$-free. Then by Corollary 2.1(a) we have $\mathrm{cl}(G_1(a)) \leq 3$ and by Lemma 2.1(b) we have $G_2(a) \not\to (3,3)$. Thus, according to Lemma 2.4 we can assume that

$$N_1(a) \supseteq V(C_5^{(1)}) \text{ and } N_1(a) \cap V(C_5^{(2)}) \text{ is independent;} \tag{2.5}$$

$$N_2(a) \supseteq V(C_5^{(3)}) \text{ and } N_2(a) \cap V(C_5^{(2)}) \text{ is not independent.} \tag{2.6}$$

It follows from (2.5) and Lemma 2.1(a) that

$$E(C_5^{(1)}) \subseteq E_2. \tag{2.7}$$

We have from the statement of the Lemma 2.5 that at least one of the cycles $C_5^{(i)}$, $i = 1, 2, 3$, is not monochromatic and since $E(C_5^{(1)}) \subseteq E_2$ it remains to consider the following two cases:

Case 1. $C_5^{(2)}$ is not monochromatic. Let for example $[u_1, u_5] \in E_1$ and $[u_1, u_2] \in E_2$. If $u_1, u_2, u_5 \in N_2(a)$ by (2.6) we have $G_2(a) \supset C_2^{(3)} + G[u_1, u_2, u_5]$. It follows from Lemma 2.2 that $G_2(a)$ contains a monochromatic 3-clique. This contradicts Lemma 2.1(b). So, at least one of the vertices u_1, u_2, u_5 belongs to $N_1(a)$. Therefore, we have the following subcases:

Subcase 1a. $u_1 \in N_1(a)$. Since there are no blue 3-cliques it follows from (2.5) that

$$N_2(u_1) \supset V(C_5^{(1)}). \tag{2.8}$$

As $[u_1, a]$, $[u_1, u_5] \in E_1$ and $\mathrm{cl}(G_1(u_1)) \leq 3$ (see Corollary 2.1(a)), the set $N_1(u_1) \cap V(C_5^{(3)})$ is independent. Therefore, $N_2(u_1) \cap V(C_5^{(3)})$ is not independent. This fact, together with $[u_1, u_2] \in E_2$ and (2.8), gives $G_2(u_1) \supset K_3 + C_5^{(1)}$, which contradicts Corollary 2.1(c).

Subcase 1b. $u_2 \in N_1(a)$ and $u_1 \in N_2(a)$. Since there are no blue 3-cliques it follows from (2.5) that

$$N_2(u_2) \supset V(C_5^{(1)}). \tag{2.9}$$

If $N_2(u_1) \cap V(C_5^{(1)})$ contains two adjacent vertices then these vertices together with u_1 and u_2 form a red 4-clique according to (2.7) and (2.9). Hence $N_2(u_1) \cap V(C_5^{(1)})$ is independent and, therefore, $N_1(u_1) \cap V(C_5^{(1)})$ is not independent. Since $u_5 \in N_1(u_1)$ and $\mathrm{cl}(G_1(u_1)) \leq 3$ (see Corollary 2.1(a)) we have $N_1(u_1) \cap V(C_5^{(3)}) = \emptyset$. Hence

$$N_2(u_1) \supset V(C_5^{(3)}). \tag{2.10}$$

By (2.6) and (2.10)

$$V(C_5^{(3)}) \subseteq N_2(u_1) \cap N_2(a).$$

Since $[a, u_1] \in E_2$ and there are no red 4-cliques we obtain that

$$E(C_5^{(3)}) \subseteq E_1. \tag{2.11}$$

As there are no blue 3-cliques from (2.11) it follows that $N_1(u_2) \cap V(C_5^{(3)})$ is independent. Therefore, $N_2(u_1) \cap V(C_5^{(3)})$ contains two adjacent vertices. This fact, together with $[u_1, u_2] \in E_2$ and (2.9), gives $G_2(u_2) \supset K_3 + C_5^{(1)}$, which contradicts Corollary 2.1(c).

Subcase 1c. $u_5 \in N_1(a)$ and $u_1, u_2 \in N_2(a)$. Since $a, u_2 \in N_2(u_1)$, it follows from Corollary 2.1(b) that at least one of the sets $N_2(u_1) \cap V(C_5^{(1)})$ and $N_2(u_1) \cap V(C_5^{(3)})$ is independent. Hence at least one of the sets $N_1(u_1) \cap V(C_5^{(1)})$, $N_1(u_1) \cap V(C_5^{(3)})$, is not independent. Assume that $N_1(u_1) \cap V(C_5^{(1)})$ is not independent. This fact, together with $u_5 \in N_1(u_1)$ and Corollary 2.1(a), implies

$$N_2(u_1) \supset V(C_5^{(3)}). \tag{2.12}$$

As $[a, u_1, u_2]$ is a red 3-clique and $[a, u_1, u_2, w_i]$ is not a red 4-clique, $i = 1, \dots, 5$, it follows from (2.6) and (2.12) that $[u_2, w_i] \in E_1$, $i = 1, \dots, 5$, i.e. $N_1(u_2) \supset$

$V(C_5^{(3)})$. We have from Lemma 2.1(a) that $E(C_5^{(3)}) \subseteq E_2$. Thus, according to (2.6) and (2.12), the vertices a and u_1 together with two adjacent vertices of $C_5^{(3)}$ form a red 4-clique, which is a contradiction.

Let us now consider the situation when $N_1(u_1) \cap V(C_5^{(3)})$ is not independent. Corollary 2.1(a) and $u_5 \in N_1(u_1)$ imply

$$N_2(u_1) \supset V(C_5^{(1)}). \tag{2.13}$$

If $N_2(u_1) \cap V(C_5^{(3)}) \neq \emptyset$ then from $a, u_2 \in N_2(u_1)$ and (2.13) it follows that $G_2(u_1) \supset K_3 + C_5^{(1)}$, which contradicts the Corollary 2.1(c). Hence $N_2(u_1) \cap V(C_5^{(3)}) = \emptyset$, i.e.

$$N_1(u_1) \supset C_5^{(3)}. \tag{2.14}$$

Since there are no blue 3-cliques, we obtain from (2.14) and Lemma 2.1(a):

$$E(C_5^{(3)}) \subseteq E_2. \tag{2.15}$$

If $N_2(u_2) \cap V(C_5^{(3)})$ is not independent then according to (2.6) and (2.15) an edge in $N_2(u_2) \cap V(C_5^{(3)})$ together with a and u_2 form a red 4-clique. Let $N_2(u_2) \cap V(C_5^{(3)})$ be independent. Then $N_1(u_2) \cap V(C_5^{(3)})$ is not independent. Thus, it follows from Corollary 2.1(a) that $N_1(u_2) \cap V(C_5^{(1)})$ is independent and $N_2(u_2) \cap V(C_5^{(1)})$ is not independent. Then an edge in $N_2(u_2) \cap V(C_5^{(1)})$, together with the vertices u_1 and u_2, form a red 4-clique, according to (2.7) and (2.13), which is a contradiction.

Case 2. $C_5^{(3)}$ is not monochromatic but $C_5^{(2)}$ is monochromatic. Without loss of generality we can assume that $[w_1, w_5] \in E_1$ and $[w_1, w_2] \in E_2$. Since $a, w_2 \in N_2(w_1)$ it follows from Corollary 2.1(b) that at least one of the sets $N_2(w_1) \cap V(C_5^{(1)})$ and $N_2(w_1) \cap V(C_5^{(2)})$ is independent. Hence at least one of the sets $N_1(w_1) \cap V(C_5^{(1)})$, $N_1(w_1) \cap V(C_5^{(2)})$ is not independent. We shall consider these possibilities:

Subcase 2a. $N_1(w_1) \cap V(C_5^{(1)})$ is not independent. Since $[w_1, w_5] \in E_1$ it follows from Corollary 2.1(a) that $N_1(w_1) \cap V(C_5^{(2)}) = \emptyset$, i.e.

$$N_2(w_1) \supset V(C_5^{(2)}). \tag{2.16}$$

By Lemma 2.1(b) $G_2(w_1)$ does not contain a monochromatic 3-clique and $G_2(w_1) \supset C_5^{(2)} + [a, w_2]$. Since $C_5^{(2)}$ is monochromatic and $[a, w_2] \in E_2$, it follows from Lemma 2.3 that

$$E(C_5^{(2)}) \subseteq E_2. \tag{2.17}$$

We see from (2.6), (2.16) and (2.17) that the vertices a and w_1, together with an edge of $C_5^{(2)}$ form a red 4-clique, which is a contradiction.

Subcase 2b. $N_1(w_1) \cap V(C_5^{(2)})$ is not independent. Since $w_5 \in N_1(w_1)$ it follows from Corollary 2.1(a) that

$$N_2(w_1) \supset V(C_5^{(1)}). \tag{2.18}$$

Corollary 2.1(c) and $G_2(w_1) \supset C_5^{(1)} + [a, w_2] = K_2 + C_5^{(1)}$ imply

$$N_1(w_1) \supset V(C_5^{(2)}). \tag{2.19}$$

Lemma 2.1(a) and (2.19) give

$$E(C_5^{(2)}) \subseteq E_2. \tag{2.20}$$

Since there are no blue 3-cliques and $[w_1, w_5] \in E_1$ it follows from (2.19) that

$$N_2(w_5) \supset V(C_5^{(2)}). \tag{2.21}$$

We see from (2.6), (2.20) and (2.21) that the vertices a and w_5 together with an edge of $C_5^{(2)}$ form a red 4-clique which is a contradiction. □

3. A PROPERTY OF THE GRAPH $C_5 + C_5 + C_5$

Let $G = C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$ where $C_5^{(i)}$, $i = 1, 2, 3$, are copies of the 5-cycle C_5. Let us consider the blue-red coloring where $E_1 = E(C_5^{(1)}) \cup E(C_5^{(2)}) \cup E(C_5^{(3)})$. It is clear that this coloring is (3,4)-free. Thus $G \not\to (3,4)$. However the following theorem holds:

Theorem 3.1. *Let $G = C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$ where $C_5^{(i)}$, $i = 1, 2, 3$, are copies of the 5-cycle C_5. Let $E(G) = E_1 \cup E_2$ be a blue-red coloring such that $E(C_5^{(1)}) \subseteq E_2$, $E(C_5^{(2)}) \subseteq E_1$ and $E(C_5^{(3)}) \subseteq E_1$. Then this coloring is not (3,4)-free.*

Proof. Assume the opposite, i.e. that there are no blue 3-cliques and no red 4-cliques. Let $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$, $C_5^{(3)} = w_1w_2w_3w_4w_5$. Since the cycles $C_5^{(2)}$ and $C_5^{(3)}$ are blue and there are no blue 3-cliques, the sets $N_1(v_i) \cap V(C_5^{(2)})$ and $N_1(v_i) \cap V(C_5^{(3)})$ are independent. Thus, we have

$$|N_2(v_i) \cap V(C_5^{(2)})| \geq 3, \quad |N_2(v_i) \cap V(C_5^{(3)})| \geq 3, \quad i = 1, \dots, 5. \tag{3.1}$$

It follows from (3.1) that

$$N_2(x) \cap N_2(y) \cap V(C_5^{(i)}) \neq \emptyset, \quad i = 2, 3, \quad x, y \in V(C_5^{(1)}). \tag{3.2}$$

Let $x, y \in V(C_5^{(1)})$. We define

$$B_1(x, y) = \{v \in V(C_5^{(2)}) \mid [x, v], [y, v] \in E_2\},$$
$$B_2(x, y) = \{v \in V(C_5^{(3)}) \mid [x, v], [y, v] \in E_2\}.$$

We see from (3.2) that

$$B_i(x, y) \neq \emptyset, \quad i = 1, 2, \quad x, y \in V(C_5^{(1)}). \tag{3.3}$$

We shall prove that

$$\text{if } [x,y] \in E(C_5^{(1)}) \text{ then } B_i(x,y) \text{ is independent, } i = 1,2. \tag{3.4}$$

Assume the opposite and let for example $u', u'' \in B_1(x,y)$ and $[u', u''] \in E(C_5^{(2)})$. By (3.3) there exists $w \in B_2(x,y)$. Since there are no blue 3-cliques, then at least one of the edges $[u', w]$, $[u'', w]$ is red. Hence $[x, y, u', w]$ or $[x, y, u'', w]$ is a red 4-clique, which is a contradiction.

Let u' and u'' be adjacent vertices in $C_5^{(2)}$. Since $[u', u''] \in E_1$ and there are no blue 3-cliques, we have

$$N_1(u') \cap N_1(u'') \cap V(C_5^{(1)}) = \emptyset.$$

Thus $|N_1(u') \cap V(C_5^{(1)})| \leq 2$ or $|N_1(u'') \cap V(C_5^{(1)})| \leq 2$. Hence

$$|N_2(u') \cap V(C_5^{(1)})| \geq 3 \text{ or } |N_2(u'') \cap V(C_5^{(1)})| \geq 3. \tag{3.5}$$

So, (3.5) holds for every two adjacent vertices in $C_5^{(2)}$. Hence $|N_2(u) \cap V(C_5^{(1)})| \geq 3$ holds for at least three vertices in $C_5^{(2)}$. Thus, there exist two adjacent vertices in $C_5^{(2)}$, for example u_1 and u_2, such that

$$|N_2(u_1) \cap V(C_5^{(1)})| \geq 3 \text{ and } |N_2(u_2) \cap V(C_5^{(1)})| \geq 3. \tag{3.6}$$

If the both inequalities in (3.6) are strict then $N_2(u_1) \cap N_2(u_2) \cap V(C_5^{(1)})$ contains two adjacent vertices v' and v''. Since $u_1, u_2 \in B(v', v'')$ then this contradicts (3.4). Thus, we may assume that $|N_2(u_1) \cap V(C_5^{(1)})| = 3$. Hence $N_2(u_1) \cap V(C_5^{(1)})$ contains two adjacent vertices, for example v_3 and v_4. Now we shall prove that the third vertex in $N_2(u_1) \cap V(C_5^{(1)})$ is the vertex v_1. Assume the opposite. Then $v_2 \in N_2(u_1) \cap V(C_5^{(1)})$ or $v_5 \in N_2(u_1) \cap V(C_5^{(1)})$. Let $v_2 \in N_2(u_1) \cap V(C_5^{(1)})$. Then $v_1, v_5 \in N_1(u_1)$. Since $v_1, v_5, u_2 \in N_1(u_1)$ it follows from Corollary 2.1(a) that $N_1(u_1) \cap V(C_5^{(3)}) = \emptyset$. Thus, $G_2(u_1)$ contains $C_5^{(3)} + [v_3, v_4] = K_2 + C_5$. According to Lemma 2.1(b) $G_2(u_1)$ does not contain monochromatic 3-cliques. As $E(C_5^{(3)}) \subseteq E_1$ and $[v_3, v_4] \in E_2$, this contradicts Lemma 2.3. We proved that $v_2 \notin N_2(u_1)$. Analogously we prove that $v_5 \notin N_2(u_1)$. So,

$$v_1, v_3, v_4 \in N_2(u_1) \text{ and } v_2, v_5 \in N_1(u_1). \tag{3.7}$$

By (3.3) we can assume that $w_1 \in B_2(v_3, v_4)$. Since $[v_3, v_4, u_1, w_1]$ is not a red 4-clique, we have

$$[u_1, w_1] \in E_1. \tag{3.8}$$

As there are no blue 3-cliques and $[u_1, v_2]$, $[u_1, v_5] \in E_1$, it follows that $[w_1, v_2]$, $[w_1, v_5] \in E_2$. Taking into consideration $w_1 \in B_2(v_3, v_4)$, we have

$$[w_1, v_i] \in E_2, \quad i = 2, 3, 4, 5. \tag{3.9}$$

By (3.3) there is $u \in B_1(v_2, v_3)$. Since $[v_2, u_1] \in E_1$ then $u \neq u_1$. We shall prove that $u = u_3$ or $u = u_4$. Assume the opposite. Then $u = u_2$ or $u = u_5$. Let, for example, $u = u_2$. Since $[v_2, v_3, u_2, w_1]$ is not a red 4-clique, it follows from (3.9) and $u_2 \in B_1(v_2, v_3)$ that $[u_2, w_1] \in E_1$. We obtained the blue 3-clique $[u_1, u_2, w_1]$, which is a contradiction. This contradiction proves that $u = u_3$ or $u = u_4$. We can assume without loss of generality that $u = u_3$. We have

$$[u_3, w_1] \in E_1, \tag{3.10}$$

because $[v_2, v_3, u_3, w_1]$ is not a red 4-clique. By (3.3) there exists $u \in B_1(v_4, v_5)$. Repeating the above considerations about $u \in B_1(v_2, v_3)$ we see that $u = u_3$ or $u = u_4$.

Case 1. $u = u_4$. Since $[v_4, v_5, w_1, u_4]$ is not a red 4-clique, we have $[u_4, w_1] \in E_1$. Hence $[u_3, u_4, w_1]$ is a blue 3-clique, which is a contradiction.

Case 2. $u = u_3$. In this case we have $u_3 \in B_1(v_2, v_3) \cap B_1(v_4, v_5)$, i.e.

$$[u_3, v_i] \in E_2, \quad i = 2, 3, 4, 5. \tag{3.11}$$

As $[v_1, w_1, u_3]$ is not a blue 3-clique, it follows from (3.10) that $[v_1, u_3] \in E_2$ or $[v_1, w_1] \in E_2$.

Subcase 2a. $[v_1, u_3] \in E_2$. By (3.11) $N_2(u_3) \supset C_5^{(1)}$. Since there are no blue 3-cliques $N_2(u_3)$ contains two adjacent vertices $w', w'' \in V(C_5^{(3)})$. Thus $G_2(u_3) \supset C_5^{(1)} + [w', w'']$. By Lemma 2.1(b) $G_2(u_3)$ contains no monochromatic 3-cliques. This contradicts Lemma 2.3 because $E(C_5^{(1)}) \subseteq E_2$ and $[w', w''] \in E_1$.

Subcase 2b. $[v_1, w_1] \in E_2$. By (3.9) we see that $N_2(w_1) \supset V(C_5^{(1)})$. Since there are no blue 3-cliques $N_2(w_1)$ contains two adjacent vertices $u', u'' \in V(C_5^{(2)})$. Hence $N_2(w_1) \supset C_5^{(1)} + [u', u'']$ which contradicts Lemma 2.3.

The theorem is proved. □

4. PROOF OF THE MAIN THEOREM

Let $C_5^{(1)} = v_1 v_2 v_3 v_4 v_5$, $C_5^{(2)} = u_1 u_2 u_3 u_4 u_5$, $C_5^{(3)} = w_1 w_2 w_3 w_4 w_5$ and $V(K_1) = \{a\}$. Assume the opposite, i.e. there exists a $(3,4)$-free blue-red coloring $E_1 \cup E_2$ of the edges of $K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$. By Lemma 2.4 we can assume that:

$$N_1(a) \supset V(C_5^{(1)}) \text{ and } N_1(a) \cap V(C_5^{(2)}) \text{ is independent;} \tag{4.1}$$

$$N_2(a) \supset V(C_5^{(3)}) \text{ and } N_2(a) \cap V(C_5^{(2)}) \text{ is not independent.} \tag{4.2}$$

We shall prove that

$$E(C_5^{(i)}) \subseteq E_2, \quad i = 1, 2, 3. \tag{4.3}$$

By (4.1) and Lemma 2.1(a), $E(C_5^{(1)}) \subseteq E_2$. According to Lemma 2.5 each of the 5-cycles $C_5^{(2)}$ and $C_5^{(3)}$ is monochromatic. By (4.2) $G_2(a) \supset C_5^{(3)} + e$ where $e \in E(C_5^{(2)})$. By Lemma 2.1(b) $G_2(a)$ contains no monochromatic 3-cliques. Thus, it follows from Lemma 2.3 that the edge e and the 5-cycle $C_5^{(3)}$ have the same color. Therefore, the 5-cycles $C_5^{(2)}$ and $C_5^{(3)}$ are monochromatic of the same color. Thus, it follows from Theorem 3.1 that $E(C_5^{(2)}) \nsubseteq E_1$ and $E(C_5^{(3)}) \nsubseteq E_1$. We proved (4.3).

Now we shall prove that

$$N_2(a) = V(C_5^{(2)}) \cup V(C_5^{(3)}). \tag{4.4}$$

Assume the opposite. Then it follows from (4.2) that $N_1(a) \cap V(C_5^{(2)}) \neq \emptyset$. Let for example $u_1 \in N_1(a) \cap V(C_5^{(2)})$, i.e. $[u_1, a] \in E_1$. We see from (4.1) that

$$[a, u_2] \in E_2. \tag{4.5}$$

As there are no blue 3-cliques by (4.1) and $[u_1, a] \in E_1$ we obtain

$$N_2(u_1) \supset V(C_5^{(1)}). \tag{4.6}$$

We see from Corollary 2.1(a) that at least one of the sets $N_2(u_2) \cap V(C_5^{(3)})$, $N_2(u_2) \cap V(C_5^{(1)})$ is not independent. If $N_2(u_2) \cap V(C_5^{(1)})$ is not independent, it follows from (4.6) and (4.3) that the vertices u_1 and u_2 together with an edge of $C_5^{(1)}$ form a red 4-clique. If $N_2(u_2) \cap V(C_5^{(3)})$ is not independent, by (4.3), (4.5) and (4.2), the vertices a and u_2 together with an edge of $C_5^{(3)}$ form a red 4-clique. This contradiction proves (4.4).

It follows from (4.4) and Lemma 2.1(b) that

$$C_5^{(2)} + C_5^{(3)} \text{ contains no monochromatic 3-cliques.} \tag{4.7}$$

Now we obtain from (4.7) and (4.3)

$$N_2(x) \cap V(C_5^{(3)}) \text{ is independent}, \quad x \in V(C_5^{(2)}); \tag{4.8}$$

$$N_2(x) \cap V(C_5^{(2)}) \text{ is independent}, \quad x \in V(C_5^{(3)}). \tag{4.9}$$

Let us note that

$$N_1(x) \cap V(C_5^{(1)}) \text{ is independent}, \quad x \in V(C_5^{(2)}) \cup V(C_5^{(3)}). \tag{4.10}$$

Indeed, let for example $x \in V(C_5^{(2)})$. By (4.8), $N_1(x) \cap V(C_5^{(3)})$ is not independent. This fact and Corollary 2.1(a) prove (4.10).

We shall prove that

$$N_1(x) \cap V(C_5^{(2)}),\ x \in V(C_5^{(1)}) \text{ is not independent} \iff N_2(x) \supset V(C_5^{(3)}); \tag{4.11}$$

$$N_1(x) \cap V(C_5^{(3)}),\ x \in V(C_5^{(1)}) \text{ is not independent} \iff N_2(x) \supset V(C_5^{(2)}). \tag{4.12}$$

The statements (4.11) and (4.12) are proved analogously. That is why we shall prove (4.11) only. Let $N_1(x) \cap V(C_5^{(2)})$, $x \in V(C_5^{(1)})$ be not independent. Since $[x, a] \in E_1$, it follows from Corollary 2.1(a) that $N_1(x) \cap V(C_5^{(3)}) = \emptyset$, i.e. $N_2(x) \supset V(C_5^{(3)})$. Now let $N_2(x) \supset V(C_5^{(3)})$, $x \in V(C_5^{(1)})$. Assume that $N_1(x) \cap V(C_5^{(2)})$ is independent. Then $N_2(x) \cap V(C_5^{(2)})$ is not independent. Since $C_5^{(1)}$ is red, $G_2(x) \supset K_3 + C_5^{(3)}$ which contradicts Corollary 2.1(c). So, (4.11) and (4.12) are proved. Using (4.11) and (4.12) we shall prove that

$$N_1(x) \cap V(C_5^{(i)}),\ i = 2, 3, \text{ is independent, } x \in V(C_5^{(1)}). \tag{4.13}$$

Assume that (4.13) is wrong and let, for example, $N_1(v_1) \cap V(C_5^{(2)})$ be not independent (remind that $C_5^{(1)} = v_1 v_2 v_3 v_4 v_5$). Then by (4.11) $N_2(v_1) \supset V(C_5^{(3)})$. If $N_2(v_2) \cap V(C_5^{(3)})$ is not independent then v_1 and v_2 together with two adjacent vertices from $N_2(v_2) \cap V(C_5^{(3)})$ form a red 4-clique, which is a contradiction. Therefore, $N_1(v_2) \cap V(C_5^{(3)})$ is not independent. Thus (4.12) gives $N_2(v_2) \supset V(C_5^{(2)})$. Repeating the above considerations about the vertex v_1 on v_2 we obtain $N_2(v_3) \supset V(C_5^{(3)})$. In the same way it follows from $N_2(v_3) \supset V(C_5^{(3)})$ that $N_2(v_4) \supset V(C_5^{(2)})$. At the end it follows from $N_2(v_4) \supset V(C_5^{(2)})$ that $N_2(v_5) \supset V(C_5^{(3)})$. So, we proved that

$$N_2(v_1) \cap N_2(v_5) \supset V(C_5^{(3)}).$$

Thus, it follows from (4.3) that v_1 and v_5, together with an edge of $C_5^{(3)}$, form a red 4-clique, which is a contradiction. This contradiction proves (4.13). According to (4.13) it follows from (4.11) and (4.12) that

$$N_2(x) \not\supset V(C_5^{(i)}),\ i = 2, 3, \quad x \in V(C_5^{(1)}). \tag{4.14}$$

Let $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. By (4.10) $|N_1(x) \cap V(C_5^{(1)})| \leq 2$. Thus, we have the following possibilities:

Case 1. $N_1(x) \cap V(C_5^{(1)}) = \emptyset$ for some vertex $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. Let, for example, $N_1(u_1) \cap V(C_5^{(1)}) = \emptyset$ (remind that $C_5^{(2)} = u_1 u_2 u_3 u_4 u_5$). Then $N_2(u_1) \supset V(C_5^{(1)})$. We see from (4.10) that $N_2(u_2) \cap V(C_5^{(1)})$ is not independent. Thus u_1 and u_2, together with two adjacent vertices from $N_2(u_2) \cap V(C_5^{(1)})$, form a red 4-clique, which is a contradiction.

Case 2. $|N_1(x) \cap V(C_5^{(1)})| = 1$ for some vertex $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. Let, for example, $|N_1(u_1) \cap V(C_5^{(1)})| = 1$. Without loss of generality we can consider $[u_1, v_1] \in E_1$ and $[u_1, v_i] \in E_2$, $i = 2, 3, 4, 5$. According to (4.14) we can assume that $[v_1, w_1] \in E_1$. Since there are no blue 3-cliques, $[u_1, w_1] \in E_2$. It follows from (4.10) that $N_2(w_1) \cap V(C_5^{(1)})$ contains two adjacent vertices. As

$$N_2(w_1) \cap V(C_5^{(1)}) \subseteq N_2(u_1) \cap V(C_5^{(1)}) = \{v_2, v_3, v_4, v_5\}$$

we see that u_1 and w_1 together with two adjacent vertices in $\{v_2, v_3, v_4, v_5\}$ form a red 4-clique, which is a contradiction.

Case 3. $|N_1(x) \cap V(C_5^{(1)})| = 2$ for every $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. According to (4.8) $N_1(u_1) \cap V(C_5^{(3)})$ is not independent. Thus, we can assume that $w_1, w_2 \in N_1(u_1) \cap V(C_5^{(3)})$, i.e.

$$[u_1, w_1], [u_1, w_2] \in E_1. \tag{4.15}$$

It follows from (4.13)

$$N_1(w_1) \cap N_1(w_2) \cap V(C_5^{(1)}) = \emptyset. \tag{4.16}$$

In the case considered we have

$$|N_1(w_1) \cap V(C_5^{(1)})| = |N_1(w_2) \cap V(C_5^{(1)})| = |N_1(u_1) \cap V(C_5^{(1)})| = 2.$$

We obtain from (4.16)

$$N_1(u_1) \cap N_1(w_1) \cap V(C_5^{(1)}) \neq \emptyset \text{ or } N_1(u_1) \cap N_1(w_2) \cap V(C_5^{(1)}) \neq \emptyset.$$

By (4.15) there is a blue 3-clique, which is a contradiction.

The Main Theorem is proved.

5. EXAMPLE OF FOLKMAN EDGE (3,5)-GRAPH WITHOUT 13-CLIQUES

Using the Main Theorem we shall prove the following

Theorem 5.1. *Let $G = K_4 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$ where $C_5^{(i)}$, $i = 1, \ldots, 4$, are copies of the 5-cycle C_5. Then $G \to (3, 5)$.*

In order to prove Theorem 5.1 we shall need the following

Lemma 5.1. *Let $E(G) = E_1 \cup E_2$ is a $(3,5)$-free blue-red coloring of $E(G)$. Then:*

(a) *$G_1(v)$, $v \in V(G)$, is a red subgraph;*

(b) *$(E(G_2(v)) \cap E_1) \cup (E(G_2(v)) \cap E_2)$ is a $(3,4)$-free blue-red coloring of $E(G_2(v))$, $v \in V(G)$. Thus, $G_2(v) \not\to (3,4)$.*

Lemma 5.1 is proved in the same way as Lemma 2.1.

Corollary 5.1. *Let $E(G) = E_1 \cup E_2$ be a $(3,5)$-free blue-red coloring of $E(G)$. Then:*

(a) $\mathrm{cl}(G_1(v)) \leq 4$, $v \in V(G)$;

(b) $\mathrm{cl}(G_2(v)) \leq 8$, $v \in V(G)$;

(c) $G_2(v) \not\supset K_4 + C_5 + C_5$;

(d) $G_2(v) \not\supset K_1 + C_5 + C_5 + C_5$.

Proof. The statement (a) follows from Lemma 5.1(a). The statement (b) follows from Lemma 5.1(b) and $K_9 \to (3,4)$, [4]. The statement (c) follows from Lemma 5.1(b) and $K_4 + C_5 + C_5 \to (3,4)$, [8]. The statement (d) follows from Lemma 5.1(b) and the Main Theorem. □

Proof of Theorem 5.1. Assume the opposite, i.e. there exists a blue-red coloring $E(G) = E_1 \cup E_2$, which is $(3,5)$-free. Let $V(K_4) = \{a_1, a_2, a_3, a_4\}$.

Case 1. There exists $a_i \in V(K_4)$ such that $|N_1(a_i) \cap V(K_4)| = 3$. Let, for example, $[a_1, a_2]$, $[a_1, a_3]$, $[a_1, a_4] \in E_1$. By Corollary 5.1(a) at most one of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, is not empty, i.e. $N_2(a_1)$ contains at least three of the cycles $C_5^{(i)}$, $i = 1,2,3,4$. Let, for example,

$$N_2(a_1) \supset V(C_5^{(2)}) \cup V(C_5^{(3)}) \cup V(C_5^{(4)}).$$

By Corollary 5.1(a) it follows that $N_1(a_1) \cap V(C_5^{(1)})$ is independent. Thus, $N_2(a_1) \cap V(C_5^{(1)}) \neq \emptyset$. We obtained that $G_2(a_1) \supset K_1 + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(d).

Case 2. There exists $a_i \in V(K_4)$ such that $|N_1(a_i) \cap V(K_4)| = 2$. Let, for example, $[a_1, a_2]$, $[a_1, a_3] \in E_1$ and $[a_1, a_4] \in E_2$. Since $[a_1, a_4] \in E_2$ if the sets $N_2(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, are not independent then $G_2(a_1) \supset K_9$, which contradicts Corollary 5.1(b). Hence, at least one of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, is not independent. Let, for example, $N_1(a_1) \cap V(C_5^{(1)})$ be not independent. According to Corollary 5.1(a) it follows from this fact and $[a_1, a_2]$, $[a_1, a_3] \in E_1$ that $N_1(a_1) \cap V(C_5^{(i)}) \emptyset$, $i = 2,3,4$, i.e. $N_2(a_1) \supset V(C_5^{(i)})$, $i = 2,3,4$. As $[a_1, a_4] \in E_2$ we have $G_2(a_1) \supset K_1 + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(d).

Case 3. There exist $a_i \in V(K_4)$ such that $|N_1(a_i) \cap V(K_4)| = 1$. Let, for example, $[a_1, a_2] \in E_1$ and $[a_1, a_3]$, $[a_1, a_4] \in E_2$. We see from Corollary 5.1(a) that at least three of the sets $N_2(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, are not independent. Let, for example, $N_2(a_1) \cap V(C_5^{(2)})$, $N_2(a_1) \cap V(C_5^{(3)})$ and $N_2(a_1) \cap V(C_5^{(4)})$ be not independent. Since $[a_1, a_3]$, $[a_1, a_4] \in E_2$ it follows from Corollary 5.1(b) that $N_1(a_1) \supset V(C_5^{(1)})$. According to Lemma 5.1(a) it follows from this fact and $[a_1, a_2] \in E_1$ that at least two of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 2,3,4$, are empty. Therefore, we can assume that $N_2(a_1) \supset V(C_5^{(3)})$ and $N_2(a_1) \supset V(C_5^{(4)})$. Since $N_2(a_1) \cap V(C_5^{(2)})$ is not independent we have $G_2(a_1) \supset K_4 + V(C_5^{(3)}) + V(C_5^{(4)})$, which contradicts Corollary 5.1(c).

Case 4. $E(K_4) \subseteq E_2$. Since $[a_1, a_i] \in E_2$, $i = 2,3,4$, it follows from Corollary 5.1(b) that at least two of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, are not independent. Let, for example, $N_1(a_1) \cap V(C_5^{(1)})$ and $N_1(a_1) \cap V(C_5^{(2)})$ be not independent. Then by Corollary 5.1(a) $N_1(a_1) \cap V(C_5^{(3)}) = \emptyset$ and $N_1(a_1) \cap V(C_5^{(4)}) = \emptyset$, i.e.

$$N_2(a_1) \supseteq V(C_5^{(3)}) \cup V(C_5^{(4)}). \tag{5.1}$$

Since $[a_1, a_i] \in E_2$, $i = 2, 3, 4$, it follows from (5.1) and Corollary 5.1(c) that $N_2(a_1) \cap V(C_5^{(1)}) = \emptyset$ and $N_2(a_1) \cap V(C_5^{(2)}) = \emptyset$. That is why, we have from (5.1)

$$N_1(a_1) = V(C_5^{(1)}) \cup V(C_5^{(2)}). \tag{5.2}$$

As the vertices a_1, a_2, a_3, a_4 are equivalent, in this case the above considerations prove that

$$N_1(a_i),\ i = 1, 2, 3, 4, \text{ is a union of two of the cycles } C_5^{(1)}, C_5^{(2)}, C_5^{(3)}, C_5^{(4)}. \tag{5.3}$$

Lemma 5.1(a) and (5.2) imply

$$C_5^{(1)} + C_5^{(2)} \text{ is a red subgraph.} \tag{5.4}$$

Since there are no red 5-cliques, we see from (5.4) that

$$N_1(a_i) \cap V(C_5^{(1)}) \neq \emptyset \text{ or } N_1(a_i) \cap V(C_5^{(2)}) \neq \emptyset, \quad i = 2, 3, 4.$$

Thus, by (5.3) we have

$$N_1(a_i) \supset V(C_5^{(1)}) \text{ or } N_1(a_i) \supset V(C_5^{(2)}), \quad i = 2, 3, 4. \tag{5.5}$$

Hence, we can assume that

$$N_1(a_2) \supset V(C_5^{(1)}) \text{ and } N_1(a_3) \supset V(C_5^{(1)}). \tag{5.6}$$

Let $C_5^{(1)} = v_1 v_2 v_3 v_4 v_5$. By (5.3) we have the following possibilities:

Subcase 4a. $N_1(a_4) \supset V(C_5^{(1)})$. According to (5.6) and (5.2) $[v_1, a_i] \in E_1$, $i = 1, 2, 3, 4$. Hence, by Corollary 5.1(a) $N_1(v_1) \cap V(C_5^{(i)}) = \emptyset$, $i = 2, 3, 4$, i.e. $G_2(v_1) \supset C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$. By (5.2) $[v_1, v_2] \in E_2$. Thus, $G_2(v_1) \supset K_1 + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(d).

Subcase 4b. $N_1(a_4) \cap V(C_5^{(1)}) = \emptyset$, i.e. $N_2(a_4) \supset V(C_5^{(1)})$. We have, from (5.2) and (5.6), $[v_1, a_i] \in E_1$, $i = 1, 2, 3$, and $[v_1, a_4] \in E_2$. By Corollary 5.1(a), at least two of the sets $N_1(v_1) \cap V(C_5^{(i)})$, $i = 2, 3, 4$, are empty. Thus, we can assume that

$$G_2(v_1) \supset C_5^{(3)} + C_5^{(4)}. \tag{5.7}$$

It follows from Corollary 5.1(a) that $N_1(v_1) \cap V(C_5^{(2)})$ is independent. Hence, $N_2(v_1) \cap V(C_5^{(2)})$ is not independent. This fact, together with $[v_1, v_2]$, $[v_1, a_4] \in E_2$ and (5.7), gives $G_2(v_1) \supset K_4 + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(c). This contradiction finishes the proof of Theorem 5.1. □

Since $\mathrm{cl}(G) = 12$ and $|V(G)| = 24$, Theorem 5.1 implies

Corollary 5.2. $F(3, 5; 13) \leq 24$.

Lin proved in [7] that $F(3, 5; 13) \geq 18$. In [9] Nenov improved this result, proving that either $K_8 + C_5 + C_5 \to (3, 5)$ or $F(3, 5; 13) \geq 19$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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A NONLOCAL BOUNDARY VALUE PROBLEM FOR A CLASS OF NONLINEAR EQUATIONS OF MIXED TYPE [1]

MARIA KARATOPRAKLIEVA

Nonlocal boundary-value problems for second order linear and nonlinear differential equations of mixed type in a bounded multidimensional cylindrical domain are considered. Uniqueness and existence of a weak solution in the linear case are established. Applying these results and Schauder's fixed point theorem existence of a weak solution in the nonlinear case is proved. A uniqueness result is also established.

Keywords: Partial differential equation of mixed type, nonlocal boundary value problem, uniqueness and existence of a weak solution.

2000 MSC: 35M10, 35D05.

1. INTRODUCTION

Let D be a bounded domain in the space $\mathbb{R}^{m-1}$ of points $x' = (x_1, \dots, x_{m-1})$, where $m \geq 2$, with a boundary $\partial D \in C^2$, if $m \geq 3$. Let $G = \{x = (x', x_m) \in \mathbb{R}^m : x' \in D,\ 0 < x_m < h\}$, $S = \{x \in \mathbb{R}^m : x' \in \partial D, 0 < x_m < h\}, h = const.$

We consider the operator

$$\mathcal{L}u = \sum_{i,j=1}^{m-1} a_{ij}(x) u_{x_i x_j} + k(x) u_{x_m x_m} + \sum_{i=1}^{m} b_i(x) u_{x_i} + c(x) u \ ,$$

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where $k, a_{ij} \in C^2(\bar{G})$, $a_{ij} = a_{ji}$ for $i,j = 1,\ldots,m-1$; $\sum_{i,j=1}^{m-1} a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_{i=1}^{m-1} \xi_i^2$ $\forall x \in \bar{G}$ and $\forall \xi' \in \mathbb{R}^{m-1}$, $a_0 = const > 0$; $k(x',0) = k(x',h) \leq 0$ $\forall x' \in \bar{D}$; $b_i \in C^1(\bar{G})$ for $i = 1,\ldots,m$; $c \in C(\bar{G})$. We denote $D_0 = \{x' \in \bar{D} : k(x',0) = 0\}$ and $D_- = \{x' \in \bar{D} : k(x',0) < 0\}$. Assume that $D_- \neq \emptyset$, $(b_m - k_{x_m})(x',h) = (b_m - k_{x_m})(x',0)$ $\forall x' \in \bar{D}$ and $(b_m - k_{x_m})(x',0) \neq 0$ $\forall x' \in D_0$. All the functions in the present paper are real-valued.

The operator $\mathcal{L}$ is elliptic, hyperbolic, parabolic at a point $x \in G$, if $k(x) > 0$, $k(x) < 0$, $k(x) = 0$, respectively. In our case $\mathcal{L}$ is an operator of mixed type in G, because there are no restrictions on the sign of $k(x)$ for $x \in G$.

First we investigate the following nonlocal boundary value problem for the linear equation

$$\mathcal{L}u = f \text{ in } G. \tag{1.1}$$

To find a function $u(x)$ defined in $\bar{G}$ which is a solution of the equation (1.1) *and satisfies the boundary conditions*

$$u = 0 \text{ on } S,\ u(x',h) = \lambda u(x',0) \text{ in } \bar{D}, \tag{1.2}$$

$$u_{x_m}(x',h) = \lambda u_{x_m}(x',0) \text{ in } D_-, \tag{1.3}$$

where $f(x)$ is a given function and $\lambda \neq 0$ is a given real constant.

In the case where $k(x',0) = k(x',h) = 0$ $\forall x' \in \bar{D}$ the problem (1.1), (1.2) was investigated in [9], [11] for $0 < |\lambda| < 1$, in [4] for $0 < \lambda \leq 1$ and in [12] for $\lambda \neq 0$. The problem (1.1) - (1.3) was investigated in [10] in the case where $k(x) \leq 0$ in $\bar{G}$ and $0 < |\lambda| < 1$. In [17] the problem (1.1) - (1.3) was considered for $0 < |\lambda| < 1$, $\lambda = 1$, $k = k(x_m)$, $b_i = 0$, $a_{ij} = \delta_i^j$, where δ_i^j is the Kroneker's symbol, $i,j = 1,\ldots,m-1$, in the following cases: $k(h) \geq 0$ and $k(0) > 0$; $k(h) \geq 0 \geq k(0)$; $k(0) \leq 0$ and $k(h) < 0$. The problem (1.1) - (1.3) with $\lambda = 1$, $b_i = 0$, $a_{ij} = -\delta_i^j$, $i,j = 1,\ldots,m-1$, $c = c(x')$ was considered in [6]. Another nonlocal boundary value problem for the equation

$$h(y)u_{yy} - u_{xx} + a(x,y)u_y + b(x,y)u = f(x,y)$$

in $\{(x,y) : -l \leq y \leq l,\ 0 \leq x \leq 1\}$, where $h(l) \geq 0 \geq h(-l)$, was investigated in [7].

The formally adjoint operator to the operator $\mathcal{L}$ is

$$\mathcal{L}^*v = \sum_{i,j=1}^{m-1} a_{ij}(x)v_{x_ix_j} + k(x)v_{x_mx_m} + \sum_{i=1}^{m} b_i^*(x)v_{x_i} + c^*(x)v\ ,$$

where $b_m^* = 2k_{x_m} - b_m$, $b_i^* = 2\sum_{j=1}^{m-1} a_{ijx_j} - b_i$, $i = 1,\ldots,m-1$, and $c^* = c - \sum_{i=1}^{m} b_{ix_i} + \sum_{i,j=1}^{m-1} a_{ijx_ix_j} + k_{x_mx_m}$. The adjoint boundary conditions to (1.2), (1.3) are

$$v = 0 \text{ on } S,\ v(x',0) = \lambda v(x',h) \text{ in } \bar{D}, \tag{1.4}$$

$$v_{x_m}(x',0) = \lambda v_{x_m}(x',h) \text{ in } D_-, \tag{1.5}$$

We denote by $\tilde{C}^2$ and $\tilde{C}^2_*$ the sets of all functions belonging to $C^2(\bar{G})$ and satisfying the conditions (1.2), (1.3) and (1.4), (1.5), respectively. Let $\tilde{W}^1$ be the closure of $\tilde{C}^2$ with respect to the norm $\|u\|_1 = (\|u\|_0^2 + \sum_{i=1}^{m} \|u_{x_i}\|_0^2)^{1/2}$ of the Sobolev space $W_2^1(G)$. We use the notations $(.,.)_0$ and $\|\cdot\|_0$ for the usual scalar product and norm of $L_2(G)$. Let $\tilde{W}^1_*$ be the closure of the set $\tilde{C}^2_*$ with respect to the norm $\|.\|_1$. Let $f \in L_2(G)$.

Definition 1.1. A function $u(x)$ is called a weak solution of the problem (1.1) - (1.3), if $u \in \tilde{W}^1$ and

$$(u, \mathcal{L}^* v) = (f, v)_0 \ \forall v \in \tilde{C}^2_*. \tag{1.6}$$

Definition 1.2. A function $u(x)$ is called a classical solution of the problem (1.1) - (1.3), if $u \in \tilde{C}^2$ and $\mathcal{L}u(x) = f(x) \ \forall x \in G$.

Denote

$$B[u,\, v] \equiv \int_G [-(kv)_{x_m} u_{x_m} - \sum_{i,j=1}^{m-1} (a_{ij} v)_{x_j} u_{x_i} + (cu + \sum_{i=1}^{m} b_i u_{x_i}) v]\, dx$$

for $u, v \in W_2^1(G)$. Let $F(x,t)$ be a given function, defined in $G \times \mathbb{R}$. We assume that $F \in$ **CAR** , i.e. $F(x,t)$ is continuous with respect to t for almost every $x \in G$ and it is measurable with respect to $x \in G$ for every $t \in \mathbb{R}$.

Further we consider the following nonlocal boundary value problem for the nonlinear equation

$$\mathcal{L}u = F(x,u) \text{ in } G. \tag{1.7}$$

To find a function $u(x)$ defined in $\bar{G}$ which is a solution of (1.7) and satisfies the boundary conditions (1.2)and (1.3).

Definition 1.3. A function $u(x)$ is called a weak solution of the problem (1.7), (1.2), (1.3), if $u \in \tilde{W}^1$ and

$$B[u,\, v] = (F(x,u),\, v)_0 \ \forall v \in \tilde{W}^1_*. \tag{1.8}$$

Nonlocal boundary value problems for different nonlinear equations of second order of mixed type are considered in [4], [6], [13].

In the present paper we consider the case $|\lambda| < 1$. In the section 2 we prove some preliminary results and Theorem 2.1 for uniqueness of a weak solution of the problem (1.1) - (1.3). In the section 3 we establish an important a priory estimate, prove Theorem 3.1 for existence of a weak solution and Theorem 3.2 for uniqueness of a classical solution of the same problem. Applying these results and Schauder's fixed point theorem, existence of a weak solution of the problem (1.7), (1.2), (1.3) is proved in section 4. Using Lemma 2.5 we get uniqueness of that solution in Theorem 4.2. Some of the results were announced in [14] without proofs.

Applying the Gauss - Ostrogradski's theorem in (1.6) we get

Lemma 2.1. *A function $u(x)$ is a weak solution of the problem* (1.1) - (1.3) *if and only if $u \in \tilde{W}^1$ and the equality*

$$B[u, v] = (f, v)_0 \ \forall v \in \tilde{W}^1_* \tag{2.1}$$

holds.

Denote $\beta_j = b_j - \sum_{i=1}^{m-1} a_{ijx_i}$ for $j = 1, \dots, m-1$ and $\nu = h^{-1} \ln \lambda^2$. Obviously $\nu < 0$.

Lemma 2.2. *Let $u \in C(\bar{G})$ and*

$$V(x) = -\int_0^{x_m} \exp(-\nu\theta) u(x', \theta)\, d\theta + \frac{\lambda}{\lambda - 1} \int_0^h \exp(-\nu\theta) u(x', \theta)\, d\theta \tag{2.2}$$

for $x \in \bar{G}$. Then a constant $\tilde{c}_0(\lambda) > 0$, depending only on λ exists such that

$$\|V\|_0 \le \tilde{c}_0(\lambda) h \|u\|_0. \tag{2.3}$$

Proof. Applying the inequality $2ab \le a^2 + b^2$ for $a, b \in \mathbb{R}$ and the Hölder inequality for integrals we obtain

$$V^2(x) \le 4\tilde{c}^2(\lambda) [\int_0^h \exp(-2\nu\theta)\, d\theta][\int_0^h u^2(x', \theta)\, d\theta],$$

where $\tilde{c}^2(\lambda) = \max(1, \lambda^2(\lambda - 1)^{-2})$. Since $exp(-2\nu\theta) \le exp(-2\nu h) = \lambda^{-4}$ for $|\lambda| < 1$, then (2.3) takes place with $\tilde{c}_0(\lambda) = 2\lambda^{-1}(1 - \lambda)^{-1}$ for $\frac{1}{2} \le \lambda < 1$ and $\tilde{c}_0(\lambda) = 2\lambda^{-2}$ for $-1 < \lambda < \frac{1}{2}$, $\lambda \ne 0$.

It is not difficult to prove the following

Lemma 2.3. *Let $u \in \tilde{C}^2$ and V be the function defined by* (2.2). *Then V, $V_{x_m} \in C^2(\bar{G})$, V satisfies the conditions* (1.4) *and $V_{x_m} = 0$ on S, $V_{x_i}(x', 0) = \lambda V_{x_i}(x', h)$ in D, $i = 1, 2, \dots, m$.*

Lemma 2.4. *For each $u \in \tilde{W}^1$ a unique element $V \in \tilde{W}^1_*$ exists with the property: if $\{u_n\}_{n=1}^{\infty} \subset \tilde{C}^2$ is a sequence convergent to u strongly in $W_2^1(G)$, and*

$$V_n(x) = -\int_0^{x_m} \exp(-\nu\theta) u_n(x', \theta)\, d\theta + \frac{\lambda}{\lambda - 1} \int_0^h \exp(-\nu\theta) u_n(x', \theta)\, d\theta \tag{2.4}$$

for $x \in \bar{G}$, $n = 1, 2, \dots$, then $V_n \underset{n \to \infty}{\longrightarrow} V$ strongly in $W_2^1(G)$. The inequality (2.3) *takes place for each $u \in \tilde{W}^1$ and its corresponding element $V \in \tilde{W}^1_*$.*

Proof. Let $u \in \tilde{W}^1$, $\{u_n\}_{n=1}^{\infty} \subset \tilde{C}^2$ and $u_n \underset{n\to\infty}{\longrightarrow} u$ strongly in $W_2^1(G)$. Further we shall omit the word "strongly". It follows from Lemma 2.2 that $\|V_n - V_s\|_0 \leq \tilde{c}_0(\lambda)h\|u_n - u_s\|_0$ $\forall n \in \mathbb{N}$, $\forall s \in \mathbb{N}$. Then $V \in L_2(G)$ exists such that $V_n \underset{n\to\infty}{\longrightarrow} V$ in $L_2(G)$. Differentiating with respect to x_i the integrals in (2.4) we calculate $\dfrac{\partial V_n}{\partial x_i}$ in $\bar{G}$ for $1 \leq i \leq m-1$. Lemma 2.2 implies that $\|\dfrac{\partial V_n}{\partial x_i} - \dfrac{\partial V_s}{\partial x_i}\|_0 \leq \tilde{c}_0(\lambda)h\|\dfrac{\partial u_n}{\partial x_i} - \dfrac{\partial u_s}{\partial x_i}\|_0$ $\forall n \in \mathbb{N}$, $\forall s \in \mathbb{N}$. Hence $\dfrac{\partial V_n}{\partial x_i} \underset{n\to\infty}{\longrightarrow} w_i$ in $L_2(G)$. Obviously $\dfrac{\partial V_n}{\partial x_m} \underset{n\to\infty}{\longrightarrow} -\exp(-\nu x_m)u$ in $L_2(G)$. Then the generalized derivatives of V are $V_{x_i} = w_i$, $i = 1, \ldots, m-1$, $V_{x_m} = -\exp(-\nu x_m)u$ (see [15], Ch. 1, Theorem 4.1). Hence $V_n \underset{n\to\infty}{\longrightarrow} V$ in $W_2^1(G)$ and $V \in \tilde{W}_*^1$ due to Lemma 2.3.

Further, if $\{\tilde{u}_n\}_{n=1}^{\infty} \subset \tilde{C}^2$ is convergent to u in $W_2^1(G)$ and

$$\tilde{V}_n(x) = -\int_0^{x_m} \exp(-\nu\theta)\tilde{u}_n(x', \theta)\, d\theta + \frac{\lambda}{\lambda - 1}\int_0^h \exp(-\nu\theta)\tilde{u}_n(x', \theta)\, d\theta$$

in $\bar{G}$, $n = 1, 2, \ldots$, then $\tilde{V}_n \underset{n\to\infty}{\longrightarrow} \tilde{V}$ in $W_2^1(G)$. The inequality

$$\|V - \tilde{V}\|_0 \leq \|V - V_n\|_0 + \tilde{c}_0(\lambda)h\|u_n - \tilde{u}_n\|_0 + \|\tilde{V}_n - \tilde{V}\|_0$$

implies that $V = \tilde{V}$ almost everywhere in G. Clearly the corresponding element V to $u \in \tilde{C}^2$ is given by (2.2).

It follows from (2.4) and Lemma 2.2 that $\|V_n\|_0 \leq \tilde{c}_0(\lambda)h\|u_n\|_0$ $\forall n \in \mathbb{N}$. Taking a limit in this inequality, we obtain (2.3) for an arbitrary $u \in \tilde{W}^1$ and its corresponding element $V \in \tilde{W}_*^1$.

Lemma 2.5. *Let the derivatives $b_{mx_mx_m}$, $k_{x_mx_mx_m}$, c_{x_m} exist and belong to $C(\bar{G})$. Let $|\lambda| < 1$, $\nu = h^{-1}\ln\lambda^2$ and the following conditions*

$$a_{ij}(x', h) = a_{ij}(x', 0) \ \forall x' \in \bar{D},\ i, j = 1, \ldots, m-1, \tag{2.5}$$

$$(2b_m - 3k_{x_m} + \nu k)(x) \geq 2\alpha_1 \ \text{in } \bar{G},\ \alpha_1 = const > 0, \tag{2.6}$$

$$\begin{cases} \displaystyle\sum_{i,j=1}^{m-1} [-\nu a_{ij}(x) - a_{ijx_m}(x)]\xi_i\xi_j \geq a_1 \sum_{i=1}^{m-1} \xi_i^2 \ \forall x \in \bar{G} \\ \text{and } \forall \xi' \in \mathbb{R}^{m-1},\ a_1 = const \geq \dfrac{2}{\alpha_1} \max_{\bar{G}} \displaystyle\sum_{j=1}^{m-1} [\beta_j(x)]^2, \end{cases} \tag{2.7}$$

$$\nu[c - (b_m - k_{x_m})_{x_m}] + c_{x_m} - (b_m - k_{x_m})_{x_mx_m} \geq \frac{2}{\alpha_1}\Big(\sum_{j=1}^{m-1} |\beta_{jx_j}|\Big)^2 \ \text{in } \bar{G}, \tag{2.8}$$

$$[c - (b_m - k_{x_m})_{x_m}](x', h) \leq [c - (b_m - k_{x_m})_{x_m}](x', 0) \ \text{in } \bar{D} \tag{2.9}$$

hold. Then for every $u \in \tilde{W}^1$ and for its corresponding element V from Lemma 2.4 one has

$$B[u, V] \geq \frac{\alpha_1}{2}\int_G \exp(-\nu x_m)u^2\, dx. \tag{2.10}$$

Proof. Let $u \in \tilde{C}^2$ and V be given by (2.2). Using the equality $u(x) = -\exp(\nu x_m)V_{x_m}(x)$ we express the first order derivatives of u by those of V up to the second order and put them in $B[u,V]$. Then, applying the Gauss - Ostrogradski's theorem , we find

$$B[u,\, V] = -\{\int_G \exp(\nu x_m)(-b_m + \frac{3}{2}k_{x_m} - \frac{\nu}{2}k)V_{x_m}^2\, dx +$$

$$+\frac{1}{2}\int_G \exp(\nu x_m)\sum_{i,j=1}^{m-1}(a_{ijx_m} + \nu a_{ij})V_{x_i}V_{x_j}\, dx -$$

$$-\int_G \exp(\nu x_m)V_{x_m}\sum_{j=1}^{m-1}\beta_j V_{x_j}\, dx - \int_G \exp(\nu x_m)VV_{x_m}\sum_{j=1}^{m-1}\beta_{jx_j}\, dx -$$

$$-\frac{1}{2}\int_G \exp(\nu x_m)[\nu(c-(b_m-k_{x_m})_{x_m}) + c_{x_m} - (b_m - k_{x_m})_{x_m x_m}]V^2\, dx +$$

$$+\frac{1}{2}\int_{\partial G}\exp(\nu x_m)[c-(b_m-k_{x_m})_{x_m}]V^2 n_m\, ds\} = \sum_{j=1}^{6} I_j\, .$$

The other integrals on ∂G are equal to zero. As usual, $(n_1,\ldots,n_m)$ is the unit normal vector of ∂G outward to G. Using the Hölder inequality for sums and the inequality

$$|ab| \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \text{ for } a,\, b\in \mathbb{R} \text{ and } \varepsilon > 0, \tag{2.11}$$

we obtain the estimate

$$|I_3 + I_4| \le \frac{1}{\alpha_1}\max_{\bar{G}}\sum_{j=1}^{m-1}[\beta_j(x)]^2\int_G \exp(\nu x_m)\sum_{j=1}^{m-1}V_{x_j}^2\, dx +$$

$$+\frac{\alpha_1}{2}\int_G \exp(\nu x_m)V_{x_m}^2\, dx + \frac{1}{\alpha_1}\int_G \exp(\nu x_m)V^2(\sum_{j=1}^{m-1}|\beta_{jx_j}|)^2\, dx.$$

Then from (2.6) - (2.9) it follows that

$$B[u,\, V] \ge \frac{\alpha_1}{2}\int_G \exp(\nu x_m)V_{x_m}^2\, dx.$$

Hence (2.10) holds for every $u \in \tilde{C}^2$ and V from (2.2). The general case of Lemma 2.5 is a consequence of the considered case and Lemma 2.4.

Theorem 2.1. *Let $|\lambda| < 1$, $\nu = h^{-1}\ln\lambda^2$ and all the assumptions of Lemma 2.5 hold. Then the problem* (1.1) - (1.3) *can have no more than one weak solution.*

Proof. If u_1 and u_2 are two weak solutions of problem (1.1) - (1.3), then $u = u_1 - u_2$ is a weak solution of that problem for $f = 0$. We apply Lemma 2.5 with V, corresponding to u according to Lemma 2.4. It follows from Lemma 2.1, (1.6) and (2.10) that $0 \geq \frac{\alpha_1}{2} \int_G \exp(-\nu x_m) u^2 \, dx$. Hence $u = 0$ almost everywhere in G, i.e. $u_1 = u_2$ almost everywhere in G.

3. EXISTENCE OF A WEAK SOLUTION OF THE LINEAR PROBLEM

Lemma 3.1. *Let the derivative c_{x_m} exist and belong to $C(\bar{G})$. Let (2.5) hold and p be a function, defined in $[0, h]$, such that*

$$p(0) \geq p(h)\lambda^2, \tag{3.1}$$

$$p \in C^1([0, h]), \; p(x_m) \neq 0 \; \forall x_m \in [0, h], \tag{3.2}$$

$$k(x', h)p(h)\lambda^2 \geq k(x', 0)p(0) \; \forall x' \in \bar{D}, \tag{3.3}$$

$$c(x', h)p(h)\lambda^2 \geq c(x', 0)p(0) \; \forall x' \in \bar{D}, \tag{3.4}$$

$$(cp' + pc_{x_m})(x) \leq 0 \; \forall x \in \bar{G}, \tag{3.5}$$

$$[(2b_m - k_{x_m})p - p'k](x) \geq 2\alpha_2 \textit{ in } \bar{G}, \; \alpha_2 = const > 0, \tag{3.6}$$

$$\begin{cases} \displaystyle\sum_{i,j=1}^{m-1} (p' a_{ij} + p a_{ij x_m})(x)\xi_i \xi_j \geq a_2 \sum_{i=1}^{m-1} \xi_i^2 \; \forall x \in \bar{G} \textit{ and} \\ \displaystyle\forall \xi' \in \mathbb{R}^{m-1}, \; a_2 = const > 0, \; a_2 \geq \frac{2}{\alpha_2} \max_{\bar{G}} \sum_{j=1}^{m-1} [p(x_m)\beta_j(x)]^2, \end{cases} \tag{3.7}$$

Then a constant $\tilde{c}_1 > 0$ exists such that the inequality

$$(\mathcal{L}u, pu_{x_m})_0 \geq \tilde{c}_1 \|u\|_1^2 \tag{3.8}$$

holds for every $u \in \tilde{C}^2$.

Proof. Let $u \in \tilde{C}^2$ and $p(x_m)$ satisfies the assumptions (3.1) - (3.7). Applying the Gauss - Ostrogradski's theorem we obtain

$$2(\mathcal{L}u, pu_{x_m})_0 = \int_G [2b_m p - (pk)_{x_m}] u_{x_m}^2 \, dx + \int_G \sum_{i,j=1}^{m-1} (pa_{ij})_{x_m} u_{x_i} u_{x_j} \, dx +$$

$$+2 \int_G pu_{x_m} \sum_{j=1}^{m-1} \beta_j u_{x_j} \, dx - \int_G (pc)_{x_m} u^2 \, dx - \int_{\partial G} p \sum_{i,j=1}^{m-1} a_{ij} u_{x_i} u_{x_j} n_m \, ds +$$

$$+ \int_{\partial G} pku_{x_m}^2 n_m \, ds + \int_{\partial G} pcu^2 n_m \, ds + \int_{\partial G} p \sum_{i,j=1}^{m-1} a_{ij} u_{x_i} n_j u_{x_m} \, ds = \sum_{l=1}^{8} J_l,$$

where $(n_1, n_2, \dots, n_m)$ is the unit outward normal vector of ∂G.

Clearly

$$J_5 = \int_D [p(0) - p(h)\lambda^2] \sum_{i,j=1}^{m-1} (a_{ij} u_{x_i} u_{x_j})(x', 0)\, dx',$$

$$J_6 = \int_D [k(x', h)p(h)\lambda^2 - k(x', 0)p(0)] u^2_{x_m}(x', 0)\, dx',$$

$$J_7 = \int_D [c(x', h)p(h)\lambda^2 - c(x', 0)p(0)] u^2(x', 0)\, dx'.$$

It follows from (1.2), (1.3), (2.5) and (3.1) - (3.5) that $J_8 = 0, J_l \geq 0, l = 4, 5, 6, 7$.

Applying the Hölder inequalities for integrals, sums and the inequality (2.11), we obtain the estimate

$$|2 \int_G p u_{x_m} \sum_{j=1}^{m-1} \beta_j u_{x_j}\, dx| \leq \alpha_2 \int_G u^2_{x_m}\, dx + \frac{1}{\alpha_2} \int_G p^2 (\sum_{j=1}^{m-1} \beta_j^2)(\sum_{j=1}^{m-1} u^2_{x_j})\, dx.$$

The Sobolev imbedding theorem ([1], 5.4) implies the inequality

$$\|u\|_0^2 \leq K_0 \sum_{j=1}^{m} \|u_{x_j}\|_0^2 \ \forall u \in \tilde{C}^2,$$

where K_0 is a positive constant depending only on $\bar{G}$. From these estimates, (3.6) and (3.7) it follows that

$$J_1 + J_2 + J_3 \geq 2\tilde{c}_1 \|u\|_1^2$$

with $\tilde{c}_1 = \dfrac{1}{2K_0} \min(\alpha_2, a_2)$. This completes the proof.

Theorem 3.1. *Let the derivative c_{x_m} exist and belong to $C(\bar{G})$. Let (2.5) hold and for some function $p(x_m)$ the assumptions (3.2), (3.4) - (3.7) be satisfied. Let*

$$p(0) = p(h)\lambda^2. \tag{3.9}$$

Then for every $f \in L_2(G)$ there exists a weak solution U of the problem (1.1) - (1.3) and the inequality

$$\|U\|_1 \leq \tilde{c}_2 \|f\|_0 \tag{3.10}$$

holds with $\tilde{c}_2 = \dfrac{p_2}{\tilde{c}_1}$, $p_2 = \max\limits_{[0,h]} |p(x_m)|$.

Proof. Let $v \in \tilde{C}^2_*$. The function

$$u_v(x) = \int_0^{x_m} p^{-1}(\theta) v(x', \theta)\, d\theta + \frac{1}{\lambda - 1} \int_0^h p^{-1}(\theta) v(x', \theta)\, d\theta$$

is the unique solution from $\tilde{C}^2$ of the equation $pu_{x_m} = v$. The condition (1.3) is valid on D because of (3.9). Obviously

$$\|v\|_0 \leq p_2 \|u_v\|_1. \tag{3.11}$$

Let $\tilde{W}^{-1}$ be the Hilbert space with negative norm constructed by the spaces $L_2(G)$ and $\tilde{W}^1$ (see [3], 1.1.1). Denote its norm by $\|.\|_{-1}$ and its inner product by $(.,.)_{-1}$. Thus

$$(\mathcal{L}u_v,\, v)_0 = (u_v,\, \mathcal{L}^* v)_0 \leq \|\mathcal{L}^* v\|_{-1} \|u_v\|_1.$$

Applying (3.8) and (3.11) to the left-hand side we obtain the estimate

$$\frac{\tilde{c}_1}{p_2} \|v\|_0 \leq \|\mathcal{L}^* v\|_{-1} \;\forall v \in \tilde{C}^2_*. \tag{3.12}$$

This estimate implies the existence of a weak solution U of the problem (1.1) - (1.3) for the given $f \in L_2(G)$ (see [3], 2.3.4).

Indeed, consider the set $Y^* = \{w \in C(\bar{G}) : w = \mathcal{L}^* v,\, v \in \tilde{C}^2_*\}$. Clearly $Y^* \subset L_2(G) \subset \tilde{W}^{-1}$ and Y^* is a linear space. The mapping $\mathcal{L}^* : \tilde{C}^2_* \to Y^*$ is one-to-one mapping, due to (3.12). Then the formula

$$\varphi(w) = (f, v)_0,\; w = \mathcal{L}^* v,$$

defines a linear functional on Y^*. The Cauchy's inequality and (3.12) imply

$$|\varphi(w)| \leq \|f\|_0 \|v\|_0 \leq \tilde{c}_2 \|f\|_0 \|w\|_{-1},\; w \in Y^*.$$

Hence φ is a bounded functional. It can be extended by the Hahn - Banach's theorem to a linear continuous functional ϕ on $\tilde{W}^{-1}$ satisfying the inequality $|\phi(w)| \leq \tilde{c}_2 \|f\|_0 \|w\|_{-1}$ $\forall w \in \tilde{W}^{-1}$. This inequality implies that $\|\phi\| \leq \tilde{c}_2 \|f\|_0$. Obviously $\phi(\mathcal{L}^* v) = (f, v)_0$, $\forall v \in \tilde{C}^2_*$.

Since $\tilde{W}^{-1}$ is a Hilbert space, the Riesz representation theorem provides the existence of a unique element $\tilde{U} \in \tilde{W}^{-1}$ such that $\|\phi\| = \|\tilde{U}\|_{-1}$ and

$$\phi(w) = (\tilde{U},\, w)_{-1} \;\forall w \in \tilde{W}^{-1}.$$

Further, there exists an element $U \in \tilde{W}^1$ with the properties (see [3], 1.1.1) $(U,\, v)_0 = (\tilde{U},\, v)_{-1}$ $\forall v \in L_2(G)$ and $\|U\|_1 = \|\tilde{U}\|_{-1}$. It follows that U satisfies (3.10) and (1.6), i.e. U is a weak solution of the problem (1.1) - (1.3).

Remark 3.1. If we take $p(x_m) = \exp(-\nu x_m)$, $\nu = h^{-1} \ln \lambda^2$ then (3.9) and (3.2) are satisfied and the conditions (3.4) - (3.7) turn to

$$\begin{cases} (2b_m - k_{x_m} + \nu k)(x) \geq 2\alpha_2 \text{ in } \bar{G},\; \alpha_2 = const > 0, \\ \displaystyle\sum_{i,j=1}^{m-1} (-\nu a_{ij} + a_{ijx_m})(x)\xi_i\xi_j \geq a_2 \sum_{i=1}^{m-1} \xi_i^2 \;\forall x \in \bar{G} \text{ and} \\ \forall \xi' \in \mathbb{R}^{m-1},\; a_2 = const > 0,\; a_2 \geq \dfrac{2}{\alpha_2} \max_{\bar{G}} \displaystyle\sum_{j=1}^{m-1} [\beta_j(x)]^2, \\ (-\nu c + c_{x_m})(x) \leq 0 \text{ in } \bar{G},\; c(x', h) \geq c(x', 0) \text{ in } \bar{D}. \end{cases} \tag{3.13}$$

In Theorem 3.1 we can also take $p(x_m) = x_m + h\lambda^2(1-\lambda^2)^{-1}$ as in [10].

Example 3.1. Consider the equation

$$(ku_{x_m})_{x_m} + \sum_{i=1}^{m-1} u_{x_i x_i} + bu_{x_m} + cu = f(x), \tag{3.14}$$

where $k(x) = -(x_m^2 - hx_m + g)$, $0 < g \le \frac{h^2}{4}$, $b = const$. If $g = \frac{h^2}{4}$, the equation (3.14) is hyperbolic - parabolic in $\bar{G}$. If $0 < g < \frac{h^2}{4}$, this is an equation of mixed type in $\bar{G}$. Let $d(x') \le 0$, $d \in C(\bar{D})$, $\nu = h^{-1}\ln\lambda^2$. We shall notice the following cases:

1/ $g = \frac{h^2}{4}$, $b > \frac{h}{2}$, $0 < |\lambda| < 1$, $c = d(x')$;

2/ $0 < g < \frac{h^2}{4}$, $b \ge \frac{h}{2}$, $e^{-1} \le |\lambda| < 1$, $c = d(x')$;

3/ $0 < g < \frac{h^2}{4}$, $\frac{h}{2} > b > \frac{h}{2} - \frac{g}{h}$, $e^{-1} \le |\lambda| < \exp(-\rho_1)$, $\rho_1 = \frac{h}{2g}(h-2b)$, $c = d(x')$;

4/ $0 < g < \frac{h^2}{4}$, $b > \frac{h}{2} - \frac{g}{h}$, $e^{-1} > |\lambda| > \exp(-\rho_2)$, $\rho_2 = h[b + (b^2 + g - \frac{h^2}{4})^{\frac{1}{2}}](\frac{h^2}{2} - 2g)^{-1}$, $c = d(x')$ or $c = d(x')\exp(2x_m(x_m - h)h^{-2})$.

In these cases the assumptions of Theorem 2.1 and (3.2), (3.9), (3.13) with $p(x_m) = \exp(-\nu x_m)$ are satisfied.

Remark 3.2. It is shown in [11] that (3.6) is very important condition for the existence of a weak solution of the problem (1.1), (1.2) in the case $k(x',0) = k(x',h) = 0\ \forall x' \in \bar{D}$.

Theorem 3.2. *Let the assumptions of Lemma 3.1 hold. Then the problem (1.1) - (1.3) can have no more than one classical solution.*

Proof. If u_1 and u_2 are two classical solutions of the problem (1.1) - (1.3), then $u = u_1 - u_2$ is a classical solution of that problem for $f = 0$. Applying Lemma 3.1 we get $0 \ge \tilde{c}_1\|u\|_1^2$. Hence $u = 0$ in $\bar{G}$, i.e. $u_1 = u_2$ in $\bar{G}$.

4. THE NONLINEAR PROBLEM

Let us consider the problem (1.7), (1.2), (1.3). Assume that there exist positive constants L, η and a function $A \in L_2(G)$ such that $1 \ge \eta\tilde{c}_2$, where $\tilde{c}_2$ is the constant from Theorem 3.1, and

$$|F(x,t) - F(x,s)| \le L|t-s|\ \forall x \in G,\ \forall t,s \in \mathbb{R}, \tag{4.1}$$

$$|F(x,t)| \le 2^{-\frac{1}{2}}\{A(x) + [(\tilde{c}_2)^{-2} - \eta^2]^{\frac{1}{2}}|t|\}\ \forall x \in G,\ \forall t \in \mathbb{R}. \tag{4.2}$$

Theorem 4.1. *Let the assumptions for $F(x,t)$ and the assumptions of Theorem 2.1 and Theorem 3.1 hold. Then the problem* (1.7), (1.2), (1.3) *has at least one weak solution.*

Proof. Let $A_0 = \|A\|_0$. Consider the set $W = \{w \in L_2(G) : \ \|w\|_0 \le A_0\eta^{-1}\}$. The inequality (4.2) gives (see [5], 12.10 and 12.11)

$$\|F(x,w)\|_0^2 \le \|A\|_0^2 + [(\tilde{c}_2)^{-2} - \eta^2]\|w\|_0^2 \le A_0^2(\tilde{c}_2\eta)^{-2}$$

for every $w \in W$. Let $w \in W$ and U_w be the unique weak solution of the problem (1.1) - (1.3) for $f(x) = F(x,w)$ due to Theorem 2.1 and Theorem 3.1. From (3.10) and the estimate for $\|F(x,w)\|_0$ it follows

$$\|U_w\|_0 \le \|U_w\|_1 \le \tilde{c}_2\|F(x,w)\|_0 \le A_0\eta^{-1}.$$

Hence $U_w \in W \cap \tilde{W}^1$.

We define an operator $T: \ W \longrightarrow W$ by the formula $Tw = U_w$. The equality (2.1) shows that $B[Tw,\, v] = (F(x,w),\, v)_0 \ \forall v \in \tilde{W}_*^1$. Applying the Schauder's fixed point theorem ([5], 30.11) we shall establish that this operator has a fixed point.

Obviously W is a bounded, closed, nonempty subset of the Hilbert space $L_2(G)$. It is a convex set, because $\|\mu w_1 + (1-\mu)w_2\|_0 \le \mu\|w_1\|_0 + (1-\mu)\|w_2\|_0 \le A_0\eta^{-1}$ for every $w_1,\ w_2 \in W$ and $0 < \mu < 1$. Consider a sequence $w_1,\ w_2,\ w_3, \ldots$ belonging to W. Let $w_0 \in W$ and $\|w_n - w_0\|_0 \underset{n\to\infty}{\longrightarrow} 0$. Denote $U_n = Tw_n,\ n = 0,\ 1,\ 2, \ldots$. We have

$$B[U_n - U_0,\, v] = (F(x,w_n) - F(x,w_0),\, v)_0 \ \forall v \in \tilde{W}_*^1,$$

i.e $U_n - U_0$ is the unique weak solution of the problem (1.1) - (1.3) for $f = F(x,w_n) - F(x,w_0)$. It follows from (3.10) and (4.1) that $\|U_n - U_0\|_0 \le \tilde{c}_2\|F(x,w_n) - F(x,w_0)\|_0 \le \tilde{c}_2 L\|w_n - w_0\|_0 \underset{n\to\infty}{\longrightarrow} 0$. Hence the operator $T: \ W \longrightarrow W$ is continuous.

In order to prove that T is a compact operator we have to show that the set $T(M)$ is a precompact set in $L_2(G)$ for every bounded set $M \in W$ (see [1], 1.16 and [5], 3.10). Since $T(M) \subset T(W)$, it is sufficient to establish that $T(W)$ is a precompact set in $L_2(G)$. Consider an arbitrary sequence $U_n = Tw_n,\ \ n = 1,\ 2, \ldots$. It is bounded in $\tilde{W}^1$, because

$$\|U_n\|_1 \le A_0\eta^{-1}, \quad n = 1,\ 2, \ldots. \tag{4.3}$$

This inequality and the Rellich - Kondrashov imbedding theorem ([1], 6.2) imply the existence of a subsequence $\{U_{n_j}\}_{j=1}^{\infty}$ strongly convergent in $L_2(G)$ to an element u. Since $\overline{T(W)} \subset W$ and W is closed, then $u \in W$. Therefore $T(W)$ is a precompact set, i.e. $\overline{T(W)}$ is a compact set in $L_2(G)$ ([8], Ch. 1, 5.1). Let us notice that $\tilde{W}^1$ is a Hilbert space with the inner product of $W_2^1(G)$. It follows from (4.3) and Theorem 1.17, [1] that the subsequence $\{U_{n_j}\}_{j=1}^{\infty}$ can be chosen to be weakly convergent in $\tilde{W}^1$ to $\tilde{u} \in \tilde{W}^1$. Then $U_{n_j} \underset{n\to\infty}{\longrightarrow} \tilde{u}$ weakly in $L_2(G)$ (see [15], Ch. 1, pp. 60 - 61). Hence $u = \tilde{u} \in \tilde{W}^1$.

The Schauder's fixed point theorem implies the existence of a fixed point $\tilde{U}$ of the operator T, i.e. $\tilde{U} = T(\tilde{U})$. Thus $\tilde{U} \in \tilde{W}^1$ and

$$B[\tilde{U}, v] = (F(x, \tilde{U}), v)_0 \ \forall v \in \tilde{W}^1_*,$$

i.e. $\tilde{U}$ is a weak solution of the problem (1.7), (1.2), (1.3).

In the proof of Theorem 4.1 we have applied the same method as in [2, 16], where local boundary value problems for nonlinear equations of mixed type in two- and three-dimensional domains have been investigated.

Theorem 4.2. *Let the assumptions for $F(x,t)$ with $L < \alpha_1[2h\tilde{c}_0(\lambda)]^{-1}$, where $\tilde{c}_0(\lambda)$ is the constant from Lemma 2.2, and the assumptions of Theorem 2.1 and Theorem 3.1 hold. Then the problem* (1.7), (1.2), (1.3) *has exactly one weak solution.*

Proof. Let $\tilde{U}_1$, $\tilde{U}_2$ be two weak solutions of (1.7), (1.2), (1.3) and $u = \tilde{U}_1 - \tilde{U}_2$. Then $u \in \tilde{W}^1$ and

$$B[u, V] = (F(x, \tilde{U}_1) - F(x, \tilde{U}_2), V)_0,$$

where $V \in \tilde{W}^1_*$ is the corresponding to u element due to Lemma 2.4. It follows from Lemma 2.5 and (4.1) that $\frac{\alpha_1}{2}\int_G \exp(-\nu x_m)u^2\,dx \le L\|u\|_0\|V\|_0$. Applying (2.3) and the inequality $1 \le \exp(-\nu x_m)$ for $x_m \ge 0$, we obtain

$$\frac{\alpha_1}{2}\|u\|_0^2 \le Lh\tilde{c}_0(\lambda)\|u\|_0^2.$$

Suppose $\|u\|_0 \neq 0$. Since $L < \alpha_1[2h\tilde{c}_0(\lambda)]^{-1}$, then $Lh\tilde{c}_0(\lambda)\|u\|_0^2 < \frac{\alpha_1}{2}\|u\|_0^2$. So we have come to a contradiction. Hence $u = 0$ almost everywhere in G.

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THE BIQUARD CONNECTION ON RIEMANNIAN QUATERNIONIC CONTACT MANIFOLDS [1]

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The conformal infinity of a quaternionic-Kähler metric on a $4n$ dimensional manifold with boundary is a codimension 3-distribution on the boundary called quaternionic contact structure. In order to study such structures O.Biquard [1] has introduced a unique connection which preserves the structure and whose torsion tensor satisfies some conditions. This paper is devoted to obtaining an explicit formula for the torsion tensor and for the connection itself.

Keywords: connection, torsion, quaternionic contact structures
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1. INTRODUCTION

The quaternionic contact structures have been introduced by O.Biquard in [1] and [2]. Namely, a quaternionic contact structure on a (4n+3)-dimensional smooth manifold X is a codimension 3 distribution V such that at each point $x \in X$ the nilpotent Lie algebra $V_x \oplus T_xX/V_x$ is isomorphic to the quaternionic Heisenberg algebra $H^m \oplus Im\mathbf{H}$, where nilpotent Lie algebra structure is defined by

$$[a,b] = \begin{cases} \pi_{T_xX/V_x}[a,b] & \text{if } a,b \in V_x \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

and the Heisenberg algebra structure is given by the formula

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$$\left[\sum_{i=1}^{m} x_i e_i, \sum_{i=1}^{m} x_i e_i\right] = Im \sum_{i=1}^{m} \bar{x}_i y_i. \tag{1.2}$$

This is equivalent to the existence of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$such that $V = Ker\ \eta$ and the three 2-forms $d\eta_{i|V}$ are the fundamental 2-forms of a quaternionic structure on V. The 1-form η is given up to the action of SO_3 on $\mathbb{R}^3$ and up to a conformal factor.

If we pick up such a 1-form η (globally defined), we obtain the quaternionic structure on V defining the three endomorphisms $I_i = (d\eta_{k|V})^{-1} \circ (d\eta_{j|V}) : V \to V$, where (i, j, k) is any cyclic permutation of $(1, 2, 3)$. Obviously, this quaternionic structure does not depend on the choice of η. We also define the metric g on V by $g(X, Y) = d\eta_s(X, I_s Y)$. This metric is given up to the conformal factor because it depends on the conformal factor of η. Further, Biquard has shown ([1]) that there exists a unique triple of vector fields $\{R_1, R_2, R_3\}$, which satisfy $\eta_s(R_k) = \delta_{sk}$, $(i_{R_s} d\eta_s)_{|V} = 0$ and $(i_{R_s} d\eta_k)_{|V} = -(i_{R_k} d\eta_s)_{|V}$. Using this triple we define the metric g on the whole $T_x X$, putting $sp\{R_1, R_2, R_3\} \perp V$ and $g(R_s, R_k) = \delta_{sk}$. This metric does not depend on the action of SO_3 but it depends on the conformal factor of η. In my exposition I will assume that this metric is fixed. Also, in order to capture the 3-Sasaki structures, I will assume that the fundamental 2-forms of a quaternionic structure on V are $\frac{1}{2} d\eta_{i|V}$ instead of $d\eta_{i|V}$. Obviously, these assumptions make no restriction to the general case.

I also assume throughout the paper that the dimension of the base manifold is $4m + 3 > 7$. The case dim X=7 needs a special approach ([3]).

The interest in quaternionic contact structures is motivated by the result of Biquard [1] on Einstein deformations of $\mathbf{H}H^m$, which asserts that if a quaternionic contact structure on S^{4m-1} is close enough to the standard one, then it is a conformal infinity of complete Einstein metric. This result of Biquard is a generalization of a Graham-Lee [4] theorem on Einstein deformations of real hyperbolic space.

In the Salamon's [5] construction of the twistor space of a quaternionic Kähler manifold one uses the Levi-Civita connection to define the horizontal space for the fibration. In the case of quaternionic contact structure, there is no canonical connection. So using the analogy with the Tanaka-Webster [6] connection in CR geometry, Biquard [1] has introduced a unique contact quaternionic connection which I will call the Biquard connection.

This paper is devoted to study the properties of the Biquard connection. Many of its properties have been proved by Biquard [1], but he did not prove an explicit formula for the torsion tensor. This I have done in Theorem 5.3 - (i), corollary 5.1 (together with Theorem 5.4) and corollaries 5.4, 5.5 and 5.6. The key point in calculating the torsion tensor is the formula for the tensor $\tilde{u}$ (see Corollary 5.1) which I have obtained redoing in completely different way the proof of the theorems 5.3 and 5.4.

Let (M, g) be an orientable Riemannian manifold of dimension $4n + 3 \geq 11$.

Definition 2.1. A triple (V, Q, φ) will be called an Almost Contact Quaternionic Structure on M if

(i) V is codimension 3 distribution on M

(ii) Q is an almost quaternionic structure on V and

(iii) φ is a linear map from $V^{\perp}$ to Q which preserves orientation and which sends the unit sphere of $V^{\perp}$ into a set of complex structures of Q.

Let $J_1, J_2, J_3 \in Q, J_1^2 = J_2^2 = J_3^2 = -1, J_1 J_2 = -J_2 J_1 = J_3$ be the usual quaternionic basis of Q. Then the set of all complex structures in Q could be thought as a two dimensional sphere $\{ \sum_i a^i J_i \mid \sum_i (a^i)^2 = 1 \}$. It is easy to see that another triple $\hat{J}_1, \hat{J}_2, \hat{J}_3 \in Q$, $\hat{J}_i = \sum_k a_i^k J_k$ forms quaternionic basis, too, if and only if the matrix $(a_i^k)_{3x3}$ belongs to $SO(3)$.

We will denote with W the 3 dimensional distribution $V^{\perp}$ and with S^2 the unit sphere in W . Let $\xi_1, \xi_2, \xi_3 \in S^2$. Then by definition $(\varphi(\xi_i))^2 = -1$ and

Lemma 2.1. *The triple $\varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3)$ forms a quaternionic basis of Q if and only if ξ_1, ξ_2, ξ_3 is orthogonal and oriented basis of W.*

We will say that the map φ originates from the exterior derivative if across any point in M one can find orthonormal local basis $\{\xi_1, \xi_2, \xi_3\}$ of W such that $g(\varphi(\xi_i)X, Y) = \frac{1}{2} d(\flat\xi_i)(X, Y)$, $X, Y \in V$, $i = 1, 2, 3$, where $\flat\xi_i(X) = g(\xi, X)$, $X \in TM$.

Definition 2.2. An almost contact quaternionic structure (V, Q, φ) on M is called contact quaternionic structure if φ originates from the exterior derivative.

3. THE STRUCTURE GROUP

We consider the space $\mathbb{R}^{4n+3} = \mathbb{R}^{4n} \times \mathbb{R}^3 = V_0 + W_0$ with standard quaternionic structure Q_0 on $V_0 = \mathbb{R}^{4n}$. Let I_0, J_0, K_0 be the standard quaternionic basis on Q_0 and $\{e_1, e_2, e_3\}$ the standard basis on $W_0 = \mathbb{R}^3$. We consider the map $\varphi_0 : W_0 \rightarrow Q_0$, $\varphi_0(e_1) = I_0$, $\varphi_0(e_2) = J_0$, $\varphi_0(e_3) = K_0$. So we obtain a constant contact quaternionic structure (V_0, Q_0, φ_0) in $\mathbb{R}^{4n+3}$.

Let G denote the group of all endomorphisms of $O(4n + 3)$ which preserve the structure (V_0, Q_0, φ_0). Obviously G is a subgroup of $SO(4n + 3)$.

Theorem 3.1. *The manifold M admits an almost contact quaternionic structure if and only if its structural group could be reduced to the subgroup of G.*

Let $A \in Sp(n)Sp(1)$ and

$$\begin{aligned} AI_0A^{-1} &= a_1^1 I_0 + a_1^2 J_0 + a_1^3 K_0 \\ AJ_0A^{-1} &= a_2^1 I_0 + a_2^2 J_0 + a_3^3 K_0 \\ AK_0A^{-1} &= a_3^1 I_0 + a_3^2 J_0 + a_3^3 K_0 \ . \end{aligned}$$

Then the matrix $(a_i^k)_{3x3}$ belongs to $SO(3)$ and we obtain a homomorphism $\tau : Sp(n)Sp(1) \to SO(3)$.

Lemma 3.1. *The group G can be represented by*

$$G = \{ \ (A, \tau(A)) \mid A \in Sp(n)Sp(1) \ \}$$

Corollary 3.1. *The group G is isomorphic to $Sp(n)Sp(1)$.*

We denote this isomorphism with $\lambda : Sp(n)Sp(1) \to G$, $\lambda(A) = (A, \tau(A))$, $A \in Sp(n)Sp(1)$.

Let g be the Lie algebra of G. We will identify $\mathbb{R}^3$ with $sp(1)$. For any matrix $A \in sp(n) \oplus sp(1)$ let $a = (a_1, a_2, a_3)$ be its projection in $sp(1)$.

Lemma 3.2. *An endomorphism $t \in gl(4n+3, \mathbb{R})$ belongs on g if and only if there exists a matrix $A \in sp(n) \oplus sp(1)$ such that*

$$t(x+y) = Ax - 2a \wedge y, \ \ x \in \mathbb{R}^n, \ \ y \in \mathbb{R}^3.$$

Proof: We compute $(\lambda_{*1} A)(x+y) = Ax + y_1[A, I_0] + y_2[A, J_0] + y_3[A, K_0] = Ax + 2a \wedge y$, where a is the $sp(1)$ component of A, considered as an element of $\mathbb{R}^3$. □

4. CONTACT QUATERNIONIC CONNECTIONS

Let on the Riemannian manifold M be fix an almost contact quaternionic structure (V, Q, φ).

Definition 4.1. A Riemannian connection is called contact quaternionic connection if its holonomy group is contained in the group G.

Theorem 4.1. *An arbitrary Riemannian connection ∇ is contact quaternionic if and only if it satisfies the conditions:*

(i) *for any vector fields $X \in TM$ and $v \in V$, $\nabla_X v \in V$ (i.e. ∇ preserves V and $W = V^\perp$),*

(ii) ∇ *preserves Q,*

(iii) $\nabla \varphi = 0$.

Note: In fact, the condition (ii) follows from the other two.

5. THE BIQUARD CONNECTION

For the rest of the paper we will assume that the fixed structure (V, Q, φ) on M is contact quaternionic, i.e. the map φ originates from the exterior derivative. This means that in some neighborhood of any point on M we can find a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, such that $V = Ker\ \eta$, the three 2-forms ω_i, defined by

$$\begin{aligned} \omega_{i|V} &= \frac{1}{2} d\eta_{i|V} \\ i_X \omega_i &= 0,\ X \in W \end{aligned} \tag{5.1}$$

are the fundamental 2-forms of the quaternionic structure Q and the three vector fields $(\#\eta_1, \#\eta_2, \#\eta_3)$ form an orthonormal basis of W such that $g(\varphi(\#\eta_i)(X), Y) = \omega_i(X, Y)$. We fix this form η and denote $\xi_i = \#\eta_i,\ I_i = \#\omega_i$. In particular we have

$$I_{i|W} = 0,\ I_i I_j = I_k,\ and\ I^2_{i|V} = -id_V \tag{5.2}$$

where (i,j,k) is any cyclic permutation of (1,2,3)

Let D denote the Levi-Civita connection on M and let π be the orthogonal projection from TM to V. We define $\nabla_X Y = \pi(D_X Y)$ for any two vector fields $X, Y \in V$. We may regard ∇ as a part of Riemannian connection which preserves the distribution V. Our purpose is to extend ∇ to the connection on all TM which preserves our contact quaternionic structure.

Let T denote the torsion of ∇. It is easy to see (Biquard [1]) that

$$T(X, Y) = -[X, Y]_W, \qquad X, Y \in V. \tag{5.3}$$

Theorem 5.1 (Biquard [1]). *∇ preserves the quaternionic structure Q of V if and only if*

(i) $i_{\xi_\alpha} d\eta_{\alpha|V} = 0,\ \alpha = 1, 2, 3$ *and*

(ii) $i_{\xi_\alpha} d\eta_{\beta|V} = -i_{\xi_\beta} d\eta_{\alpha|V},\ \alpha \neq \beta.$

More precisely, if these two conditions hold, we have

$$\nabla_X \omega_\alpha = -d\eta_\alpha(\xi_\beta, X)\omega_\beta + d\eta_\gamma(\xi_\alpha, X)\omega_\gamma, \tag{5.4}$$

where $X \in V$ and (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

For any p-form ω we denote $\pi\omega(X_1, ..., X_p) = \omega(\pi X_1, ..., \pi X_p)$.

Lemma 5.1. *We have*

$$2\omega_i = \pi d\eta_i = d\eta_i - \sum_{s=1}^{3} \eta_s \wedge (i_{\xi_s} d\eta_i) + \sum_{1 \le s < t \le 3} d\eta_i(\xi_s, \xi_t)\eta_s \wedge \eta_t$$

Proof: We have $2\omega_i(X,Y) = d\eta_i(X - \eta(X), Y - \eta(Y)) =$
$= d\eta_i(X,Y) - d\eta_i(X,\eta(Y)) - d\eta_i(\eta(X),Y) + d\eta_i(\eta(X),\eta(Y))$, etc. □

From now on we assume that the conditions of the Theorem 5.1 hold. Using the equation $\nabla\varphi = 0$, we are able to determine (Biquard [1]) the covariant derivative $\nabla_X\xi$, $X \in V, \xi \in W$. We have

$$\nabla_X\xi = [X,\xi]_W \tag{5.5}$$

Let H be a subgroup of $Gl(4n,\mathbb{R})$ and h be the corresponding Lie algebra. Suppose that on V is given an H-structure and an extension of our connection in form $\nabla_\xi X$ wich preserves the H-structure. Then, for the torsion T, we obtain

$$T(\xi,X) = \nabla_\xi X - \nabla_X\xi - [\xi,X] = \nabla_\xi X - [\xi,X]_V \ \in V \tag{5.6}$$

In particular, we may regard $T(\xi,.)$ as an endomorphism T_ξ of V.

Lemma 5.2 (Biquard [1]). *For any H-structure on V there exists a unique extension of our connection in form $\nabla_\xi X$ which preserves this structure and such that*

$$T_\xi \in h^\perp, \ \xi \in W.$$

Proof: Let $\hat{\nabla}$ be an arbitrary extension of the covariant derivative which preserves the H-structure. Then for any other extension ∇ which preserves the H-structure we have $\nabla_\xi X = \hat{\nabla}_\xi X + a_\xi(X)$, where a_ξ is an endomorphism of V and $a_\xi \in h$. We obtain

$$T(\xi,Y) = \hat{T}(\xi,X) + a_\xi(X), \ \xi \in W, X \in V.$$

Obviously the tenzor $a_\xi(X)$ might be chosen in a unique way. □

It follows the main theorem.

Theorem 5.2 (Biquard [1]). *If the conditions*
(i) $i_{\xi_\alpha} d\eta_{\alpha|V} = 0, \ \alpha = 1,2,3$
(ii) $i_{\xi_\alpha} d\eta_{\beta|V} = -i_{\xi_\beta} d\eta_{\alpha|V}, \ \alpha \neq \beta$
are satisfied, there exists a unique contact quaternionic connection ∇ with torsion T such that
(i) $T(X,Y) = -[X,Y]_W, \ X,Y \in V$
(ii) $T_\xi \in (sp(n) \oplus sp(1))^\perp, \ \xi \in W$

We call this connection the Biquard connection.

One may decompose the tensor T_ξ(we regard it as an endomorphism of V which by the definition belongs to $(sp(n) \oplus sp(1))^\perp$) in two components: T^0_ξ - the symmetric one and a_ξ - the anti-symmetric. We have

$$T_\xi = T^0_\xi + a_\xi \tag{5.7}$$

Note: Through the Lemma 5.2 one can construct connection ∇^0 using the group $H = SO(4n)$ instead of $H = Sp(n)Sp(1)$. Then, according to the Lemma, $T'_\xi \in so(4n)^\perp$, where T' is the torsion of ∇^0. We have

$$T_\xi(X) = T'_\xi(X) + b_\xi(X), \ \xi \in W, X \in V \tag{5.8}$$

and since ∇ and ∇' both preserve the metric, $b_\xi \in so(V)$. So we obtain again the decomposition (5.7) and in particular $T^0_\xi = T'_\xi$.

My next aim is to calculate the torsion tensor T of the Biquard connection. Theorem 5.3 (ii) and (iii) and Theorem 5.4 were originally proved by Biquard [1], but in order to obtain an explicit formula for T I will remake there proofs in completely different way.

We will use the following well known lemma:

Lemma 5.3 (Biquard [1]). *Any endomorphism u of V might be decomposed uniquely:*

$$u = u^{+++} + u^{+--} + u^{-+-} + u^{--+},$$

where u^{+++} commutes with all three I_i, u^{+--} commutes with I_1 and anti-commutes with the others two and etc. In fact we have

$$4u^{+++} = u - I_1uI_1 - I_2uI_2 - I_3uI_3.$$

$$4u^{+--} = u - I_1uI_1 + I_2uI_2 + I_3uI_3.$$

$$4u^{-+-} = u + I_1uI_1 - I_2uI_2 + I_3uI_3.$$

$$4u^{--+} = u + I_1uI_1 + I_2uI_2 - I_3uI_3.$$

We define $L'_X(Y) = \pi L_X(Y)$, $X, Y \in TM$, where L denotes the Lee differentiation. If we regard the distribution V as a vector bundle over M, then we may regard L'_X and ∇_X as two differentiations of the tensor algebra of this vector bundle. In fact, for any differentiation of V we have the following useful lemma.

Lemma 5.4. *Let D be any differentiation of the tensor algebra of V. Then we have*

(i) $D(I_i)I_i = -I_iD(I_i)$, $i = 1, 2, 3$

(ii) $I_1D(I_1)^{-+-} = I_2D(I_2)^{+--}$ (*The other two identities could be obtained through cyclic permutation of* $(1, 2, 3)$).

Proof: We calculate

$$0 = D(-Id_V) = D(I_iI_i) = D(I_i)I_i + I_iD(I_i)$$

and we obtain (i). To get (ii), we calculate

$$0 = D(I_1I_2 + I_2I_1) = I_1D(I_2) + D(I_1)I_2 + D(I_2)I_1 + I_2D(I_1) =$$
$$= I_2(D(I_1) - I_2D(I_1)I_2) + I_1(D(I_2) - I_1D(I_2)I_1) =$$
$$= I_2D(I_1)^{-+-} + I_1D(I_2)^{+--}. \qquad \square$$

Theorem 5.3. *For any $X,Y\in V$ the symmetric component T^0 of the torsion T satisfies:*

(i) $g(T^0_{\xi_i}(X),Y)=\frac{1}{2}L_{\xi_i}g(X,Y),\ i=1,2,3;$

(ii) (Biquard [1]) $T^0_{\xi_i}(I_iX)=-I_iT^0_{\xi_i}(X),\ i=1,2,3;$

(iii) (Biquard [1]) $I_2(T^0_{\xi_2})^{+--}=I_1(T^0_{\xi_1})^{-+-}$ (*The other two identities could be obtained through cyclic permutation of* (1,2,3)).

Lemma 5.5.

$$L'_{\xi_1}I_2=-2T^{0\ --+}_{\xi_1}I_2-2I_3\tilde{u}+d\eta_1(\xi_2,\xi_1)I_1+\frac{1}{2}(d\eta_1(\xi_2,\xi_3)-d\eta_2(\xi_3,\xi_1)-d\eta_3(\xi_1,\xi_2))I_3 \tag{5.9}$$

$$L'_{\xi_2}I_1=-2T^{0\ --+}_{\xi_2}I_1+2I_3\tilde{u}+d\eta_2(\xi_1,\xi_2)I_2-\frac{1}{2}(-d\eta_1(\xi_2,\xi_3)+d\eta_2(\xi_3,\xi_1)-d\eta_3(\xi_1,\xi_2))I_3 \tag{5.10}$$

$$L'_{\xi_1}I_1=-2T^0_{\xi_1}I_1+d\eta_1(\xi_1,\xi_2)I_2+d\eta_1(\xi_1,\xi_3)I_3 \tag{5.11}$$

and six more identities which may be obtained through cyclic permutation of $(1,2,3)$. *Here $\tilde{u}$ is symmetric endomorphism of V which commutes with I_1, I_2 and I_3.*

Proof: (Theorem 5.3 and Lemma 5.5) Let $X,Y\in V$. Using (5.6), we calculate

$$g(T^0_{\xi_i}X,Y)=\frac{1}{2}(g(T_{\xi_i}X,Y)+g(T_{\xi_i}Y,X))=$$

$$=\frac{1}{2}(g(\nabla_{\xi_i}X-[\xi_iX]_V,Y)+g(\nabla_{\xi_i}Y-[\xi_iY]_V,X))=$$

$$=\frac{1}{2}(\xi_ig(X,Y)-g([\xi_iX]_V,Y)-g([\xi_iY]_V,X))=\frac{1}{2}L_{\xi_i}g(X,Y).$$

We also have

$$L_{\xi_i}\omega_j(X,Y)=\xi_ig(I_jX,Y)-g(I_j[\xi_i,X],Y)-g(I_jX,[\xi_i,Y])=$$

$$=L_{\xi_i}g(I_jX,Y)+g((L_{\xi_i}I_j)X,Y)$$

which leads to

$$\#L'_{\xi_i}\omega_j=2T^0_{\xi_i}I_j+L'_{\xi_i}I_j \tag{5.12}$$

On the other hand, using Lemma 5.1 and the well known identity $L_{\xi_i}\omega_i=i_{\xi_i}d\omega_i+di_{\xi_i}\omega_i$, we compute

$$L_{\xi_i}\omega_{i|V}=d\eta_i(\xi_i,\xi_j)\omega_j+d\eta_i(\xi_i,\xi_k)\omega_k, \tag{5.13}$$

where (i,j,k) is cyclic permutation of $(1,2,3)$. So we obtain

$$L'_{\xi_i} I_i = -2T^0_{\xi_i} I_i + d\eta_i(\xi_i, \xi_j) I_j + d\eta_i(\xi_i, \xi_k) I_k \tag{5.14}$$

Now we apply Lemma 5.4 (i) for $D = L'$ and this completes the proof of Theorem 5.3, (i) and (ii).

We use the well known formula $L_{\xi_1}\omega_2 = i_{\xi_1} d\omega_2 + d i_{\xi_1}\omega_2$ and Lemma 5.1 to compute

$$(L_{\xi_1}\omega_2)_{|V} = \frac{1}{2}(d(i_{\xi_1} d\eta_2) - i_{\xi_1} d\eta_3 \wedge i_{\xi_3} d\eta_2)_{|V}. \tag{5.15}$$

Next we apply the condition (ii) of Theorem 5.1 to obtain

$$(L_{\xi_1}\omega_2 + L_{\xi_2}\omega_1)_{|V} = \frac{1}{2}(d(i_{\xi_1} d\eta_2) + d(i_{\xi_2} d\eta_1))_{|V} =$$

$$= d\eta_1(\xi_2, \xi_1)\omega_1 + d\eta_2(\xi_1, \xi_2)\omega_2 + (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_1, \xi_3))\omega_3 \tag{5.16}$$

On the other hand, (5.12) leads to

$$2T^0_{\xi_1} I_2 + L'_{\xi_1} I_2 + 2T^0_{\xi_2} I_1 + L'_{\xi_2} I_1 =$$

$$= d\eta_1(\xi_2, \xi_1) I_1 + d\eta_2(\xi_1, \xi_2) I_2 + (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_1, \xi_3)) I_3. \tag{5.17}$$

Now we decompose (5.17) according to Lemma 5.3 to get

$$L'_{\xi_1} I_2{}^{+--} = -2T^0_{\xi_1}{}^{--+} I_2 + d\eta_1(\xi_2, \xi_1) I_1 \tag{5.18}$$

$$L'_{\xi_2} I_1{}^{-+-} = -2T^0_{\xi_2}{}^{--+} I_1 + d\eta_2(\xi_1, \xi_2) I_2 \tag{5.19}$$

$$(L'_{\xi_1} I_2 + L'_{\xi_2} I_1)^{--+} = (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_1, \xi_3)) I_3. \tag{5.20}$$

$$T^0_{\xi_1}{}^{-+-} I_2 + T^0_{\xi_2}{}^{+--} I_1 = 0 \tag{5.21}$$

Obviously, (5.21) completes the proof of Theorem 5.3. Using (5.20), we define

$$2\tilde{u} = I_3 L'_{\xi_1} I_2{}^{--+} + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) Id_V = \tag{5.22}$$

$$= -I_3 L'_{\xi_2} I_1{}^{--+} + \frac{1}{2}(-d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) Id_V.$$

Applying Lemma 5.4 for $D = L'$, we obtain the formulas in Lemma 5.5.

Now we shall show that $\tilde{u}$ is symmetric. For any $X, Y \in V$ according to (5.12) we have

$$L_{\xi_1}\omega_2(X, Y) = 2g(T^0_{\xi_1} I_2 X, Y) + g(L_{\xi_1} I_2 X, Y).$$

But $L_{\xi_1}\omega_2(X,Y)$ is skew-symmetric and applying (5.9) we get

$$0 = symm(2T^0_{\xi_1}I_2 + L'_{\xi_1}I_2) = -I_3 antisymm(2\tilde{u}) \qquad \square$$

Let (i,j,k) be any cyclic permutation of $(1,2,3)$. We define three 2-forms

$$A_i = \frac{1}{2}\pi\{d(\pi(i_{\xi_j}d\eta_k)) + (i_{\xi_i}d\eta_j)\wedge(i_{\xi_i}d\eta_k)\} = \tag{5.23}$$

$$= \frac{1}{2}\pi\{d(i_{\xi_j}d\eta_k) + (i_{\xi_i}d\eta_j)\wedge(i_{\xi_i}d\eta_k)\} - d\eta_k(\xi_j,\xi_k)\omega_k + d\eta_k(\xi_i,\xi_j)\omega_i$$

We put this into (5.15) to get

$$(L_{\xi_1}\omega_2)_{|V} = A_3 + d\eta_2(\xi_1,\xi_2)\omega_2 - d\eta_2(\xi_3,\xi_1)\omega_3$$

On the other hand, using (5.9) and (5.12) we calculate

$$\#L'_{\xi_1}\omega_2 = 2T^0_{\xi_1}{}^{-+-}I_2 - 2I_3\tilde{u} + d\eta_1(\xi_2,\xi_1)I_1 + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))I_3$$

We decompose the last two identities to obtain

Lemma 5.6.

$$\begin{aligned}
\#A_3^{+++} &= 2T^0_{\xi_1}{}^{-+-}I_2\\
\#A_3^{+--} &= d\eta_1(\xi_2,\xi_1)I_1\\
\#A_3^{-+-} &= -d\eta_2(\xi_1,\xi_2)I_2\\
\#A_3^{--+} &= -2I_3\tilde{u} + \tfrac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))I_3
\end{aligned}$$

Analogous formulas for A_1 and A_2 may be obtained through cyclic permutation of $(1,2,3)$.

Corollary 5.1. *For the symmetric tensor $\tilde{u}$ we have*

$$2\tilde{u} = I_1\#A_1^{+--} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2))Id_V =$$

$$= I_2\#A_2^{-+-} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2))Id_V =$$

$$= I_3\#A_3^{--+} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))Id_V$$

and also

$$tr(\tilde{u}) = \frac{1}{2}tr(I_1\#A_1) + n(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)) =$$

$$= \frac{1}{2}tr(I_2\#A_2) + n(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)) =$$

$$= \frac{1}{2}tr(I_3\#A_3) + n(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)).$$

Theorem 5.4 (Biquard [1]). *For any $i = 1, 2, 3$ we have*

$$T_{\xi_i} = T^0_{\xi_i} + I_i u.$$

Here $u = \tilde{u} - \dfrac{tr(\tilde{u})}{4n} Id_V$ *and* $\tilde{u}$ *is given in Corollary 5.1.*

Proof: First we denote with Σ^2 the space of symmetric endomorphisms of V and with "ant" the projection

$$ant : End(V) = \#\Sigma^2(V) \oplus \#\Lambda^2(V) \to \#\Lambda^2(V).$$

We have

$$4[T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} = 3ant(T_{\xi_i}) + I_1 ant(T_{\xi_i}) I_1 + I_2 ant(T_{\xi_i}) I_2 + I_3 ant(T_{\xi_i}) I_3 = \tag{5.24}$$

$$= \sum_{s=1}^{3} (ant(T_{\xi_i}) + I_s ant(T_{\xi_i}) I_s).$$

We apply (5.6) and for any $X, Y \in V$, we obtain

$$g(4[T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} X, Y) = -\sum_{s=1}^{3} g((\nabla_{\xi_i} I_s)X, I_s Y) + \tag{5.25}$$

$$+\frac{1}{2}\sum_{s=1}^{3} \{g((L_{\xi_i} I_s)X, I_s Y) - g((L_{\xi_i} I_s)Y, I_s X)\}.$$

We have also

$$T_{\xi_i} = [T_{\xi_i}]_{(sp(n)\oplus sp(1))^\perp} = T^0_{\xi_i} + [T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} - [T_{\xi_i}]_{sp(1)}.$$

Now we apply Lemma 5.5 and the theorem follows. □

Corollary 5.2. $I_2(T_{\xi_2})^{+--} = I_1(T_{\xi_1})^{-+-}$ (*The other two identities could be obtained through cyclic permutation of* $(1, 2, 3)$).

Corollary 5.3. *For any* $X, Y \in V$

$$g(\nabla_{\xi_i} X, Y) = \frac{1}{2} L_{\xi_i} g(X, Y) + g([\xi_i, X], Y) + g(I_i u X, Y).$$

Corollary 5.4.

$$\begin{aligned}
\nabla_{\xi_1} I_1 &= -d\eta_1(\xi_3, \xi_1) I_3 + d\eta_1(\xi_1, \xi_2) I_2 \\
\nabla_{\xi_1} I_2 &= -d\eta_1(\xi_1, \xi_2) I_1 + (-\tfrac{tr(\tilde{u})}{2n} + \tfrac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2))) I_3 \\
\nabla_{\xi_1} I_3 &= -(-\tfrac{tr(\tilde{u})}{2n} + \tfrac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2))) I_2 + d\eta_1(\xi_3, \xi_1) I_1
\end{aligned}$$

$$\nabla_{\xi_2} I_1 = -(-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)))I_3 + d\eta_2(\xi_1,\xi_2)I_2$$
$$\nabla_{\xi_2} I_2 = -d\eta_2(\xi_1,\xi_2)I_1 + d\eta_2(\xi_2,\xi_3)I_3$$
$$\nabla_{\xi_2} I_3 = -d\eta_2(\xi_2,\xi_3)I_2 + (-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)))I_1$$

$$\nabla_{\xi_3} I_1 = -d\eta_3(\xi_3,\xi_1)I_3 + (-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)))I_2$$
$$\nabla_{\xi_3} I_2 = -(-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)))I_1 + d\eta_3(\xi_2,\xi_3)I_3$$
$$\nabla_{\xi_3} I_3 = -d\eta_3(\xi_2,\xi_3)I_2 + d\eta_3(\xi_3,\xi_1)I_1$$

Of course we may write all this formulas briefly as follows

$$\nabla_{\xi_s} I_i = -\alpha_j(\xi_s)I_k + \alpha_k(\xi_s)I_j, \tag{5.26}$$

where $\alpha_i(\xi_s) = -\delta_{is}(\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2))) + d\eta_s(\xi_j,\xi_k)$, $s = 1,2,3$ *and* (i,j,k) *is any cyclic permutation of* $(1,2,3)$.

Proof: According to (5.6) we have

$$\nabla_{\xi_s} I_i = [T_{\xi_s}, I_i] + L'_{\xi_s} I_i = [T^0_{\xi_s}, I_i] + u[I_s, I_i] + L'_{\xi_s} I_i.$$

We apply Lemma 5.5 to get the corollary . □

Corollary 5.5.

$$\nabla_{\xi_s}\xi_i = -\alpha_j(\xi_s)\xi_k + \alpha_k(\xi_s)\xi_j,$$

Here $\alpha_i(\xi_s)$ *is the same as in* (5.26), (i,j,k) *is any cyclic permutation of* $(1,2,3)$ *and* $s = 1,2,3$.

Corollary 5.6.

$$T(\xi_i,\xi_j) = -\frac{tr(\tilde{u})}{n}\xi_k - [\xi_i,\xi_j]_V.$$

Here i,j,k *is any cyclic permutation of* $(1,2,3)$.

Proof: Using Corollary 5.5 we compute

$$T(\xi_i,\xi_j) = \nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i - [\xi_i,\xi_j] = -\frac{tr(\tilde{u})}{n}\xi_k - [\xi_i,\xi_j]_V. \quad \square$$

6. THE 3-SASAKIAN CASE

Let on the Riemannian manifold (M,g) be given a 3-Sasakian structure. This means there are given three Killing vector fields $\{\xi_1,\xi_2,\xi_3\}$, which satisfy

(i) $g(\xi_i,\xi_j) = \delta_{ij},\ i,j = 1,2,3$

(ii) $[\xi_i,\xi_j] = -2\xi_k$, for any cyclic permuatation (i,j,k) of $(1,2,3)$

(iii) $(D_X\tilde{I}_i)Y = g(\xi_i,Y)X - g(X,Y)\xi_i,\ i = 1,2,3,\ X,Y \in TM$. Where $\tilde{I}_i(X) = D_X\xi_i$ and D denotes the Levi-Civita connection.

We denote $V = \{\xi_1,\xi_2,\xi_3\}^{\perp}$. We shall use without proof the next well known

Lemma 6.1. *Let (i,j,k) be any cyclic permutation of $(1,2,3)$. We have*

$$\tilde{I}_i(\xi_j) = \xi_k; \tag{6.1}$$

$$\tilde{I}_i \circ \tilde{I}_j(X) = \tilde{I}_k X, \ \ X \in V; \tag{6.2}$$

$$\tilde{I}_i \circ \tilde{I}_i(X) = -X, \ \ X \in V; \tag{6.3}$$

$$d\eta_i(X,Y) = 2g(\tilde{I}_i X, Y), \ \ X, Y \in V. \tag{6.4}$$

If we define $W = space\{\xi_1, \xi_2, \xi_3\}$, $I_{i|V} = \tilde{I}_{i|V}$, $I_{i|W} = 0$ and $\varphi(\xi_i) = I_i$ we clearly obtain a contact quaternionic structure $(V, Q = \{I_1, I_2, I_3\}, \varphi)$ on M. In this case it is easy to calculate

Lemma 6.2.

$$i_{\xi_i} d\eta_{j|V} = 0 \ \ \textit{for all } i,j = 1,2,3; \tag{6.5}$$

$$d\eta_1(\xi_2,\xi_3) = 2, \ d\eta_1(\xi_1,\xi_3) = d\eta_1(\xi_1,\xi_2) = 0; \tag{6.6}$$

$$A_1 = A_2 = A_3 = 0; \tag{6.7}$$

$$\tilde{u} = \frac{1}{2} Id_V. \tag{6.8}$$

Theorem 6.1. *The contact quaternionic structure (V, Q, φ) satisfies the conditions of the Theorem 5.2 and therefore it admits the Biquard connection ∇. We have*

(i) $\nabla_X I_i = 0, X \in V$,
(ii) $\nabla_{\xi_i} I_i = 0$,
(iii) $\nabla_{\xi_i} I_j = -2I_k$, $\nabla_{\xi_j} I_i = 2I_k$, *here (i,j,k) is cyclic permutation of* $(1,2,3)$.
(iv) $T(\xi_i, \xi_j) = -2\xi_k$,
(v) $T(\xi_i, X) = 0, \ X \in V$.

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CYCLIC CODES AS INVARIANT SUBSPACES

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The description of the linear cyclic codes as ideals in the algebra $\mathfrak{F}_n = F[x]/(x^n - 1)$, where F is a finite field, is well known in the coding theory. The map cyclic shift is a linear operator in F^n. Our aim is to consider a new method of describing the cyclic codes as invariant subspaces of F^n regarding this operator.

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1. INTRODUCTION

The linear cyclic codes are traditionally described using the methods of the commutative algebra (see [2] and [3]). Since the linear codes have the structure of linear subspaces of F^n, the description of the linear cyclic codes in terms of the linear algebra is natural.

The main purpose of this paper is to study some properties of the cyclic codes as invariant linear subspaces. Some generalizations for consta-cyclic codes are considered.

2. SOME LINEAR ALGEBRA

Let $F = \mathrm{GF}(q)$ and let F^n be the n-dimensional vector space over F with standard basis $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$.

Let

$$\varphi : \begin{cases} F^n \to F^n \\ (x_1, x_2, \ldots, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1}) \end{cases}.$$

Then $\varphi \in \operatorname{Hom} F^n$ and has the following matrix

$$A = \begin{pmatrix} 0\,0\,0 \ldots & 1 \\ 1\,0\,0 \ldots & 0 \\ 0\,1\,0 \ldots & 0 \\ \vdots\,\vdots\,\vdots\,\ddots & \vdots \\ 0\,0\,0 \ldots & 0 \end{pmatrix}$$

in the basis $e_1, e_2, \ldots, e_n$. Note that $A^t = A^{-1}$ and $A^n = E$. The characteristic polynomial of A is

$$f_A(x) = \begin{vmatrix} -x & 0 & 0 & \ldots & 1 \\ 1 & -x & 0 & \ldots & 0 \\ 0 & 1 & -x & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -x \end{vmatrix} = (-1)^n(x^n - 1).$$

Let us denote it by $f(x)$.

We show the following well known fact.

Proposition 2.1. *Let U be a φ-invariant subspace of V and* $\dim_F V = n$. *Then $f_{\varphi|_U}(x)$ divides $f_\varphi(x)$. In particular, if $V = U \oplus W$ and W is φ-invariant subspace of F^n then $f_\varphi(x) = f_{\varphi|_U}(x) f_{\varphi|_W}(x)$.*

Proof: Let $\dim_F U = k$ and let $g_1, \ldots, g_k$ be a basis of U over F. We complement this basis to a basis $g_1, \ldots, g_k, g_{k+1}, \ldots, g_n$ of V. Then the coordinates of the vectors $\varphi(g_1), \ldots, \varphi(g_k)$ from the $(k+1)$-th vanish and hence in this basis φ has the matrix

$$A' = \begin{pmatrix} \alpha_{11} \ldots \alpha_{1k} & \alpha_{1,k+1} & \ldots & \alpha_{1n} \\ \cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \alpha_{k1} \ldots \alpha_{kk} & \alpha_{k,k+1} & \ldots & \alpha_{kn} \\ 0 \;\ldots\; 0 & \alpha_{k+1,k+1} & \ldots & \alpha_{k+1,n} \\ \cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ 0 \;\ldots\; 0 & \alpha_{n,k+1} & \ldots & \alpha_{n,n} \end{pmatrix} = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}.$$

The matrix $A_1 = \begin{pmatrix} \alpha_{11} \ldots \alpha_{1k} \\ \cdots\cdots\cdots \\ \alpha_{k1} \ldots \alpha_{kk} \end{pmatrix}$ is obviously the matrix of φ in $g_1, \ldots, g_k$. Then

$$f_\varphi(x) = \det(A' - xE) = \det \begin{pmatrix} A_1 - xE & B \\ 0 & A_2 - xE \end{pmatrix} =$$

$$= f_{\varphi|_U}(x) \det(A_2 - xE). \qquad \square$$

Let m be the multiplicative order of q modulo n, i.e., m is the smallest natural number with the property that n divides $q^m - 1$. Then $\mathrm{GF}(q^m)$ is the splitting field of $f(x)$ over F. Let $f(x) = (-1)^n f_1(x) \dots f_t(x)$ be the factorization of $f(x)$ into irreducible factors. We assume that $(n, q) = 1$. In that case $f(x)$ has distinct factors $f_i(x)$, $i = 1, \dots, t$, which are monic.

Let denote by U_i the space of the solutions of the homogeneous system with matrix $f_i(A)$ for each $i = 1, \dots, t$, i.e., $U_i = \operatorname{Ker} f_i(\varphi)$.

Theorem 2.1. *The subspaces U_i of F^n satisfy the following conditions:*

1) *U_i is a φ-invariant subspace of F^n;*

2) $F^n = U_1 \oplus \cdots \oplus U_t$;

3) $\dim U_i = \deg f_i = k_i$;

4) $f_{\varphi|_{U_i}}(x) = (-1)^{k_i} f_i(x)$;

5) *U_i is a minimal φ-invariant subspace of F^n.*

Proof: 1) Let $u \in U_i$, i.e., $f_i(A)u = \mathbf{0}$. Then $f_i(A)\varphi(u) = f_i(A)Au = Af_i(A)u = \mathbf{0}$, so that $\varphi(u) \in U_i$.

2) Let $\hat{f}_i(x) = \frac{f(x)}{f_i(x)}$ for $i = 1, \dots, t$. Since $(\hat{f}_1(x), \dots, \hat{f}_t(x)) = 1$, by the Euclidean algorithm there are polynomials $a_1(x), \dots, a_t(x) \in F[x]$ such that

$$a_1(x)\hat{f}_1(x) + \cdots + a_t(x)\hat{f}_t(x) = 1.$$

Then for every vector $v \in V$ the condition $v = a_1(A)\hat{f}_1(A)v + \cdots + a_t(A)\hat{f}_t(A)v$ holds. Let $v_i = a_i(A)\hat{f}_i(A)v$. Then $f_i(A)v_i = a_i(A)f(A)v = \mathbf{0}$ so that $v_i \in U_i$. Hence

$$F^n = U_1 + \cdots + U_t.$$

Assume that $v \in U_i \cap \sum_{j \neq i} U_j$, then $f_i(A)v = \mathbf{0}$, $\hat{f}_i(A)v = \mathbf{0}$. Since $(f_i, \hat{f}_i) = 1$, there are polynomials $a(x), b(x) \in F[x]$, such that $a(x)f_i(x) + b(x)\hat{f}_i(x) = 1$. Hence $a(A)f_i(A)v + b(A)\hat{f}_i(A)v = v = \mathbf{0}$, so that $U_i \cap \sum_{j \neq i} U_j = \{\mathbf{0}\}$. Thus

$$F^n = U_1 \oplus \cdots \oplus U_t.$$

3) Let $g \in U_i$ be an arbitrary nonzero vector and let $k \geq 1$ be the smallest natural number with the property that the vectors $g, \varphi(g), \dots, \varphi^{k-1}(g)$ are linearly independent. Then there are elements $c_0, \dots, c_{k-1} \in F$, at least one of which is nonzero, such that

$$\varphi^k(g) = c_0 g + c_1\varphi(g) + \cdots + c_{k-1}\varphi^{k-1}(g).$$

Consider the polynomial $t(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_0 \in F[x]$. Since $(t(\varphi))(g) = (f_i(\varphi))(g) = \mathbf{0}$, it follows that $[(t(x), f_i(x))(\varphi)](g) = \mathbf{0}$. But $(t(x), f_i(x))$ is 1 or $f_i(x)$. Hence $(t(x), f_i(x)) = f_i(x)$ and $f_i(x)$ divides $t(x)$. Thus $k_i = \deg f_i(x) \leq$

$\deg t(x) = k$. On the other hand, the vectors $g, \varphi(g), \ldots, \varphi^{k_i}(g)$ are linearly dependent, since $(f_i(\varphi))(g) = \mathbf{0}$, and from the minimality of k we obtain $k = k_i$. Then $\dim U_i \geq k_i$. Therefore

$$n = \dim{}_F F^n = \sum_{i=1}^{t} \dim{}_F U_i \geq \sum_{i=1}^{t} k_i = \sum_{i=1}^{t} \deg f_i = \deg f = n$$

and $\dim{}_F U_i = k_i$.

4) Let $g_1^{(i)}, \ldots, g_{k_i}^{(i)}$ be a basis of U_i over F, $i = 1, \ldots, t$, and let A_i be the matrix of $\varphi|_{U_i}$ in that basis. Let $\tilde{f}_i = f_{\varphi|_{U_i}}$. Suppose $(\tilde{f}_i, f_i) = 1$. Hence there are polynomials $a(x), b(x) \in F[x]$, such that $a(x)\tilde{f}_i(x) + b(x)f_i(x) = 1$. Then $a(A_i)\tilde{f}_i(A_i) + b(A_i)f_i(A_i) = E$. Therefore $b(A_i)f_i(A_i) = E$. We will show that $f_i(A_i) = \mathbf{0}$, which contradicts the last equation.

By the property 2) we obtain that $g_1^{(1)}, \ldots, g_{k_1}^{(1)}, \ldots, g_1^{(t)}, \ldots, g_{k_t}^{(t)}$ is the basis of F^n and the matrix of φ in that basis is

$$A' = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_t \end{pmatrix}.$$

Beside this $A' = T^{-1}AT$, where T is the change basis matrix from the standard basis of F^n to that one. Then

$$f_i(A') = \begin{pmatrix} f_i(A_1) & & & \\ & f_i(A_2) & & \\ & & \ddots & \\ & & & f_i(A_t) \end{pmatrix} = f_i(T^{-1}AT) = T^{-1}f_i(A)T.$$

Let $g_j^{(i)} = \lambda_{j1}^{(i)}e_1 + \cdots + \lambda_{jn}^{(i)}e_n$, $j = 1, \ldots, k_i$. Since $g_j^{(i)} \in U_i$, we obtain

$$f_i(A') \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = T^{-1}f_i(A)T \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = T^{-1}f_i(A) \begin{pmatrix} \lambda_{j1}^{(i)} \\ \vdots \\ \lambda_{jn}^{(i)} \end{pmatrix} = \mathbf{0},$$

where 1 is on the $(k_1 + \cdots + k_{i-1} + j)$–th position. According to the last equation, $f_i(A_i) = \mathbf{0}$. Therefore $(f_i, \tilde{f}_i) \neq 1$. Since f_i and $\tilde{f}_i$ are polynomials of the same degree k_i and f_i is monic and irreducible, we obtain $\tilde{f}_i = (-1)^{k_i} f_i$.

5) Let $\{\mathbf{0}\} \neq U \subseteq U_i$. Then by Proposition 2.1 we obtain $f_{\varphi|_U}$ divides f_i. Since the polynomial f_i is irreducible, $\dim{}_F U = \dim{}_F U_i$ and $U = U_i$. □

Proposition 2.2. *Let U be a φ–invariant subspace of F^n. Then U is a direct sum of some minimal φ–invariant subspaces U_i of F^n.*

Proof: Let $\widetilde{U}_i = U \cap U_i$, $i = 1, \dots, t$. Then $\widetilde{U}_i$ is $\{\mathbf{0}\}$ or U_i, since U_i are minimal. Therefore

$$U = U \cap F^n = U \cap (U_1 \oplus \cdots \oplus U_t) = \widetilde{U}_1 \oplus \cdots \oplus \widetilde{U}_t = \bigoplus_{U_i \leq U} U_i. \qquad \square$$

3. LINEAR CYCLIC CODES

Definition 3.1. A code C with length n over F is called cyclic, if whenever $x = (c_1, c_2, \dots, c_n)$ is in C, so is its cycle shift $y = (c_n, c_1, \dots, c_{n-1})$.

The following statement is clear from the definitions.

Proposition 3.1. *A linear code C with length n over F is cyclic iff C is a $\varphi-$invariant subspace of F^n.*

Theorem 3.1. *Let C be a linear cyclic code with length n over F. Then the following facts hold.*

1) $C = U_{i_1} \oplus \cdots \oplus U_{i_s}$ for some minimal $\varphi-$invariant subspaces U_{i_r} of F^n and $\dim_F C = k_{i_1} + \cdots + k_{i_s} = k$;

2) $f_{\varphi|_C}(x) = (-1)^k f_{i_1}(x) \dots f_{i_s}(x) = g(x)$;

3) $c \in C$ *iff* $g(A)c = \mathbf{0}$;

4) the polynomial $g(x)$ has the smallest degree with the property 3;

5) $\mathrm{r}\,(g(A)) = n - k$.

Proof: 1) This follows from Proposition 2.2.

2) Let $g_1^{(i_r)}, \dots, g_{k_{i_r}}^{(i_r)}$ be a basis of U_{i_r} over F, $r = 1, \dots, s$. Then $g_1^{(i_1)}, \dots, g_{k_{i_1}}^{(i_1)}, \dots, g_1^{(i_s)}, \dots, g_{k_{i_s}}^{(i_s)}$ is a basis of C over F and $\varphi|_C$ has a matrix

$$\begin{pmatrix} A_{i_1} & & & \\ & A_{i_2} & & \\ & & \ddots & \\ & & & A_{i_s} \end{pmatrix}$$

in that basis. Hence

$$f_{\varphi|_C}(x) = \tilde{f}_{i_1}(x) \dots \tilde{f}_{i_s}(x) = (-1)^{k_{i_1} + \cdots + k_{i_s}} f_{i_1}(x) \dots f_{i_s}(x).$$

Note that A_{i_r} and $\tilde{f}_{i_r}(x)$ are defined as in Theorem 2.1.

3) Let $c \in C$. Then $c = u_{i_1} + \cdots + u_{i_s}$ for some $u_{i_r} \in U_{i_r}$, $r = 1, \dots, s$ and $g(A)c = (-1)^k[(f_{i_1} \dots f_{i_s})(A)u_{i_1} + \cdots + (f_{i_1} \dots f_{i_s})(A)u_{i_s}] = \mathbf{0}$.

Conversely, suppose $g(A)c = \mathbf{0}$ for some $c \in F^n$ and let $c = u_1 + \cdots + u_t$, $u_i \in U_i$. Then $g(A)c = (-1)^k[(f_{i_1} \dots f_{i_s})(A)u_1 + \cdots + (f_{i_1} \dots f_{i_s})(A)u_t] = \mathbf{0}$, so that $g(A)[u_{j_1} + \cdots + u_{j_l}] = \mathbf{0}$, where $\{j_1, \dots j_l\} = \{1, \dots, t\} \setminus \{i_1, \dots, i_s\}$. Let $v = u_{j_1} + \cdots + u_{j_l}$ and

$$h(x) = \frac{(-1)^n [x^n - 1]}{g(x)} = \frac{f(x)}{g(x)}.$$

Since $(h(x), g(x)) = 1$, there are polynomials $a(x)$, $b(x) \in F[x]$ such that $a(x)h(x) + b(x)g(x) = 1$. Hence $a(A)h(A)v + b(A)g(A)v = v = \mathbf{0}$ and $c = u_{i_1} + \cdots + u_{i_s} \in C$.

4) Suppose $b(x) \in F[x]$ is a nonzero polynomial of smallest degree such that $b(A)c = \mathbf{0}$ for all $c \in C$. By the division algorithm in $F[x]$ there are polynomials $q(x), r(x)$ such that $g(x) = b(x)q(x) + r(x)$, where $\deg r(x) < \deg b(x)$. Then for each vector $c \in C$ we have $g(A)c = q(A)b(A)c + r(A)c$ and hence $r(A)c = \mathbf{0}$. But this contradicts the choice of $b(x)$ unless $r(x)$ is identically zero. Thus, $b(x)$ divides $g(x)$. If $\deg b(x) < \deg g(x)$, then $b(x)$ is a product of some of the irreducible factors of $g(x)$ and without loss of generality we can suppose $b(x) = (-1)^{k_{i_1} + \cdots + k_{i_q}} f_{i_1} \dots f_{i_q}$ and $q < s$. Let us consider the code $C' = U_{i_1} \oplus \cdots \oplus U_{i_q} \subset C$. Then $b(x) = f_{\varphi|_{C'}}$ and by the equation $g(A)c = \mathbf{0}$ for all $c \in C$ we obtain $C \subsetneq C'$. This contradiction proves the statement.

5) By the property 3) C is the space of the solutions of the homogeneous system with matrix $g(A)$. Then $\dim_F C = k = n - \mathrm{r}\,(g(A))$, which proves the statement.□

Definition 3.2. Let $x = (x_1, \dots, x_n)$ and $y = (y_1 \dots, y_n)$ be two vectors in F^n. We define an inner product over F by $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$. If $\langle x, y \rangle = 0$, we say that x and y are orthogonal to each other.

Definition 3.3. Let C be a linear code over F. We define the dual of C (which is denoted by $C^\perp$) to be the set of all vectors which are orthogonal to all codewords in C, i.e.,

$$C^\perp = \{v \in F^n \mid \langle v, c \rangle = 0 \text{ for all } c \in C\}.$$

It is well known that if C is k–dimensional, then $C^\perp$ is $(n-k)$–dimensional.

Proposition 3.2. *The dual of a linear cyclic code is also cyclic.*

Proof: Let $h = (h_1, \dots, h_n) \in C^\perp$ and $c = (c_1, \dots, c_n) \in C$. We show that $\varphi(h) = (h_n, h_1, \dots, h_{n-1}) \in C^\perp$. We have

$$\langle \varphi(h), c \rangle = c_1 h_n + \cdots + c_n h_{n-1} = \langle h, \varphi^{-1}(c) \rangle = \langle h, \varphi^{n-1}(c) \rangle = \mathbf{0},$$

which proves the statement. □

Proposition 3.3. *The matrix H, whose rows are arbitrary $n-k$ linear independent rows of $g(A)$, is a parity check matrix of C.*

Proof: The proof follows from the equation $g(A)c = \mathbf{0}$ for every vector $c \in C$ and the fact that $\mathrm{r}\,(g(A)) = n - k$. □

Let $g_{l_1}, \dots, g_{l_{n-k}}$ be a basis of $C^\perp$, where g_{l_r} is a l_r–th vector row of $g(A)$. By the equation $g(A)h(A) = \mathbf{0}$ we obtain that $\langle g_{l_r}, h_i \rangle = 0$ for each $i = 1, \dots, n$, $r =$

$1, \ldots, n-k$. The last equation gives us that the columns h_i of $h(A)$ are codewords in C.

We show that $\mathrm{r}(h(A)) = k$. By Sylvester's inequality we obtain $\mathrm{r}(\mathbf{0}) = 0 \geq \mathrm{r}(g(A)) + \mathrm{r}(h(A)) - n$. Since $\mathrm{r}(h(A)) \leq n - \mathrm{r}(g(A)) = n-(n-k) = k$. On the other hand, Sylvester's inequality, applied to the product $h(A) = (-1)^{n-k} f_{j_1}(A) \ldots f_{j_l}(A)$, gives $\mathrm{r}(h(A)) \geq r_{j_1} + \cdots + r_{j_l} - n(l-1) = nl - k_{j_1} - \cdots - k_{j_l} - nl + n = n - (k_{j_1} + \cdots + k_{j_l}) = n - (n - k_{i_1} - \cdots - k_{i_s}) = n-(n-k) = k$. Therefore $\mathrm{r}(h(A)) = k$. Thus we have proved the following:

Proposition 3.4. *The matrix G, whose rows are arbitrary k linear independent rows of $(h(A))^t$, is a generator matrix of the code C.*

Lemma 3.1. *If $g(x) \in F[x]$, then $g(A^{-1}) = g(A^t) = (g(A))^t$. In particular, if n divides $\deg g(x)$, then $g^*(A) = (g(A))^t$, where $g^*(x)$ is the reciprocal polynomial of $g(x)$.*

Proof: Let $g(x) = g_0 x^k + g_1 x^{k-1} + \cdots + g_{k-1}x + g_k$, then $g(A) = g_0 A^k + g_1 A^{k-1} + \cdots + g_{k-1}A + g_k E$. Transposing both sides of the last equation, we obtain $(g(A))^t = g_0 (A^k)^t + g_1 (A^{k-1})^t + \cdots + g_{k-1} A^t + g_k E = g_0 (A^t)^k + g_1 (A^t)^{k-1} + \cdots + g_{k-1} A^t + g_k E = g(A^t)$.

In particular, if $\deg g(x) = ns$ for some $s \in \mathbb{N}$, then $g^*(A) = A^{ns} g(A^{-1}) = A^{ns} g(A^t) = g(A^t) = (g(A))^t$. $\square$

Let $f_{\varphi|_{C^\perp}}(x) = \tilde{h}$. By Theorem 3.1 it follows that $\tilde{h}$ is the polynomial of the smallest degree such that $\tilde{h}(A)u = \mathbf{0}$ for every $u \in C^\perp$. Let $h^*(x) = \tilde{h}(x)q(x) + r(x)$, where $\deg r(x) < \deg \tilde{h}(x)$. Then by Lemma 3.1 $h^*(A) = A^{n-k}(h(A))^t = \tilde{h}(A)q(A) + r(A)$, hence for every vector $u \in C^\perp$ the assertion $A^{n-k}(h(A))^t u = q(A)\tilde{h}(A)u + r(A)u$ holds, so that $r(x) = 0$. Thus $\tilde{h}(x)$ divides $h^*(x)$. Since both are polynomials of the same degree , $h^*(x) = a\tilde{h}(x)$, where $a \in F$ is the leading coefficient of the product $f^*_{j_1}(x) \ldots f^*_{j_l}(x)$. Thus

$$\tilde{h} = \frac{1}{a} h^* = (-1)^{n-k} \frac{1}{a} f^*_{j_1} \cdots f^*_{j_l} = \prod_{r=1}^{l} \frac{1}{a_{j_r}} f^*_{j_r} = (-1)^{n-k} f_{s_1} \cdots f_{s_l},$$

where a_{j_r} is the leading coefficient of $f^*_{j_r}(x)$. Note that the polynomials $f_{s_r}(x) = \frac{1}{a_{j_r}} f^*_{j_r}(x)$ are monic irreducible and divide $f(x) = (-1)^n [x^n - 1]$.

Now we show that $C^\perp = U_{s_1} \oplus \cdots \oplus U_{s_l}$. By Theorem 3.1 $C^\perp$ is the space of the solutions of the homogeneous system with matrix $\tilde{h}(A)$. Let $u \in U = U_{s_1} \oplus \cdots \oplus U_{s_l}$ and let $u = u_{s_1} + \cdots + u_{s_l}$ for $u_{s_r} \in U_{s_r}$, $r = 1, \ldots, l$. Then

$$\tilde{h}(A)u = (-1)^{n-k}[(f_{s_1} \ldots f_{s_l})(A)u_{s_1} + \cdots + (f_{s_1} \ldots f_{s_l})(A)u_{s_l}] = \mathbf{0}.$$

Hence $U \leq C^\perp$. Since $\dim_F U = \dim_F C^\perp$, then

$$C^\perp = U_{s_1} \oplus \cdots \oplus U_{s_l}.$$

Thus we have proved the following:

Theorem 3.2. *Let $C = U_{i_1} \oplus \cdots \oplus U_{i_s}$ be a linear cyclic code over F and $\{j_1, \ldots, j_l\} = \{1, \ldots, t\} \setminus \{i_1, \ldots, i_s\}$. Then the dual code of C is given by $C^{\perp} = U_{s_1} \oplus \cdots \oplus U_{s_l}$ and $\tilde{f}_{s_r}(x) = (-1)^{k_{s_r}} f_{s_r}(x) = (-1)^{k_{s_r}} \frac{1}{a_{j_r}} f^*_{j_r}(x)$, where $f^*_{j_r}(x)$ is the reciprocal polynomial of $f_{j_r}(x)$ with leading coefficient equals to a_{j_r}, $r = 1, \ldots, l$.*

Let $C \subset F^n$ be an arbitrary, not necessary linear, cyclic code. Let us consider the action of the group $G = \langle \varphi \rangle = \{\mathrm{id}, \varphi, \ldots, \varphi^{n-1}\} \cong \mathbb{C}_n$ over F^n. Then the following theorem holds:

Theorem 3.3. *$C = \Omega_1 \cup \ldots \cup \Omega_s$, where Ω_i are G-orbits and $k_i = |\Omega_i|$ is a divisor of $|G| = n$. In particural, $|C| = \sum\limits_{i=1}^{s} k_i$.*

4. CONSTA-CYCLIC CODES

In this section we give a generalization of the results obtained in the previous sections.

Definition 4.1. Let a be a nonzero element of F. A code C with length n over F is called consta-cyclic with respect to a, if whenever $x = (c_1, c_2, \ldots, c_n)$ is in C, so is $y = (ac_n, c_1, \ldots, c_{n-1})$.

Let $a \in F$. We consider the linear operator $\psi_a \in \mathrm{Hom}\, F^n$

$$\psi_a : (x_1, x_2, \ldots, x_n) \mapsto (ax_n, x_1, \ldots, x_{n-1}).$$

Its matrix in the standard basis $e_1, e_2, \ldots e_n$ of F^n is

$$B_a = \begin{pmatrix} 0 & 0 & 0 & \ldots & a \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

The relations $B_a^{-1} = B^t_{\frac{1}{a}}$ and $B_a^n = aE$ hold. The characteristic polynomial of B_a is $f_{B_a}(x) = (-1)^n(x^n - a)$. Let denote it by $f_a(x)$. We assume that $(n, q) = 1$. The polynomial f_a has no multiple roots and splits to distinct irreducible monic factors $f_a(x) = (-1)^n f_1(x) \ldots f_t(x)$. Let $U_i = \mathrm{Ker}\, f_i(\psi_a)$. It's easy to see that Theorem 2.1 and Proposition 2.2 are true in this case, too.

The following statement is clear from the definition.

Proposition 4.1. *A linear code C with length n over F is consta-cyclic iff C is a ψ_a-invariant subspace of F^n.*

The next theorem is analogous to Theorem 3.1 and we omit its proof.

Theorem 4.1. *Let C be a linear consta-cyclic code with length n over F. Then the following facts hold.*

1) $C = U_{i_1} \oplus \cdots \oplus U_{i_s}$ for some minimal ψ_a-invariant subspaces U_{i_r} of F^n and $\dim_F C = k_{i_1} + \cdots + k_{i_s} = k$;

2) $f_{\psi_a|_C}(x) = (-1)^k f_{i_1}(x) \dots f_{i_s}(x) = g(x)$;

3) $c \in C$ iff $g(B_a)c = \mathbf{0}$;

4) the polynomial $g(x)$ has the smallest degree with the property 3);

5) $\mathrm{r}\,(g(B_a)) = n - k$.

Proposition 4.2. *The dual of a linear consta-cyclic code with respect to a is consta-cyclic with respect to $\frac{1}{a}$.*

Proof: The proof follows from the equality

$$\langle \psi_a(c), h \rangle = \langle B_a c, h \rangle = \langle c, B_a^t h \rangle = \langle c, B_{\frac{1}{a}}^{-1} h \rangle = a \langle c, \psi_{\frac{1}{a}}^{n-1}(h) \rangle = 0$$

for every $c \in C$ and $h \in C^{\perp}$. □

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CYCLIC CODES WITH LENGTH DIVISIBLE BY THE FIELD CHARACTERISTIC AS INVARIANT SUBSPACES

DIANA RADKOVA, ASEN BOJILOV

In the theory of cyclic codes it is a common practice to require $(n, q) = 1$, where n is the word length and F_q is the alphabet. However, much of the theory also goes through without this restriction on n and q. We observe that the cyclic shift map is a linear operator in F_q^n. Our approach is to consider cyclic codes as invariant subspaces of F_q^n with respect to this operator and thus obtain a description of cyclic codes in this more general setting.

Keywords: Cyclic codes, invariant subspaces.

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1. INTRODUCTION

The main purpose of this paper is the study of some properties of the cyclic codes as linear subspaces without the requirement that the field characteristic is coprime with n. We already considered the case of coprime field characteristic and word length in [4].

The linear cyclic codes are traditionally described using the methods of commutative algebra (see [2] and [3]). Since the linear codes have the structure of linear subspaces of F^n, where F is a finite field, the description of linear cyclic codes in terms of the linear algebra is natural.

2. SOME LINEAR ALGEBRA

Let $F = \mathrm{GF}(q)$ and let F^n be the n-dimensional vector space over F with the standard basis $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,\ldots,0), \ldots, e_n = (0,0,\ldots,1)$.

Let $\varphi : F^n \to F^n$ be the linear map given by the formula $\varphi(x_1, x_2, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$.

Then φ has the following matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

in the basis $e_1, e_2, \ldots, e_n$. Note that $\varphi(e_1) = e_2$, $\varphi(e_2) = e_3, \ldots$, $\varphi(e_{n-1}) = e_n$, $\varphi(e_n) = e_1$.

We observe that $A^t = A^{-1}$ and $A^n = E$. The characteristic polynomial of A is

$$f_A(x) = \begin{vmatrix} -x & 0 & 0 & \ldots & 1 \\ 1 & -x & 0 & \ldots & 0 \\ 0 & 1 & -x & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -x \end{vmatrix} = (-1)^n(x^n - 1).$$

We will denote the polynomial $f_A(x)$ by $f(x)$.

We will assume that $(n,q) = p^s = d$ and $n = dn_1$, $(p, n_1) = 1$, where $p = \Gamma F$. Let $x^{n_1} - 1 = f_1(x) \ldots f_t(x)$ be the factorization of $x^{n_1} - 1$ into irreducible monic factors over F. Then the factorization of $f(x)$ is

$$f(x) = (-1)^n(x^n - 1) = (-1)^n\Big(x^{n_1} - 1\Big)^d = (-1)^n\Big(f_1(x)\Big)^d\Big(f_2(x)\Big)^d \ldots \Big(f_t(x)\Big)^d.$$

Let us denote by U_i the space of all solutions of the homogeneous system with matrix $f_i^d(A)$ for $i = 1, \ldots, t$, i.e. $U_i = \operatorname{Ker} f_i^d(\varphi)$.

Theorem 2.1. *The subspaces U_i of F^n satisfy the following conditions:*

1) U_i is a φ-invariant subspace of F^n;

2) $F^n = U_1 \oplus \cdots \oplus U_t$;

3) $f_i^d(x)$ is the monic polynomial of minimal degree in $F[x]$ such that $f_i^d(A)u = \mathbf{0}$ for all $u \in U_i$;

4) $f_{\varphi|_{U_i}} = (-1)^{d \deg f_i} f_i^d$. In particular, $\dim U_i = \deg f_{\varphi|_{U_i}} = d \deg f_i$*;*

5) There exist a vector $u_i \in U_i$ such that the vectors

$$u_i, \varphi(u_i), \ldots, \varphi^{\dim U_i - 1}(u_i)$$

are basis of U_i;

6)For each vector u in U_i there exists a polynomial $g \in F[x]$ such that $u = \Big(g(A)\Big)(u_i)$.

Proof: 1) Let $u \in U_i$, i.e. $f_i^d(A)u = \mathbf{0}$. Then $f_i^d(A)\varphi(u) = f_i^d(A)Au = Af_i^d(A)u = \mathbf{0}$, so that $\varphi(u) \in U_i$.

2) Let $\hat{f}_i(x) = \frac{f(x)}{f_i^d(x)}$ for $i = 1, \dots, t$. Since $(\hat{f}_1(x), \dots, \hat{f}_t(x)) = 1$, then by the Euclidean algorithm there are polynomials $a_1(x), \dots, a_t(x) \in F[x]$ such that

$$a_1(x)\hat{f}_1(x) + \cdots + a_t(x)\hat{f}_t(x) = 1.$$

Then for every vector $v \in V$ the condition $v = a_1(A)\hat{f}_1(A)v + \cdots + a_t(A)\hat{f}_t(A)v$ holds. Let $v_i = a_i(A)\hat{f}_i(A)v$. Then $f_i^d(A)v_i = a_i(A)f(A)v = \mathbf{0}$, so that $v_i \in U_i$. Hence

$$F^n = U_1 + \cdots + U_t.$$

Let us assume that $v \in U_i \cap \sum_{j \neq i} U_j$. Then $f_i^d(A)v = \mathbf{0}$ and $\hat{f}_i(A)v = \mathbf{0}$. Since $(f_i^d, \hat{f}_i) = 1$, there are polynomials $a(x), b(x) \in F[x]$, such that $a(x)f_i^d(x) + b(x)\hat{f}_i(x) = 1$. Hence $a(A)f_i^d(A)v + b(A)\hat{f}_i(A)v = v = \mathbf{0}$ and we conclude that $U_i \cap \sum_{j \neq i} U_j = \{\mathbf{0}\}$. Thus

$$F^n = U_1 \oplus \cdots \oplus U_t.$$

3) Let $m_i(x) \in F[x]$ be the monic polynomial of smallest degree such that $m_i(A)u = \mathbf{0}$ for all $u \in U_i$. By the division algorithm in $F[x]$ there are polynomials $q_i(x), r_i(x)$ such that $f_i^d(x) = m_i(x)q_i(x) + r_i(x)$, where $\deg r_i(x) < \deg m_i(x)$. Then for each vector $u \in U_i$ we have $f_i^d(A)u = q_i(A)m_i(A)u + r_i(A)u$ and hence $r_i(A)u = \mathbf{0}$. But this contradicts the choice of $m_i(x)$ unless $r_i(x)$ is identically zero. Thus, $m_i(x)$ divides $f_i^d(x)$ for all $i = 1, \dots, t$. Therefore there are numbers $0 \leq s_i \leq d$ such that $m_i(x) = f_i^{s_i}(x)$. Set $m(x) = m_1(x) \dots m_t(x)$. Since $m(A)u = \mathbf{0}$ for all $u \in F^n$ and $m(x)$ divides the minimal polynomial $x^n - 1$ of A, we conclude that $x^n - 1 = m(x)$. Then

$$f_1^d(x) \dots f_t^d(x) = x^n - 1 = f_1^{s_1}(x) \dots f_t^{s_t}(x).$$

Now the statement follows from the uniqueness of the factorization of a polynomial into irreducible factors.

4) Let $k_i = \dim U_i$, $i = 1, \dots, t$ and let $\tilde{f}_i(x) = f_{\varphi|_{U_i}}$. We choose a basis $g_1^{(i)}, \dots, g_{k_i}^{(i)}$ of U_i over F, $i = 1, \dots, t$. Denote by A_i the matrix of $\varphi|_{U_i}$ in that basis.

By property 2) we obtain that $g_1^{(1)}, \dots, g_{k_1}^{(1)}, \dots, g_1^{(t)}, \dots, g_{k_t}^{(t)}$ is a basis of F^n and the matrix of φ in that basis is

$$A' = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_t \end{pmatrix}.$$

Besides $A' = T^{-1}AT$, where T is the change of basis matrix from the standard basis of F^n to that one. Then

$$\tilde{f}_i(A') = \begin{pmatrix} \tilde{f}_i(A_1) & & & \\ & \tilde{f}_i(A_2) & & \\ & & \ddots & \\ & & & \tilde{f}_i(A_t) \end{pmatrix} = \tilde{f}_i(T^{-1}AT) = T^{-1}\tilde{f}_i(A)T.$$

Note that $\tilde{f}_i(A_i) = \mathbf{0}$. Let $g_j^{(i)} = \lambda_{j1}^{(i)}e_1 + \cdots + \lambda_{jn}^{(i)}e_n$, $j = 1, \ldots, k_i$. Since $g_j^{(i)} \in U_i$, we obtain

$$\tilde{f}_i(A)\begin{pmatrix} \lambda_{j1}^{(i)} \\ \vdots \\ \lambda_{jn}^{(i)} \end{pmatrix} = T\tilde{f}_i(A')T^{-1}\begin{pmatrix} \lambda_{j1}^{(i)} \\ \vdots \\ \lambda_{jn}^{(i)} \end{pmatrix} = T\tilde{f}_i(A')\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0},$$

where 1 is on the $(k_1 + \cdots + k_{i-1} + j)$−th position. Therefore $f_i^d(x)$ divides $\tilde{f}_i$ for all $i = 1, \ldots, t$. Let $\tilde{f}_i(x) = f_i^d(x)g_i(x)$. Then

$$f(x) = \tilde{f}_1(x)\ldots\tilde{f}_t(x) = f_1^d(x)\ldots f_t^d(x)g_1(x)\ldots g_t(x).$$

It follows from the last identity that $g_i(x) = (-1)^{d \deg f_i(x)}$.

5) Let $e_1 = u_1 + u_2 + \cdots + u_t$ for $u_i \in U_i$, $i = 1, \ldots, t$. Then

$$\begin{array}{lcccccc} e_2 = & \varphi(e_1) & = & \varphi(u_1) & + & \varphi(u_2) & + \cdots + & \varphi(u_t) \\ e_3 = & \varphi(e_2) & = & \varphi^2(u_1) & + & \varphi^2(u_2) & + \cdots + & \varphi^2(u_t) \\ \multicolumn{8}{c}{\dotfill} \\ e_n = & \varphi(e_{n-1}) & = & \varphi^{n-1}(u_1) & + & \varphi^{n-1}(u_2) & + \cdots + & \varphi^{n-1}(u_t) \end{array}.$$

Let v be an arbitrary vector from F^n. Then

$$\begin{aligned} v = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = \\ = \lambda_1(u_1 + u_2 + \cdots + u_t) + \lambda_2(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_t)) + \\ + \cdots + \lambda_n(\varphi^{n-1}(u_1) + \varphi^{n-1}(u_2) + \cdots + \varphi^{n-1}(u_t)) = \\ = (\lambda_1 u_1 + \lambda_2\varphi(u_1) + \cdots + \lambda_n\varphi^{n-1}(u_1)) + \\ + \cdots + (\lambda_1 u_t + \lambda_2\varphi(u_t) + \cdots + \lambda_n\varphi^{n-1}(u_t)) \end{aligned}.$$

Hence $v_i = \lambda_1 u_i + \lambda_2\varphi(u_i) + \cdots + \lambda_n\varphi^{n-1}(u_i)$ holds for each vector $v_i \in U_i$ and all $i = 1, \ldots, t$. Therefore $U_i = l\{u_i, \varphi(u_i), \ldots, \varphi^{n-1}(u_i)\}$. Since $\dim U_i = k_i$, the vectors

$$u_i, \varphi(u_i), \ldots, \varphi^{k_i-1}(u_i)$$

are a basis of U_i.

6)This follows from 5). □

Theorem 2.2. *Let U be a φ-invariant subspace of U_i for some $1 \le i \le t$. Then there exists a number $0 \le k \le d$ such that $U = \operatorname{Im} f_i^k(\varphi_{|U_i}) = \operatorname{Ker} f_i^{d-k}(\varphi_{|U_i}) = \operatorname{Ker} f_i^{d-k}(\varphi)$.*

Proof: Let the vector $u_i \in U_i$ be as in Theorem 2.1 and let us consider the set

$$J = \{g \in F[x] \mid \big(g(A)\big)(u_i) \in U\}.$$

It is easy to verify that J is a principal ideal in $F[x]$. Then there exists a monic polynomial $h \in F[x]$ such that $J = (h)$. We are going to show that $U = \operatorname{Im} h(\varphi_{|U_i})$. First, let $u \in U$. Then $u = g(A)u_i$ for a suitable polynomial $g(x) \in F[x]$ by Theorem 2.1, 6). Since $g(x) \in J$ then $g(x) = h(x)g_1(x)$. Hence $u = (hg_1)(A)u_i = h(A)g_1(A)u_i = h(A)v_i$, where $v_i \in U_i$. Thus $u \in \operatorname{Im} h(\varphi_{|U_i})$. Conversely, suppose $u \in \operatorname{Im} h(\varphi_{|U_i})$, i.e. $u = h(A)v$ for some $v \in U_i$. Then $v = g(A)u_i$ for a suitable polynomial $g(x) \in F[x]$ and hence $u = h(A)g(A)u_i = (hg)(A)u_i$. Since $h(x)g(x) \in J$, we conclude that $u \in U$.

Now we are going to show that $h(x) = f_i^k(x)$ for some $0 \le k \le d$. Since $f_i^d(A)u_i = \mathbf{0}$, then $f_i^d(x) \in J$. Therefore $h(x)$ divides $f_i^d(x)$. Since $f_i(x)$ is an irreducible polynomial, $h(x) = f_i^k(x)$ for some $0 \le k \le d$. Hence $U = \operatorname{Im} f_i^k(\varphi_{|U_i})$. It remains to prove that $U = \operatorname{Ker} f_i^{d-k}(\varphi_{|U_i})$. We have

$$f_i^{d-k}(A_i)f_i^k(A_i) = f_i^d(A_i) = \mathbf{0},$$

where A_i is the matrix of $\varphi_{|U_i}$.

Since each column of $f_i^k(A_i)$ is a solution of the homogeneous system with matrix $f_i^{d-k}(A_i)$, then $U = \operatorname{Im} f_i^k(\varphi_{|U_i}) \subseteq \operatorname{Ker} f_i^{d-k}(\varphi_{|U_i})$. It is easy to verify that $\operatorname{Ker} f_i^{d-k}(\varphi_{|U_i}) = \operatorname{Ker} f_i^{d-k}(\varphi)$. Now suppose $u \in \operatorname{Ker} f_i^{d-k}(\varphi)$, i.e. $f_i^{d-k}(A)u = \mathbf{0}$. Then $u \in \operatorname{Ker} f_i^d(\varphi) = U_i$ and $u = g(A)u_i$ for a suitable polynomial $g(x) \in F[x]$. Hence $f_i^{d-k}(A)g(A)u_i = \mathbf{0}$. Since $f_i^d(x)$ is the minimal polynomial with the property $f_i^d(A)u_i = \mathbf{0}$ we conclude that $f_i^k(x)$ divides $g(x)$. Thus $g(x) \in J$ and $u \in U$, which proves the statement. $\square$

Proposition 2.1. *Let U be a φ-invariant subspace of F^n. Then U is a direct sum of subspaces of F^n of the form $\operatorname{Ker} f_i^{s_i}(\varphi)$, where $0 \le s_i \le d$.*

Proof: Let $\widetilde{U}_i = U \cap U_i$, $i = 1, \dots, t$. Then $\widetilde{U}_i = \operatorname{Ker} f_i^{s_i}(\varphi)$ for some $0 \le s_i \le d$. Therefore

$$U = U \cap F^n = U \cap (U_1 \oplus \cdots \oplus U_t) = \widetilde{U}_1 \oplus \cdots \oplus \widetilde{U}_t. \qquad \square$$

3. LINEAR CYCLIC CODES

Definition 3.1. *A code C with length n over F is called cyclic, if whenever $x = (c_1, c_2, \dots, c_n)$ is in C, so is its cyclic shift $y = (c_n, c_1, \dots, c_{n-1})$.*

The following statement is clear from the definition.

Proposition 3.1. *A linear code C with length n over F is cyclic iff C is a $\varphi-$invariant subspace of F^n.*

Theorem 3.1. *Let C be a linear cyclic code with length n over F. Then the following facts hold.*

1) $C = \widetilde{U}_{i_1} \oplus \cdots \oplus \widetilde{U}_{i_m}$ for some $\varphi-$invariant subspaces $\widetilde{U}_{i_r} = \operatorname{Ker} f_{i_r}^{s_r}(\varphi)$ of F^n, $0 < s_r \leq d$, and $\dim_F C = \sum_{r=1}^{m} s_r \deg f_{i_r} = k$;

2) $f_{\varphi|_C}(x) = (-1)^k f_{i_1}^{s_1}(x) \dots f_{i_m}^{s_m}(x) = g(x)$;

3) $c \in C$ iff $g(A)c = \mathbf{0}$;

4) the polynomial $g(x)$ has the smallest degree with the property 3);

5) $\mathrm{r}(g(A)) = n - k$.

Proof: 1) The first part of the statement follows from Proposition 2.1. Now we are going to show that $\dim_F \operatorname{Ker} f_{i_r}^{s_r} = s_r \deg f_{i_r}$. Let us consider the following chain of linear subspaces of F^n

$$\operatorname{Ker} f_{i_r}(\varphi) \subset \operatorname{Ker} f_{i_r}^2(\varphi) \subset \cdots \subset \operatorname{Ker} f_{i_r}^d(\varphi) = U_{i_r}.$$

Since the characteristic polynomial of the restriction of φ to $\operatorname{Ker} f_{i_r}^l(\varphi)$ divides $f_{\varphi|_{U_{i_r}}} = (-1)^{d \deg f_{i_r}} f_{i_r}^d$ for all $l = 1, \dots d$, then for the dimensions of the respective subspaces we obtain the following inequalities of natural numbers

$$l_1 \deg f_{i_r} < l_2 \deg f_{i_r} < \cdots < l_d \deg f_{i_r} = d \deg f_{i_r}.$$

Thus $l_i = i$ for $i = 1, \dots, d$, which proves the statement. In particular, it follows from the proof that $f_{\varphi|_{\widetilde{U}_{i_r}}}(x) = (-1)^{s_r deg f_{i_r}} f_{i_r}^{s_r}(x)$.

2) Let us denote $\alpha_{i_r} = \dim \widetilde{U}_{i_r} = s_r \deg f_{i_r}$. We choose a basis $u_1^{(i_r)}, \dots, u_{\alpha_{i_r}}^{(i_r)}$ of $\widetilde{U}_{i_r}$ over F, $r = 1, \dots, m$ and denote by B_{i_r} the matrix of $\varphi|_{\widetilde{U}_{i_r}}$ in that basis. Then $u_1^{(i_1)}, \dots, u_{\alpha_{i_1}}^{(i_1)}, \dots, u_1^{(i_m)}, \dots, u_{\alpha_{i_m}}^{(i_m)}$ is a basis of C over F and $\varphi|_C$ has a matrix

$$\begin{pmatrix} B_{i_1} & & & \\ & B_{i_2} & & \\ & & \ddots & \\ & & & B_{i_m} \end{pmatrix}$$

in that basis. Hence

$$f_{\varphi|_C}(x) = f_{\varphi|_{\widetilde{U}_{i_1}}}(x) \dots f_{\varphi|_{\widetilde{U}_{i_m}}}(x) = (-1)^k f_{i_1}^{s_1}(x) \dots f_{i_m}^{s_m}(x).$$

3) Let $c \in C$. Then $c = u_{i_1} + \cdots + u_{i_m}$ for some $u_{i_r} \in \widetilde{U}_{i_r}$, $r = 1, \dots, m$ and $g(A)c = (-1)^k[(f_{i_1}^{s_1} \dots f_{i_m}^{s_m})(A)u_{i_1} + \cdots + (f_{i_1}^{s_1} \dots f_{i_m}^{s_m})(A)u_{i_m}] = \mathbf{0}$.

Conversely suppose that $g(A)c = \mathbf{0}$ for some $c \in F^n$ and let $c = u_1 + \cdots + u_t$, $u_i \in U_i$. Then $g(A)c = (-1)^k[(f_{i_1}^{s_1} \ldots f_{i_m}^{s_m})(A)u_1 + \cdots + (f_{i_1}^{s_1} \ldots f_{i_m}^{s_m})(A)u_t] = \mathbf{0}$, so that $g(A)[u_{j_1} + \cdots + u_{j_l}] = \mathbf{0}$, where $\{j_1, \ldots j_l\} = \{1, \ldots, t\} \backslash \{i_1, \ldots, i_m\}$. Set $v_{j_r} = g(A)u_{j_r}$, for all $r = 1, \ldots, l$. Hence $v_{j_r} \in U_{j_r}$ and $v_{j_1} + \cdots + v_{j_l} = \mathbf{0}$. Therefore $v_{j_r} = \mathbf{0}$ for all $r = 1, \ldots l$. Since $(g, f_{j_r}^d) = 1$ there are polynomials $a(x), b(x) \in F[x]$, such that $a(x)g(x) + b(x)f_{j_r}^d(x) = 1$. Then $u_{j_r} = a(A)g(A)u_{j_r} + b(A)f_{j_r}^d(A)u_{j_r} = \mathbf{0}$. Thus $c = u_{i_1} + \cdots + u_{i_m} \in C$.

We omit the proofs of 4) and 5), since they are clear. □

Definition 3.2. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1 \ldots, y_n)$ be two vectors in F^n. We define an inner product over F by $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$. If $\langle x, y \rangle = 0$, we say that x and y are orthogonal to each other.*

Definition 3.3. *Let C be a linear code over F. We define the dual of C (which is denoted by $C^\perp$) to be the set of all vectors which are orthogonal to all codewords in C, i.e.*

$$C^\perp = \{v \in F^n \mid \langle v, c \rangle = 0 \; for \; all \, c \in C\}.$$

It is well known that if C is k-dimensional, then $C^\perp$ is $(n-k)$-dimensional. Besides the dual of a linear cyclic code is also cyclic.

Proposition 3.2. *The matrix H, which rows are arbitrary $n-k$ linearly independent rows of $g(A)$, is a parity check matrix of C.*

Proof: The proof follows from the equation $g(A)c = \mathbf{0}$ for every vector $c \in C$ and the fact that $\mathrm{r}\,(g(A)) = n - k$. □

Let us denote

$$h(x) = \frac{f(x)}{g(x)} = (-1)^{n-k} f_1^{d-s_1}(x) \ldots f_t^{d-s_t}(x),$$

where $0 \leq s_r \leq d$ for all $r = 1, \ldots, t$.

Let $g_{l_1}, \ldots, g_{l_{n-k}}$ be a basis of $C^\perp$, where g_{l_r} is a l_r-th vector row of $g(A)$. By the equation $g(A)h(A) = \mathbf{0}$ we obtain that $\langle g_{l_r}, h_i \rangle = 0$ for each $i = 1, \ldots, n$, $r = 1, \ldots, n-k$. The last equation gives us that the columns h_i of $h(A)$ are codewords in C.

We show that $\mathrm{r}\,(h(A)) = k$. By Sylvester's inequality we obtain that $\mathrm{r}\,(\mathbf{0}) = 0 \geq \mathrm{r}\,(g(A)) + \mathrm{r}\,(h(A)) - n$. Thus $\mathrm{r}\,(h(A)) \leq n - \mathrm{r}\,(g(A)) = n - (n-k) = k$.

On the other hand, Sylvester's inequality, applied to the product $h(A) = (-1)^{n-k} f_1^{d-s_1}(A) \ldots f_t^{d-s_t}(A)$, gives us that $\mathrm{r}\,(h(A)) \geq r(f_1^{d-s_1}(A)) + \cdots + r(f_t^{d-s_t}(A)) - n(t-1) = nt - d\sum_{i=1}^t \deg f_i + \sum_{i=1}^t s_i \deg f_i - nt + n = k$. Therefore $\mathrm{r}\,(h(A)) = k$. Thus we have proved the following:

Proposition 3.3. *The matrix G, which rows are arbitrary k linearly independent rows of $(h(A))^t$, is a generator matrix of the code C.*

Let $f_{\varphi|_{C^\perp}}(x) = \tilde{h}$. By Theorem 3.1 it follows that $\tilde{h}$ is the polynomial of the smallest degree such that $\tilde{h}(A)u = \mathbf{0}$ for every $u \in C^\perp$. Let $h^*(x) = \tilde{h}(x)q(x)+r(x)$, where $\deg r(x) < \deg \tilde{h}(x)$. Then $h^*(A) = A^{n-k}(h(A^t)) = \tilde{h}(A)q(A) + r(A)$, hence for every vector $u \in C^\perp$ the assertion $A^{n-k}(h(A))^t u = q(A)\tilde{h}(A)u + r(A)u$ holds, so that $r(x) = 0$. Thus $\tilde{h}(x)$ divides $h^*(x)$. Since both are polynomials of the same degree , $h^*(x) = a\tilde{h}(x)$, where $a \in F$ is the leading coefficient of the product $(f_1^*(x))^{d-s_1} \dots (f_t^*(x))^{d-s_t}$. Thus

$$\tilde{h} = \frac{1}{a}h^* = (-1)^{n-k}\frac{1}{a}(f_1^*(x))^{d-s_1} \dots (f_t^*(x))^{d-s_t} =$$

$$(-1)^{n-k}\prod_{i=1}^{t}\frac{1}{a_i}(f_i^*(x))^{d-s_i} = (-1)^{n-k}\prod_{i=1}^{t} f_{n_i}^{d-s_i}(x),$$

where a_i is the leading coefficient of $(f_i^*(x))^{d-s_i}$. Note that the polynomials $f_{n_i}(x)$ are monic irreducible and divide $f(x) = (-1)^n(x^n - 1)$.

Now we show that $C^\perp = \overline{U_{n_1}} \oplus \dots \oplus \overline{U_{n_t}}$, where $\overline{U_{n_i}} = \operatorname{Ker} f_{n_i}^{d-s_i}(\varphi)$. By Theorem 3.1 $C^\perp$ is the space of the solutions of the homogeneous system with matrix $\tilde{h}(A)$. Let $u \in U = \overline{U_{n_1}} \oplus \dots \oplus \overline{U_{n_t}}$ and let $u = u_{n_1} + \dots + u_{n_t}$ for $u_{n_r} \in U_{n_r}$, $r = 1, \dots, t$. Then

$$\tilde{h}(A)u = (-1)^{n-k}[(f_{n_1}^{d-s_1} \dots f_{n_t}^{d-s_t})(A)u_{n_1} + \dots + (f_{n_1}^{d-s_1} \dots f_{n_t}^{d-s_t})(A)u_{n_t}] = \mathbf{0}.$$

Hence $U \leq C^\perp$. Since $\dim_F U = \dim_F C^\perp$, then

$$C^\perp = \overline{U_{n_1}} \oplus \dots \oplus \overline{U_{n_t}}.$$

Thus we have proved the following:

Theorem 3.2. *Let $C = \widetilde{U}_1 \oplus \dots \oplus \widetilde{U}_t$ be a linear cyclic code over F, where $\widetilde{U}_i = \operatorname{Ker} f_i^{s_i}(\varphi)$, $0 \leq s_i \leq d$. Then the dual code of C is given by $C^\perp = \overline{U_{n_1}} \oplus \dots \oplus \overline{U_{n_t}}$ and $f_{\varphi|_{\overline{U_i}}}(x) = (-1)^{d-s_i}\frac{1}{a_i}(f_i^*(x))^{d-s_i} = (-1)^{d-s_i} f_{n_i}^{d-s_i}(x)$ where $(f_i^*(x))^{d-s_i}$ is the reciprocal polynomial of $f_i^{d-s_i}(x)$ with leading coefficient equals to a_i, $i = 1, \dots, t$.*

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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MEASURES RELATED TO OPERATOR IDEALS AND SPARR'S INTERPOLATION METHODS

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We establish an estimate for the outer measure $\gamma_I(T)$ and the inner measure $\beta_I(T)$ of an operator T acting between some intermediate spaces constructed for n-tuples of Banach spaces. We also show that many operator ideals have the strong interpolation property for Sparr's interpolation methods.

Keywords: Interpolation of compact operators, operator ideals

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1. INTRODUCTION

The behaviour of compact linear operators under real interpolation for Banach couples or Banach n-tuples has been extensively studied by many authors. The class of compact operators between Banach spaces is an injective surjective closed operator ideal in the sense of Pietsch [11]. It is therefore natural to investigate whether the similar results are valid for such ideals. There are two measures, $\gamma_I(T)$ (outer measure) and $\beta_I(T)$ (inner measure) of an operator $T \in \mathcal{L}(A, B)$, introduced, respectively, by Astala [1] and by Tylli [14], which show the deviation of T from the ideal I.

It is known that

if I is surjective and closed, then $\gamma_I(T_{A,B}) = 0$ if and only if $T \in I(A, B)$ (see [1])

and, analogously that

if I is injective and closed, then $\beta_I(T_{A,B}) = 0$ if and only if $T \in I(A,B)$ (see [14]).

Particularizing the operator ideal I, measures γ_I and β_I coincide with well-known notion. For example, when I is the ideal of compact operator k, $\gamma_k(T)$ is equal to the measure of non-compactness of T and $\beta_k(T)$ turns out to be the limit of the Gelfand numbers of T.

The behaviour under real interpolation of measures γ_I and β_I in the case of Banach couples has been pointed out by Cobos, Manzano and A. Martinez [3] and Cobos, Cwikel and Matos [2]. They have derived estimates for the measures γ_I and β_I provided that one of the couples degenerated into a single Banach space, or that the ideal I satisfied the so-called Σp-condition (see [8]), without assuming any condition on the Banach couples.

In this paper we establish an estimate for the measures $\gamma_I(T)$ and $\beta_I(T)$ of an operator T acting between some intermediate spaces constructed for n-tuples of Banach spaces. We consider here 4-tuples of Banach spaces and Sparr's interpolation method. We obtain similar results for $n \geq 5$.

We also show that weakly compact operators, Rosenthal operators, Banach-Saks operators and Radon-Nikodym operators have the strong interpolation property for Sparr's interpolation methods.

2. PRELIMINARIES

Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, that is to say, a family of 4 Banach spaces A_j all of them continuously embedded in a common linear Hausdorff space. If $\bar{t} = (t_1, t_2, t_3)$ and $\bar{s} = (s_1, s_2, s_3)$ are triples of positive numbers, we set

$$\bar{t}\bar{s} = (t_1 s_1, t_2 s_2, t_3 s_3), \quad 2^{\bar{t}} = (2^{t_1}, 2^{t_2}, 2^{t_3}), \quad |\bar{t}| = t_1 t_2 t_3$$

$$K(\bar{t}, a, \overline{A}) = \inf \left\{ \|a_0\|_{A_0} + \sum_{i=1}^{3} t_i \|a_i\|_{A_i},\ a = \sum_{i=0}^{3} a_i,\ a_i \in A_i \right\},$$

$$a \in \Sigma(\overline{A}) := A_0 + A_1 + A_2 + A_3$$

and

$$J(\bar{t}, a, \overline{A}) = \max_{1 \leq i \leq 3} \{\|a\|_{A_0},\ t_i \|a\|_{A_i}\},\ a \in \Delta(\overline{A}) := A_0 \cap A_1 \cap A_2 \cap A_3$$

If A and B are Banach spaces, we denote by $\mathcal{L}(A,B)$ the space of all bounded linear operators between A an B with the usual norm. Given two Banach 4-tuples $\overline{A} = (A_0, A_1, A_2, A_3)$, $\overline{B} = (B_0, B_1, B_2, B_3)$, we write $T \in \mathcal{L}(\overline{A}, \overline{B})$ or $T : \overline{A} \to \overline{B}$, meaning that T is linear operators from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j $(j = 0,1,2,3)$. For each $T \in \mathcal{L}(\overline{A}, \overline{B})$ we consider the norm:

$$\|T\|_{\overline{A},\overline{B}} := \max_{0 \leq i \leq 3} \{\|T\|_{A_i, B_i}\}.$$

If one of the 4-tuples $\overline{A}$ or $\overline{B}$ reduces to single Banach spaces, i.e. if $A_0 = A_1 = A_2 = A_3 = A$, or if $B_0 = B_1 = B_2 = B_3 = B$, then we write $T \in \mathcal{L}(A, \overline{B})$ or, respectively $T \in \mathcal{L}(\overline{A}, B)$.

A Banach space A is said to be an intermediate space with respect to $\overline{A}$ if

$$\Delta(\overline{A}) \hookrightarrow A \hookrightarrow \Sigma(\overline{A}),$$

where the notation $\hookrightarrow$ means continuous inclusion.

An intermediate space A is said to be interpolation for the 4-tuple $\overline{A}$ if, for all operators $T \in \mathcal{L}(\overline{A}, \overline{B})$, there exists a constant $C = C(A, \overline{A})$ such that

$$\|T\|_{A,A} \leq C\|T\|_{\overline{A},\overline{A}}.$$

If we consider only the one-dimensional operator T, i.e.

$$Tx = f(x)a, \quad a \in \Delta(\overline{A}), \quad f \in (\Sigma(\overline{A}))^*$$

then the space A is called party interpolation, or rank-one interpolation.

Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuples and let A be an intermediate space with respect to $\overline{A}$. For the triple $\overline{t} = (t_1, t_2, t_3)$ of positive numbers set

$$\psi(\overline{t}) = \psi(\overline{t}, A, \overline{A}) = \sup\left\{K(\overline{t}, a, \overline{A}) : \ \|a\|_A = 1\right\}$$

and

$$\rho(\overline{t}) = \rho(\overline{t}, A, \overline{A}) = \inf\left\{J(\overline{t}, a, \overline{A}) : \ a \in \Delta(\overline{A}), \ \|a\|_A = 1\right\}.$$

Proposition 2.1. [6] *Let $\overline{A}$ be a Banach 4-tuple and let A be an intermediate space with respect to $\overline{A}$. Then A is party interpolation space if and only if there exists a constant $C = C(A, \overline{A})$ such that*

$$\psi(\overline{t}) \leq C\rho(\overline{t}), \quad \text{for all } \overline{t} = (t_1, t_2, t_3) \in (0, \infty)^3.$$

Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple. Then $\Delta(\overline{A}) \hookrightarrow A_0$ and $A_0 \hookrightarrow \Sigma(\overline{A})$. We denote by A_0^0 the closure of $\Delta(\overline{A})$ in A_0 and by $\widetilde{A}_0$ the completion of A_0 with respect to $\Sigma(\overline{A})$ (or the Gagliardo completion of A_0 in $\Sigma(\overline{A})$).

Proposition 2.2. [6] *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple and let A be a party interpolation space with respect to $\overline{A}$*

(i) If $\lim\limits_{\overline{t}\to\overline{0}} \psi(\overline{t}) > 0$, then $A_0^0 \hookrightarrow A$

(ii) If $\lim\limits_{\overline{t}\to\overline{\infty}} (1/\rho(\overline{t})) > 0$, then $A \hookrightarrow \widetilde{A}_0$.

Let A be an intermediate space with respect to Banach 4-tuple $\overline{A}$. We say that A is of class $\mathcal{C}_K(\overline{\theta}, \overline{A})$, where $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in [0,1]^3$, $\theta_1 + \theta_2 + \theta_3 \leq 1$ if there is a constant c such that for all $\overline{t} = (t_1, t_2, t_3) \in (o, \infty)^3$ and $a \in A$

$$|\overline{t}^{-\overline{\theta}}|K(\overline{t}, a, \overline{A}) \leq C\|a\|_A$$

and of class $\mathcal{C}_J(\overline{\theta}, \overline{A})$ if there is a constant C such that for all $\overline{t} = (t_1, t_2, t_3) \in (0, \infty)^3$ and $a \in \Delta(\overline{A})$

$$\|a\|_A \le C|\overline{t}^{-\overline{\theta}}|J(\overline{t}, a, \overline{A}).$$

An important example of spaces of class $\mathcal{C}_K(\overline{\theta}, \overline{A})$ is the real interpolation K-space $\overline{A}_{\overline{\theta},p,K} := (A_0, A_1, A_2, A_3)_{\overline{\theta},p,K}$ (or Sparr's K-space). We remind that for $1 \le p \le \infty$ and $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ the space $\overline{A}_{\overline{\theta},p,K}$ consists of all $a \in \Sigma(\overline{A})$, which have a finite norm:

$$\|a\|_{\overline{\theta},p,K} = \begin{cases} \left(\sum\limits_{\overline{n} \in \mathbb{Z}^3} (|2^{-\overline{n}\overline{\theta}}| K(2^{\overline{n}}, a, \overline{A}))^p \right)^{1/p} & \text{if } 1 \le p < \infty \\ \sup\limits_{\overline{n} \in \mathbb{Z}^3} \{ |2^{-\overline{n}\overline{\theta}}| K(2^{\overline{n}}, a, \overline{A}) \} & \text{if } p = \infty. \end{cases}$$

On the other hand, the real interpolation J-space $\overline{A}_{\overline{\theta},p,J} := (A_0, A_1, A_2, A_3)_{\overline{\theta},p,J}$ (or Sparr's J-space) is an important example of space of class $\mathcal{C}_J(\overline{\theta}, \overline{A})$. We remind that for $1 \le p \le \infty$ and $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ the space $\overline{A}_{\overline{\theta},p,J}$ consists of all $a \in \Sigma(\overline{A})$ which can be represented in the form:

$$a = \sum_{\overline{n} \in \mathbb{Z}^3} u_{\overline{n}} \qquad (\text{convergence in } \Sigma(\overline{A}))$$

with $(u_{\overline{n}})_{\overline{n} \in \mathbb{Z}^3} \subset \Delta(\overline{A})$ and

$$\sum_{\overline{n} \in \mathbb{Z}^3} (|2^{-\overline{n}\overline{\theta}}| J(2^{\overline{n}}, u_{\overline{n}}, \overline{A}))^p < \infty.$$

(The sum should be replaced by supremum if $p = \infty$.)
Moreover

$$\|a\|_{\overline{\theta},p,J} = \inf \left\{ \left(\sum_{\overline{n} \in \mathbb{Z}^3} (|2^{-\overline{n}\overline{\theta}}| J(2^{\overline{n}}, u_{\overline{n}}, \overline{A}))^p \right)^{1/p} : a = \sum_{\overline{n}} u_{\overline{n}} \right\}$$

defines a norm on $\overline{A}_{\overline{\theta},p,J}$.

An operator ideal I is any subclass of the class $\mathcal{L}$ of all bounded linear operators between arbitrary Banach spaces such that the components $I(A, B) = I \cap \mathcal{L}(A, B)$ satisfy the following conditions:

(i) $I(A, B)$ is a linear subspace of $\mathcal{L}(A, B)$;

(ii) $I(A, B)$ contains the finite rank operators;

(iii) if $R \in \mathcal{L}(X, A)$, $T \in I(A, B)$ and $S \in \mathcal{L}(B, Y)$ then $STR \in I(X, Y)$.

The operator ideal I is injective if for every isomorphic embedding $J \in \mathcal{L}(B, Y)$ one has that $T \in \mathcal{L}(A, B)$ and $JT \in I(A, Y)$ imply $T \in I(A, B)$, it is surjective if for every surjection $Q \in \mathcal{L}(X, A)$ we have $T \in \mathcal{L}(A, B)$ and $TQ \in I(X, B)$ imply $T \in I(A, B)$. The ideal I is closed if the components $I(A, B)$ are closed subspaces of $\mathcal{L}(A, B)$ (see Pietsch [11]).

The outer measure of $T \in \mathcal{L}(A, B)$ is denoted by $\gamma_I(T) = \gamma_I(T_{A,B})$ and is the infimum of all positive numbers σ such that $T(U_A) \subset \sigma U_B + R(U_Z)$ for some Banach space Z and some operators $R \in I(Z, B)$ (where U_X denotes the closed unit ball of X). The inner measure of the same operator is denoted by $\beta_I(T) = \beta_I(T_{A,B})$ and is the infimum of all positive numbers σ such that for some Banach space Z and some operators $R \in I(A, Z)$ the inequality

$$\|Tx\|_B \leq \sigma \|x\|_A + \|Rx\|_Z$$

holds for all $x \in A$.

The ideal I possesses the strong interpolation property for a method $\mathcal{F}$ of interpolation if the interpolated operator $T_{\mathcal{F}} : \mathcal{F}(\overline{A}) \to \mathcal{F}(\overline{B})$ belongs to I when the induced operators $T_{IS} : \Delta(\overline{A}) \to \Sigma(\overline{B})$ is in I.

3. ESTIMATES FOR THE OUTER MEASURE $\gamma_I(T)$

In this section we establish an estimate for the measure $\gamma_I(T)$ when one of the Banach 4-tuples reduces to a single Banach space.

Theorem 3.1. *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, let A be an intermediate space with respect to $\overline{A}$, and let B be an arbitrary Banach space. Let I be an operator ideal. Then for each $T \in \mathcal{L}(\overline{A}, B)$ we have*

$$\gamma_I(T) \leq \psi(t_1, t_2, t_3) \left[\gamma_I(T_{A_0,B}) + \frac{1}{t_1}\gamma_I(T_{A_1,B}) + \frac{1}{t_2}\gamma_I(T_{A_2,B}) + \frac{1}{t_3}\gamma_I(T_{A_3,B})\right] \tag{3.1}$$

for all $t_1, t_2, t_3 > 0$.

Proof. In view of the definition of $\gamma_I(T_{A_i,B})$, for $i = 0, 1, 2, 3$ and for each $\varepsilon > 0$ there exist Banach spaces Z_i and operators $S_i \in I(Z_i, B)$ such that

$$T(U_{A_i}) \subset (\varepsilon + \gamma_I(T_{A_i,B}))U_B + S_i(U_{Z_i}) \tag{3.2}$$

Now, consider an arbitrary element $a \in U_A$ and fixed positive numbers t_1, t_2, t_3 and δ. Since

$$K(t_1, t_2, t_3, a, \overline{A}) < \delta + \psi(t_1, t_2, t_3) \tag{3.3}$$

there exists a decomposition $a = a_0 + a_1 + a_2 + a_3$, with $a_i \in A_i$ $(i = 0, 1, 2, 3)$ such that

$$\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + t_2\|a_2\|_{A_2} + t_3\|a_3\|_{A_3} < \delta + \psi(t_1, t_2, t_3).$$

Thus $a_0 \in (\delta + \psi(t_1, t_2, t_3))U_{A_0}$, $a_1 \in t_1^{-1}(\delta + \psi(t_1, t_2, t_3))U_{A_1}$, $a_2 \in t_2^{-1}(\delta + \psi(t_1, t_2, t_3))U_{A_2}$ and $a_3 \in t_3^{-1}(\delta + \psi(t_1, t_2, t_3))U_{A_3}$.

From this and (3.2) it follows that

$$
\begin{aligned}
T(U_A) &\subset (\delta+\psi(t_1,t_2,t_3))\left[T(U_{A_0})+\frac{1}{t_1}T(U_{A_1})+\frac{1}{t_2}T(U_{A_2})+\frac{1}{t_3}T(U_{A_3})\right]\\
&\subset (\delta+\psi(t_1,t_2,t_3))\left[\varepsilon+\gamma_I(T_{A_0,B})+\frac{\varepsilon}{t_1}+\frac{1}{t_1}\gamma_I(T_{A_1,B})+\frac{\varepsilon}{t_2}+\right.\\
&+ \left.\frac{1}{t_2}\gamma_I(T_{A_2,B})+\frac{\varepsilon}{t_3}+\frac{1}{t_3}\gamma_I(T_{A_3,B})\right]U_B+S_0'(U_{Z_0})+S_1'(U_{Z_1})+\\
&+ S_2'(U_{Z_2})+S_3'(U_{Z_3})
\end{aligned}
$$

where $S_0'=(\delta+\psi(t_1,t_2,t_3))S_0$, $S_1'=\frac{1}{t_1}(\delta+\psi(t_1,t_2,t_3))S_1$, $S_2'=\frac{1}{t_2}(\delta+\psi(t_1,t_2,t_3))S_2$ and $S_3'=\frac{1}{t_3}(\delta+\psi(t_1,t_2,t_3))S_3$ are operators belonging to $I(Z_0,B)$, $I(Z_1,B)$, $I(Z_2,B)$ and $I(Z_3,B)$, respectively. Let Z be the Banach space $Z=Z_0\oplus Z_1\oplus Z_2\oplus Z_3$ with norm $\|(x,y,z,w)\|=\max(\|x\|_{Z_0},\|y\|_{Z_1},\|z\|_{Z_2},\|w\|_{Z_3})$ and define $S:Z\to B$ by $S(x,y,z,w)=S_0'x+S_1'y+S_2'z+S_3'w$. Then $S(U_Z)=S_0'(U_{Z_0})+S_1'(U_{Z_1})+S_2'(U_{Z_2})+S_3'(U_{Z_3})$ and using the ideal properties of I and the projection operators from Z onto Z_i, $i=0,1,2,3$, we have $S\in I(Z,B)$. Consequently

$$
\gamma_I(T_{A,B})\leq\psi(t_1,t_2,t_3)\left[\gamma_I(T_{A_0,B})+\frac{1}{t_1}\gamma_I(T_{A_1,B})+\frac{1}{t_2}\gamma_I(T_{A_1,B})+\frac{1}{t_3}\gamma_I(T_{A_3,B})\right].\square
$$

Corollary 3.2. *If A is a Banach space of class $\mathcal{C}_K(\overline{\theta},\overline{A})$ with constant C, for some $\overline{\theta}=(\theta_1,\theta_2,\theta_3)\in(0,1)^3$, $\theta_1+\theta_2+\theta_3<1$ then*

$$
\begin{aligned}
\gamma_I(T_{A,B}) &\leq C(1-\theta_1-\theta_2-\theta_3)^{\theta_1+\theta_2+\theta_3-1}\theta_1^{-\theta_1}\theta_2^{-\theta_2}\theta_3^{-\theta_3}\gamma_I(T_{A_0,B})^{1-\theta_1-\theta_2-\theta_3}\\
&\cdot \gamma_I(T_{A_1,B})^{\theta_1}\gamma_I(T_{A_2,B})^{\theta_2}\gamma_I(T_{A_3,B})^{\theta_3}
\end{aligned}
\tag{3.4}
$$

Proof. Let $\sigma_i>\gamma_I(T_{A_i,B})$, $i=0,1,2,3$. By the definition of $\gamma_I(T_{A_i,B})$ for $i=0,1,2,3$ and for each $\varepsilon>0$ there exist Banach spaces Z_i and operators $S_i\in I(Z_i,B)$ such that

$$
T(U_{A_i})\subset\sigma_iU_B+S_i(U_{Z_i}).
$$

Since $A\in\mathcal{C}_K(\overline{\theta},\overline{A})$, given any $\varepsilon>0$, $t_1>0$, $t_2>0$, $t_3>0$ and $a\in A$ with $\|a\|_A\leq 1$ we can find $a_i\in A_i$, $i=0,1,2,3$, so that $a=a_0+a_1+a_2+a_3$ and $\|a_0\|_{A_0}\leq(1+\varepsilon)Ct_1^{\theta_1}t_2^{\theta_2}t_3^{\theta_3}$, $\|a_1\|_{A_1}\leq(1+\varepsilon)Ct_1^{\theta_1-1}t_2^{\theta_2}t_3^{\theta_3}$, $\|a_2\|_{A_2}\leq(1+\varepsilon)Ct_1^{\theta_1}t_2^{\theta_2-1}t_3^{\theta_3}$, $\|a_3\|_{A_3}\leq(1+\varepsilon)Ct_1^{\theta_1}t_2^{\theta_2}t_3^{\theta_3-1}$. The proof proceeds now in the same way as in Theorem 3.1 to obtain the inequality

$$
\gamma_I(T_{A,B}) \leq C\inf_{t_1>o,t_2>0,t_3>0}t_1^{\theta_1}t_2^{\theta_2}t_3^{\theta_3}\left[\gamma_I(T_{A_0,B})+\frac{1}{t_1}\gamma_I(T_{A_1,B})+\frac{1}{t_2}\gamma_I(T_{A_2,B})\right.
$$

$$+ \quad \frac{1}{t_3}\gamma_I(T_{A_3,B})\Bigg].$$

This inequality implies the result. □

Corollary 3.3. *If I is a surjective closed operator ideal and $T \in \mathcal{L}(\overline{A}, B)$ is such that for some i, say $i = 0$, $T \in I(A_0, B)$ then $T \in I(\overline{A}_{\overline{\theta},p,K}, B)$.*

Proof. Since $\gamma_I(T) = 0$ if and only if $T \in I$ and $\overline{A}_{\overline{\theta},p,K}$ is of class $\mathcal{C}_K(\overline{\theta}, \overline{A})$, form (3.4) the result follows. □

Remark 3.4. When $I = K$ the ideal of compact operators, $\gamma_K(T)$ coincides with the measures of non-compactness of T, so we recover well-known Lions' and Peetre's compactness results [see [9], [4], [6], [10]).

Theorem 3.5. *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, let A be a party interpolation respect to $\overline{A}$, and let B be another Banach space. Let I be a surjective closed operator ideal and $T \in \mathcal{L}(\overline{A}, B)$ such that $T \in I(A_i, B)$, $i = 1, 2, 3$. Then at least one of the following conditions must hold*

(i) $T \in I(A, B)$;

(ii) $A_0^0 \hookrightarrow A$.

Proof. Since $\gamma_I(T_{A_i,B}) = 0$, $(i = 1, 2, 3)$ from (3.1) we have

$$\gamma_I(T_{A,B}) \leq \gamma_I(T_{A_0,B}) \varliminf_{\overline{t}\to\overline{0}} \psi(t_1, t_2, t_3, A, \overline{A}).$$

Consequently, either $T \in I(A, B)$ (i.e. $\gamma_I(T_{A,B}) = 0$), or, alternatively, $\varliminf_{\overline{t}\to\overline{0}} \psi(t_1, t_2, t_3, A, \overline{A}) > 0$, which, by Proposition 2.3, implies that $A_0^0 \hookrightarrow A$. □

4. ESTIMATES FOR THE INNER MEASURE $\beta_I(T)$

In this section we establish an estimate for the measure $\beta_I(T)$ when one of the Banach 4-tuples reduces to a single Banach space.

Theorem 4.1. *Let $\overline{B} = (B_0, B_1, B_2, B_3)$ be a Banach 4-tuple, let B be an intermediate space with respect to $\overline{B}$ and let A be an arbitrary Banach space. Let I be an operator ideal. Then, for each $T \in \mathcal{L}(A, \overline{B})$ we have*

$$\beta_I(T) \leq \frac{1}{\rho(t_1, t_2, t_3)} \max\{\beta_I(T_{A,B_0}), t_1\beta_I(T_{A,B_1}), t_2\beta_I(T_{A,B_2}), t_3\beta_I(T_{A,B_3})\} \quad (4.1)$$

for all $t_1 > 0$, $t_2 > 0$, $t_3 > 0$.

Proof. By the definition of $\beta_I(T_{A,B_i})$, for each $\varepsilon > 0$ there exist Banach spaces Z_i and operators $S_i \in I(A, Z_i)$ such that

$$\|Ta\|_{B_i} \leq (\varepsilon + \beta_I(T_{A,B_i}))\|a\|_A + \|S_i a\|_{Z_i}, \quad \text{for all } a \in A,\ i = 0,1,2,3. \tag{4.2}$$

Since $\|b\|_B \leq \dfrac{J(t_1,t_2,t_3,b,\overline{B})}{\rho(t_1,t_2,t_3)}$ for all $b \in \Delta(\overline{B})$ and all $t_1 > 0$, $t_2 > 0$, $t_3 > 0$ and using (4.2) we obtain:

$$\begin{aligned}
\|Ta\|_B &\leq \frac{1}{\rho(t_1,t_2,t_3)} \max(\|Ta\|_{B_0}, t_1\|Ta\|_{B_1}, t_2\|Ta\|_{B_2}, t_3\|Ta\|_{B_3}) \\
&\leq \frac{1}{\rho(t_1,t_2,t_3)} \max[\varepsilon + \beta_I(T_{A,B_0}), t_1(\varepsilon + \beta_I(T_{A,B_1})), t_2(\varepsilon + \beta_I(T_{A,B_2})), \\
&\qquad t_3(\varepsilon + \beta_I(T_{A,B_3}))]\|a\|_A + \|S_0'a\|_{Z_0} + \|S_1'a\|_{Z_1} + \|S_2'a\|_{Z_2} + \|S_3'a\|_{Z_3}
\end{aligned}$$

where $S_0' = \dfrac{S_0}{\rho(t_1,t_2,t_3)}$, $S_i' = \dfrac{t_i S_i}{\rho(t_1,t_2,t_3)}$, $(i = 1,2,3)$

Let Z be the Banach space $Z = Z_0 \oplus Z_1 \oplus Z_2 \oplus Z_3$ with the norm $\|(x,y,z,w)\| = \|x\|_{Z_0} + \|y\|_{Z_1} + \|z\|_{Z_2} + \|w\|_{Z_3}$ and let the operator $S : A \to Z$ defined by $Sa = (S_0'a, S_1'a, S_2'a, S_3'a)$. Using the ideal properties of I and the canonical embeddings of Z_i into Z $(i = 0,1,2,3)$ we have $S \in I(A,Z)$ and

$$\begin{aligned}
\|Ta\|_B &\leq \frac{1}{\rho(t_1,t_2,t_3)} \max[\varepsilon + \beta_I(T_{A,B_0}), t_1(\varepsilon + \beta_I(T_{A,B_1})), t_2(\varepsilon + \beta_I(T_{A,B_2})), \\
&\qquad t_3(\varepsilon + \beta_I(T_{A,B_3}))]\|a\|_A + \|Sa\|_Z, \text{ for all } a \in A.
\end{aligned}$$

This implies (4.1). □

Corollary 4.2. *If* B *is a Banach space of class* $\mathcal{C}_J(\overline{\theta}, \overline{B})$ *with constant* C, *for some* $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ *and* $\beta_I(T_{A,B_i}) > 0$, $(i = 0,1,2,3)$ *then*

$$\beta_I(T_{A,B}) \leq C\beta_I(T_{A,B_0})^{1-\theta_1-\theta_2-\theta_3}\beta_I(T_{A,B_1})^{\theta_1}\beta_I(T_{A,B_2})^{\theta_2}\beta_I(T_{A,B_3})^{\theta_3}. \tag{4.3}$$

Proof. Since $B \in \mathcal{C}_J(\overline{\theta}, \overline{B})$ and using (4.2) we obtain the inequality

$$\begin{aligned}
\|Ta\|_B &\leq Ct_1^{-\theta_1}t_2^{-\theta_2}t_3^{-\theta_3} \max\{\beta_I(T_{A,B_0}), t_1\beta_I(T_{A,B_1}), t_2\beta_I(T_{A,B_2}), \\
&\qquad t_3\beta_I(T_{A,B_3})\}\|a\|_A + \|Sa\|_Z, \text{ for all } a \in A \text{ and } t_1 > 0, t_2 > 0, t_3 > 0.
\end{aligned}$$

Taking $t_i = \dfrac{\beta_I(T_{A,B_0})}{\beta_I(T_{A,B_i})}$ $(i = 1, 2, 3)$, we get (4.3). □

Corollary 4.3. *If i is an injective closed operator ideal and $T \in \mathcal{L}(A, \overline{B})$ is such that for some i, say $T \in I(A, B_0)$, $T \in I(A, \overline{B}_{\overline{\theta},p,J})$.*

Proof. Since $\beta_I(T) = 0$ if and only if $T \in I$ and $\overline{B}_{\overline{\theta},p,J}$ is of class $\mathcal{C}_J(\overline{\theta}, \overline{A})$, the result follows from (4.3). □

Theorem 4.4. *Let $\overline{B} = (B_0, B_1, B_2, B_3)$ be a Banach 4-tuple, let B be a party interpolation space with respect to $\overline{B}$ and let A be another Banach space. Let I be an injective closed operator ideal and $T \in \mathcal{L}(A, \overline{B})$ such that $T \in I(A, B_i)$, $(i = 1, 2, 3)$. Then at least one of the following conditions must hold:*
(i) $T \in I(A, B)$,
(ii) $B \hookrightarrow \widetilde{B}_0$.

Proof. Since $\beta_I(T_{A,B_i}) = 0$, $(i = 1, 2, 3)$ from (4.1) we have

$$\beta_I(T_{A,B}) \leq \beta_I(T_{A,B_0}) \lim_{\bar{t}\to\overline{\infty}} \frac{1}{\rho(t_1, t_2, t_3)}.$$

Consequently, either $T \in I(A, B)$ (i.e. $\beta_I(T_{A,B}) = 0$), or, alternatively $\lim_{\bar{t}\to\overline{\infty}} \dfrac{1}{\rho(t_1, t_2, t_3)} > 0$, which, by Proposition 2.2, implies $B \hookrightarrow \widetilde{B}_0$. □

5. THE STRONG INTERPOLATION PROPERTY

In this section we show that many classes of operators ideals possess the strong interpolation property with respect to Sparr's interpolation method. To obtain the strong interpolation property for the ideal I (without assuming any condition on the Banach 4-tuples), we require the operator ideal I to satisfy the so-called $\sum p$-condition (which was introduced by Heinrich [8]).

Given any sequence of Banach spaces $(E_{\overline{m}})_{\overline{m}\in\mathbb{Z}^3}$ we denote by $l_p(E_{\overline{m}})$ the vector-valued l_p space defined by

$$l_p(E_{\overline{m}}) = \left\{ x = (x_{\overline{m}}) : x_{\overline{m}} \in E_{\overline{m}} \text{ and } \|x\|_{l_p(E_{\overline{m}})} = \Big(\sum_{\overline{m}\in\mathbb{Z}^3} (\|x_{\overline{m}}\|_{E_{\overline{m}}})^p\Big)^{1/p} < \infty \right\}.$$

Denote by $Q_{\overline{k}} : l_p(E_{\overline{m}}) \to E_{\overline{k}}$ the projection $Q_{\overline{k}}(x_{\overline{m}}) = x_{\overline{k}}$ and by $I_{\overline{n}} : E_{\overline{n}} \to l_p(E_{\overline{m}})$ the natural (isometric) embedding $I_{\overline{n}}y = \big(\delta^{\overline{n}}_{\overline{m}}y\big)$, where $\delta^{\overline{n}}_{\overline{m}}$ is the Kronecker symbol.

The operator ideal I satisfies the $\sum p$-condition if for any two sequences $(E_{\overline{m}})_{\overline{m}\in\mathbb{Z}^3}$ and $(F_{\overline{m}})_{\overline{m}\in\mathbb{Z}^3}$ of Banach spaces the following holds:

if $T \in \mathcal{L}(l_p(E_{\overline{m}}), l_p(F_{\overline{m}}))$ and $Q_{\overline{k}} T I_{\overline{n}} \in I(E_{\overline{n}}, F_{\overline{k}})$ for any $\overline{n}, \overline{k} \in \mathbb{Z}^3$, then $T \in I(l_p(E_{\overline{m}}), l_p(F_{\overline{m}}))$.

Theorem 5.1. *Let $1 < p < \infty$, $\overline{\theta} = (\theta_1, \theta_2, \theta_3) \in (0,1)^3$, $\theta_1 + \theta_2 + \theta_3 < 1$ and let I be a closed injective and surjective operator ideal which satisfies the $\sum p$-condition. Suppose that $\overline{A}, \overline{B}$ are Banach 4-tuples and $T \in \mathcal{L}(\overline{A}, \overline{B})$. Let T_{IS} be the induced operator from $\Delta(\overline{A})$ into $\sum(\overline{B})$. If $T_{IS} \in I(\Delta(\overline{A}), \sum(\overline{B}))$ then $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$.*

Proof. Define on $\Delta(\overline{A})$ and $\sum(\overline{B})$ the following equivalent norms

$$\|a\|_{\overline{m}} = 2^{-\theta_1 m_1 - \theta_2 m_2 - \theta_3 m_3} J(2^{m_1}, 2^{m_2}, 2^{m_3}, a, \overline{A}), a \in \Delta(\overline{A}), \overline{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$$

$$\|b\|_{\overline{m}} = 2^{-\theta_1 m_1 - \theta_2 m_2 - \theta_3 m_3} K(2^{m_1}, 2^{m_2}, 2^{m_3}, b, \overline{B}), b \in \Sigma(\overline{B}), \overline{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$$

Denote by $A_{\overline{m}}$ the space $(\Delta(\overline{A}), \|\cdot\|_{\overline{m}})$ and by $B_{\overline{m}}$ the space $(\sum(\overline{B}), \|\cdot\|_{\overline{m}})$. In view of the definition of the space $\overline{A}_{\overline{\theta},p,J}$ there is a surjection Q from $l_p(A_{\overline{m}})$ onto $A_{\overline{\theta},p,J}$ defined by

$$Q((x_{\overline{m}})_{\overline{m}}) = \sum_{\overline{m} \in \mathbb{Z}^3} x_{\overline{m}} \quad (\text{convergence in } \textstyle\sum(\overline{A})).$$

By the definition of the space $\overline{B}_{\overline{\theta},p,K}$ there is an (isomorphic) embedding J from $\overline{B}_{\overline{\theta},p,K}$ into $l_p(B_{\overline{m}})$ defined by

$$J(y) = (\ldots y, y, y, \ldots)$$

Then the operator $Q_{\overline{k}} J T Q J_{\overline{n}}$ is the operator T_{IS} from $A_{\overline{n}} = \Delta(\overline{A})$ into $B_{\overline{k}} = \sum(\overline{B})$. So, it is an operator of the class I. Since I satisfies the $\sum p$-condition the operator JTQ belongs to $I(l_p(A_{\overline{m}}), l_p(B_{\overline{m}}))$. Now, the injectivity and surjectivity of I implies $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$. □

Corollary 5.2. *Let $T \in \mathcal{L}(\overline{A}, \overline{B})$ such that for some i, say $i = 0$, $T \in I(A_0, B_0)$, then $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$.*

Proof. Using the commutative diagram

$$\begin{array}{ccc}
\Delta(\overline{A}) & \xrightarrow{T_{IS}} & \sum(\overline{B}) \\
\downarrow & & \uparrow \\
A_0 & \xrightarrow{T} & B_0
\end{array}$$

and the ideal properties of I, we obtain $T_{IS} \in I(\Delta(\overline{A}), \sum(\overline{B}))$. Consequently $T \in I(\overline{A}_{\overline{\theta},p,J}, \overline{B}_{\overline{\theta},p,K})$. □

Remark 5.3. For the case of Banach couples Heinrich [8] has proved rezults like those from theorem 5.1. He has also shown that weakly compact operators, Rosenthal operators, Banach-Saks operators and dual Radon-Nikodym operator satisfy the $\sum p$-condition, for $1 < p < \infty$ (these operators ideals are also injective, surjective and closed) but the above condition is not satisfied by the compact operators. So, Theorem 5.1 does not apply to compact operators, though we have a similar result as in Corollary 5.2 for compact operators (see [4], [6]).

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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ACCURACY OF THE PLANAR COMPLIANT MECHANISMS

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In the recent years a class of devices called compliant mechanisms is in the focus of many investigations. Their use in the design of modern devices, especially in micro-electro-mechanical systems (MEMS), is inevitable because of the difficulty in fabricating rigid-body joints and assembling parts. Compliant mechanisms rely upon elastic deformation to perform their function of transmitting and/or transforming motion and force. Flexural pivot-based designs use narrow sections connecting relatively rigid segments. Thus, compliance is lumped to a few portions of the mechanism. The introduction of the elastic pivots instead of the rigid-body joints leads to certain deviations in the performance of the compliant mechanisms compared with the analogous rigid-body linkages. These deviations are the object of study in the paper. Based on the graph theory, a method for effective estimation of the accuracy of compliant mechanisms with flexural pivots is elaborated and practical examples are considered.

Keywords: Accuracy, Compliant Mechanisms, Flexure Hinges.

2000 MSC: 70B15

1. INTRODUCTION

The definition of *compliant mechanisms* can be found in the literature, based either on the output motion, or on their design. Compliant mechanisms derive a part or whole of the relative motion between its members from intentional elastic deformation of the members rather than from conventional rigid body kinematic pairs alone [1]. A compliant mechanism can be also defined, as a single-piece flexible structure that delivers the desired motion by undergoing elastic deformation as opposed to the rigid body motions in a conventional mechanism [2].

Such mechanisms may be considered for use in a particular application for a variety of reasons. The advantages of compliant mechanisms are considered in two categories: cost reduction (part-count reduction, reduced assembly time, and simplified manufacturing processes) and increased performance (increased precision, increased reliability, reduced wear, reduced weight, and reduced maintenance).

Generally, the categories of compliant mechanisms can be divided into three kinds:

— Fully compliant mechanisms.
— Compliant mechanisms in which only the joints are compliant.
— Compliant mechanisms in which only the links are compliant.

Our interest is in the second one, in which the flexure hinges (flexure pivots) act as of joints. A flexure hinge is a thin member that provides the relative rotation between two adjacent rigid members through flexing (bending) where a conventional rotational joint is compared to a flexure hinge [3]. Flexure hinge is a typical simple and ingenious mechanical structure. Being made up of a monolithic material, it possesses many outstanding properties which ordinary hinge does not have, and can satisfies the demands for high accuracy and stability measurement and movement [4].

The flexure hinges are incorporated in a large number of applications, both civil and military, including translation micro-positioning stages, piezoelectric actuators and motors, high-accuracy alignment devices for optical fibers, missile-control devices, displacement amplifiers, robotic micro-displacement mechanisms and so on. Recently, increasing applications of monolithic flexure hinge mechanism have been made to guide motions with precision. Micro-motion stages utilizing the flexure hinge mechanism can have many advantages: negligible backlash and stick-slip friction, smooth and continuous displacement, adequate for magnifying the output displacement of actuation, and inherently infinite resolution [5].

One kind of the flexure hinges is called super elastic hinges. These hinges are made of a super elastic material such as shape memory alloy (SMA) having an effect of super elasticity, so that they have the capacity to perform large bending displacements [6].

Generally, the accuracy of a mechanical system is the quality of the system characterizing closeness of the results of the execution of certain operations by the mechanical system to the result of the execution of the same operations by the ideal mechanical system. In this paper the performance of mechanism with super elastic hinges is compared with the performance of mechanism with normal joints, considered as an ideal system. A mathematical model and compact analytical expressions allowing the exact estimation of the deflections in link positions of the mechanism with super elastic hinges are presented. Widely used mechanisms are considered as examples for application of the theory presented.

2. MATRIX DESCRIPTION OF THE INTERCONNECTION STRUCTURE

Let us consider a planar mechanism consisting of $n+1$ links interconnected by m rotational hinges. We replace each rotational hinge by a super elastic plate and in this way we arrive to a mechanism with compliance for which the interconnection structure of the links is the same but the hinges are not more rotational pairs. The deviation in the position of an arbitrary link of the new (compliant) mechanism in the absolute plane with respect to the position of the same link of the primary (rigid) mechanism is the object of study in the paper.

There are always two basic links in each real mechanism: the stationary base (fixed link or frame) and another link, which plays a special role in the mechanism and performs a preliminary given motion. This is the motion for which the mechanism is actually designed. This link is called characteristic link. In the formalism developed further each link can be considered as a characteristic one if its motion is of special interest. The fixed link will be considered as a link number 0 and the characteristic link gets number i^*. The fixed link together with the characteristic link determine the basic open chain (possibly not the only one) in the mechanism. This basic chain is unambiguously determined for some mechanisms like industrial robots and manipulators but for others the basic chain may be chosen under some possibilities. The links belonging to the basic chain get numbers $1, 2, \ldots, i^*$ starting from the link next to the fixed link.

We represent the mechanism structure by a graph, whose vertices s_i $(i = 0, 1, \ldots, n)$ and edges u_a $(a = 1, \ldots, m)$ symbolize respectively the links and the hinges of the system. The labeling of the links and vertices, as well as the hinges and edges is identical and it will be clear from the context when there is a question of link or vertex, respectively of hinge or edge. We are talking about rotational pair and more generally about hinge when two links ar interacting directly, i.e. each rotational pair (hinge) connects exactly two links. The three links given in Fig. 1 sharing one rotational axis define in this way two rotational pairs.

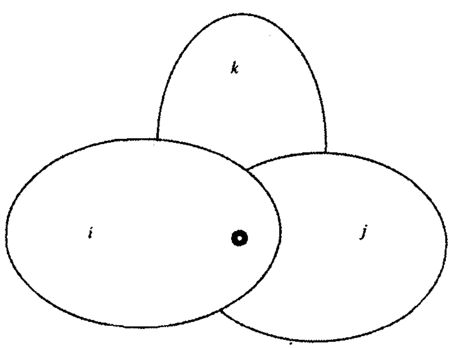

Fig. 1

The edge of the basic chain, which is incident with the vertex number i ($i = 1, \ldots, i^*$) gets the same number. The system graph is generally an arbitrary graph and its transformation into a graph with a tree-like structure (so called *skeleton tree*) can be reduced to the removing of $\hat{n} = m - n$ appropriately selected edges from the graph. We assume that the removed edges do not belong to the basic chain. In the tree obtained each pair of vertices is connected with one and only one simple chain in which every vertex appears only one time. We label the vertices in such a way, that the numbers of the vertices belonging to each simple chain beginning from the vertex s_0 form a monotonously increasing sequence. Such labeling is called *regular*. In this labeling the numbers from 1 to n are assigned to the edges of the skeleton tree in such a way that one of the two vertices connected by the edge number a has the same number $i = a$ and besides, this edge belongs to the simple chain connecting s_0 with s_i. The nonsceleton edges get numbers from $(n+1)$ to m. The simple chains which connect the vertices of the skeleton tree s_i with the vertex s_0 will be called *direct paths* and denoted by the symbol $[s_0, s_i]$ [7].

When describing the relative motion in the hinge number a it must be specified unambiguously which motion relative to which link is meant. As a basic link when describing the relative motion in hinge number a we choose the link with the smaller number. After the choice is completed we can define two functions $i^+(a)$ and $i^-(a)$ ($a = 1, \ldots, m$) where $i^+(a)$ means the number of the reference link and $i^-(a)$ is the number of the contiguous link. From the chosen rule of links labeling it follows obviously that $i^-(a) = a$ for $= 1, 2, \ldots, n$. By introducing the functions $i^+(a)$ and $i^-(a)$ we obtain the possibility to give a sense of direction to every edge and in this way to transform it to arc assuming that $i^+(a)$ is the number of the vertex from which the arc u_a is pointing away, and $i^-(a)$ is the number of the vertex toward which the arc u_a is pointing. The graph obtained is called *oriented* graph.

One of the basic matrices describing the structure of the introduced graphs is the *incidence* matrix of the oriented graph $\underline{I} = (S_{ia})$ ($i = 0, 1, \ldots, n$, $a = 1, \ldots, m$), where

$$S_{ia} = \begin{cases} 1 & \text{if } i = i^+(a), \\ -1 & \text{if } i = i^-(a), \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

i.e. $S_{ia} = 1$, if the arc u_a starts at a vertex s_i, $S_{ia} = -1$, if the arc u_a ends at the vertex s_i and $S_{ia} = 0$ otherwise [8]. Obviously each column of the incidence matrix contains exactly one element $+1$ and one element -1 because exactly two vertices define each arc. The incidence matrix allows reconstructing entirely the system graph and describes in this way unambiguously the system structure.

We introduce also the matrix $\underline{I}^+ = (S^+_{ia})$ ($i = 0, 1, \ldots, n$; $a = 1, \ldots, m$) according to the rule

$$S^+_{ia} = \begin{cases} 1 & \text{if } i = i^+(a), \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

This matrix is gained, obviously, from the matrix $\underline{S}$ replacing in it all -1 through zero.

Let us represent now the incidence matrix in the form

$$\underline{I} = \begin{bmatrix} \overset{\vee}{\underline{S}}_0 & \overset{\wedge}{\underline{S}}_0 \\ \overset{\vee}{\underline{S}} & \overset{\wedge}{\underline{S}} \end{bmatrix} = \begin{bmatrix} \underline{S}_0 \\ \underline{S} \end{bmatrix},$$

where:

$$\overset{\vee}{\underline{S}}_0 = (S_{0a}) \quad (a = 1, \dots, n); \quad \overset{\wedge}{\underline{S}}_0 = (S_{0a}) \quad (a = n+1, \dots, m);$$

$$\underline{S}_0 = (S_{0a}) \quad (a = 1, \dots, m); \quad \overset{\vee}{\underline{S}} = (S_{ia}) \quad (i, a = 1, \dots, n);$$

$$\overset{\wedge}{\underline{S}} = (S_{ia}) \quad (i = 1, \dots, n; \ a = n+1, \dots, m);$$

$$\underline{S} = (S_{ia}) \quad (i = 1, \dots, n; \ a = 1, \dots, m).$$

Another important matrix is the *fundamental loops matrix* (*cyclomatic matrix*) $\underline{\Phi}$ [8]. Let $\Phi_{n+1}, \Phi_{n+2}, \dots, \Phi_m$ be the fundamental loops determined by the nonskeleton arcs $u_{n+1}, u_{n+2}, \dots, u_m$. We choose the direction of the arc u_{n+i} as a positive direction in the loop Φ_{n+i}. The cyclomatic matrix is determined then as a $\hat{n} \times m$-matrix $\underline{\Phi} = (\varphi_{n+i,b})$ $(i = 1, \dots, \hat{n},\ b = 1, \dots, m)$ in the following way:

$$\varphi_{n+i,b} = \begin{cases} 1, & \text{if } u_b \in \Phi_{n+i} \text{ and has the direction of } u_{n+i}, \\ -1, & \text{if } u_b \in \Phi_{n+i} \text{ and has the opposite direction of } u_{n+i}, \\ 0, & \text{otherwise .} \end{cases} \tag{2.3}$$

The last structure matrix we introduce is the *matrix of direct paths* $\underline{\Psi} = (\psi_{ai})$ $(a = 1, \dots, m;\ i = 1, \dots, n)$ [7], where

$$\psi_{ai} = \begin{cases} 1 & \text{if } u_a \in [s_0, s_i] \text{ and is directed towards } s_0, \\ -1 & \text{if } u_a \in [s_0, s_i] \text{ and is directed from } s_0, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix has the form

$$\underline{\Psi} = \begin{bmatrix} \underline{T} \\ \underline{0}_{\hat{n} \times n} \end{bmatrix}; \quad \hat{n} = m - n, \quad \underline{T} = \overset{\vee}{\underline{S}}{}^{-1}, \quad \underline{T} = (\tau_{ai}) \quad (a, i = 1, \dots, n)$$

because of the introduced regular labeling. Here and further $\underline{0}_{k \times s}$ denotes matrix with all elements equal to zero.

Let us consider as a first example the four-bar mechanism with coupler point presented in Fig. 2. The four-bar linkage is the simplest possible closed-loop mechanism, and has numerous uses in industry and for simple devices found in automobiles, toys, etc. The device gets its name from its four distinct links (or bars). Link 0 is the ground link (the frame or fixed link), and is assumed to be motionless. Links 1 and 3 each rotate relative to the ground link about fixed pivots (A_0 and B_0). Link 2 is called coupler link, and is the only link a point C of which can trace paths of different shape (because the link is not rotating about a fixed pivot). Usually one of the "grounded links" (link 1 or 3) serves as the input link, which is the link which may either be turned by hand, or perhaps driven by an electric motor or a hydraulic or pneumatic cylinder.

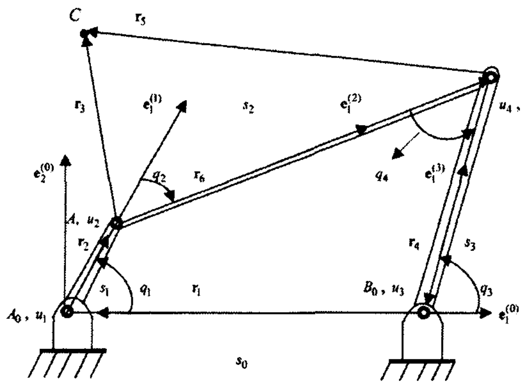

Fig. 2. Four-bar mechanism with coupler point C

For the given mechanism one possible choice of the functions $i^+(a)$ and $i^-(a)$ is represented in the following table

a	1	2	3	4
$i^+(a)$	0	1	0	2
$i^-(a)$	1	2	3	3

The corresponding oriented graph is given in Fig. 3.

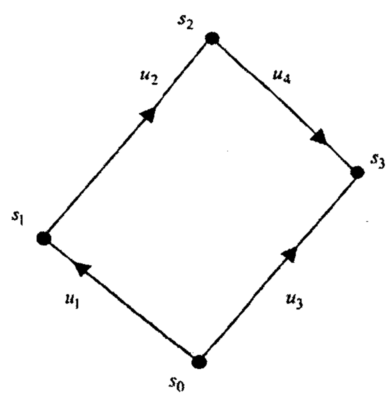

Fig. 3. Graph of the four-bar mechanism

The structure matrices $\underline{I}$, $\underline{\Phi}$ and $\underline{\Psi}$ have the form, respectively:

$$\underline{I} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \ \underline{\Phi} = \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}, \ \underline{\Psi} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As a second example we consider the six-bar Stephenson-I mechanism (Fig. 4).

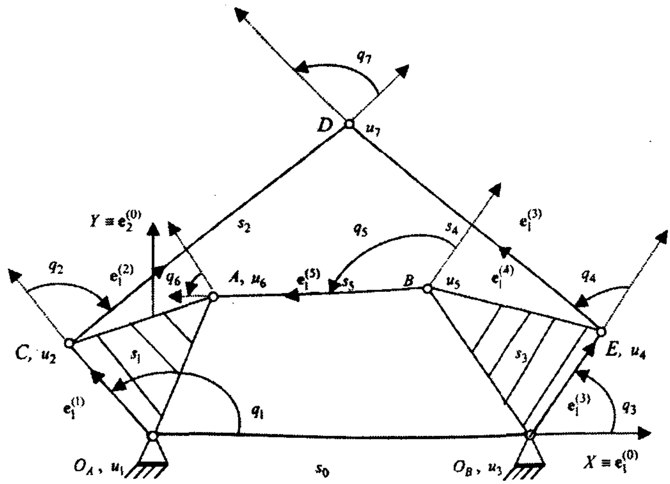

Fig. 4. Six-bar Stephenson-I mechanism with coordinate systems

The six-bar mechanism is considered as a multibody system consisting of six bodies (including the frame) interconnected with seven revolute joints as shown in Fig. 4. The moving links are numbered from 1 to 5 while the frame gets the number 0. The joints are numbered from 1 to 7. One possible choice of the functions $i^+(a)$ and $i^-(a)$ is given through the following table

a	1	2	3	4	5	6	7
$i^+(a)$	0	1	0	3	3	1	2
$i^-(a)$	1	2	3	4	5	5	4

The graph of the six-bar mechanism is a cyclic graph (Fig. 5). It can be reduced to a graph with a tree-like structure by cutting exactly two appropriately chosen arcs, for instance u_6 and u_7.

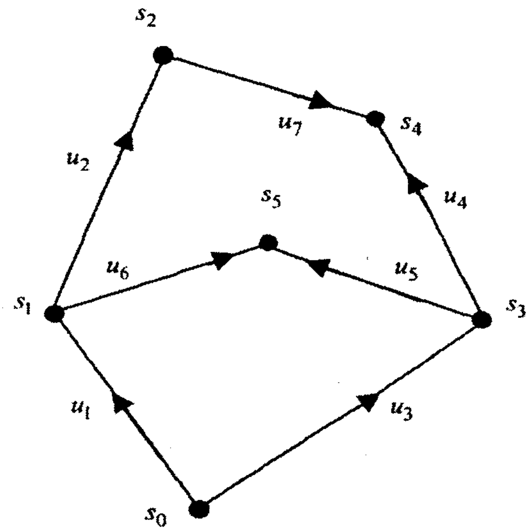

Fig. 5. Graph of the six-bar mechanism

The structure matrices have the form:

$$\underline{I} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}, \quad \underline{\Phi} = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix},$$

$$\underline{\Psi} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last example is the mechanism shown in Fig. 6 with nine links and ten revolute joints (planar platform).

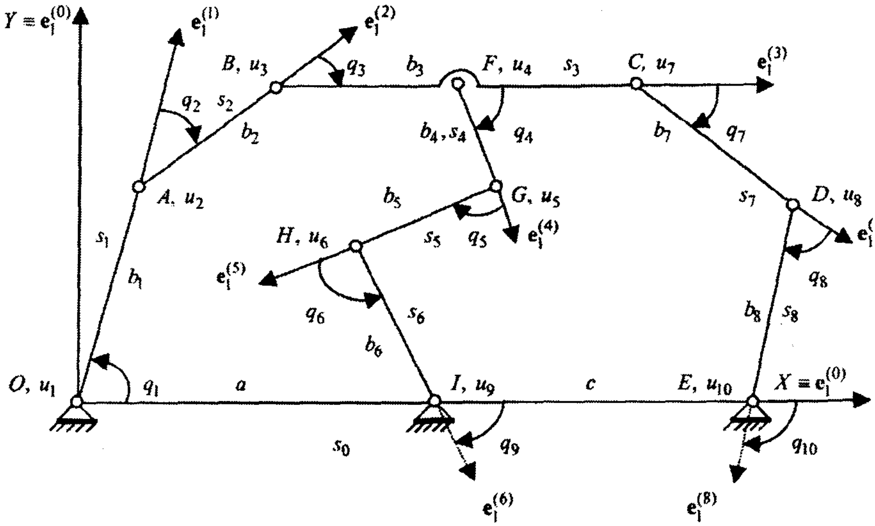

Fig. 6. Nine-bar mechanism with the coordinate systems

One possible choice of the functions $i^+(a)$ and $i^-(a)$ is given in the following table:

a	$i^+(a)$	$i^-(a)$
1	0	1
2	1	2
3	2	3
4	3	4
5	4	5
6	5	6
7	3	7
8	7	8
9	0	6
10	0	8

The corresponding oriented graph is given in Fig. 7.

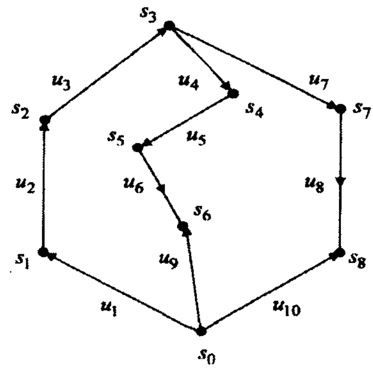

Fig. 7. Graph of the nine-bar mechanism

The graph of the mechanism is a cyclic graph and can be reduced to a graph with a tree-like structure by cutting exactly two appropriately chosen arcs, for instance u_9 and u_{10}. The corresponding structure matrices have the following form:

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix},$$

$$\underline{\Phi} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\underline{\Psi} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In link number i ($i = 0, 1, \ldots, n$) we chose a coordinate system $O_i x_i y_i z_i$ in the following way. The axis z_i is the rotation axis of link i with respect to the previous link in the direct path from s_0 to s_i. All axes z_i are parallel and orthogonal to the common motion plain of the links. If the link number i is an inner one for the skeleton tree and besides it is connected with one or more following links with numbers $j, k, \ldots, l$ $(j < k < \cdots < l)$, then the axis x_i is a common normal of z_i and z_j in the motion plain, directed towards z_j. The axis x_i intersects the axes z_i and z_j in points O_i and O_j, respectively. We chose the first point as an origin of the coordinate system $O_i x_i y_i z_i$ in which the axis y_i lays in the motion plane and complements the axes x_i and z_i to right-hand system. Analogously, on each of the axes belonging to the link i with numbers $j < k < \cdots < l$ points $O_j, O_k, \ldots, O_l$ are defined (Fig. 8).

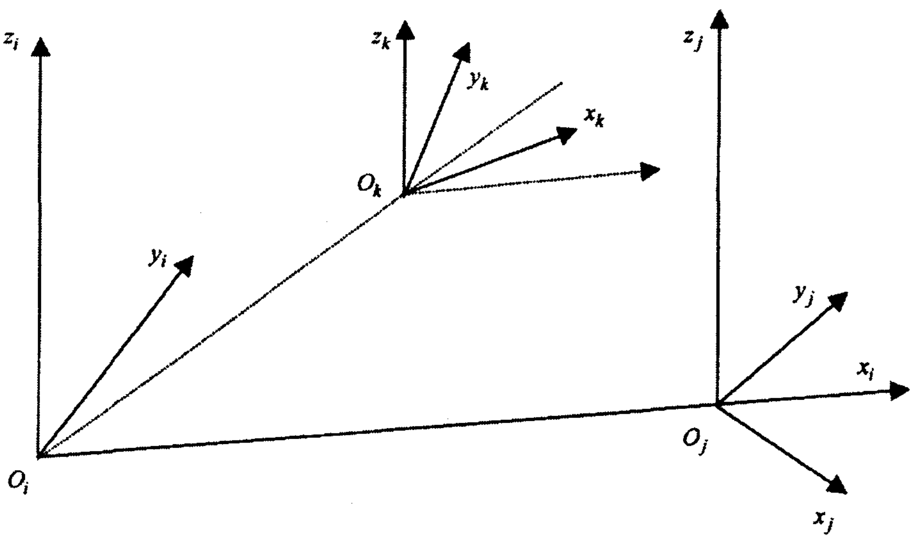

Fig. 8

In the peripheral links the axes x and y of the coordinate systems are chosen arbitrarily but so that they built right-hand systems with the rotation axis z of the peripheral link with respect to the previous one. In the fixed (zero) link the axis z_0 is chosen to coincide with the axis z_1 and the axes x_0 and y_0 are chosen arbitrarily. In addition to the coordinate system $O_i x_i y_i z_i$ in each of the links except the zero one we will use a coordinate system with an origin in an arbitrarily chosen point C_i of the link and parallel axes with unit vectors $\mathbf{e}_1^{(i)}$, $\mathbf{e}_2^{(i)}$, $\mathbf{e}_3^{(i)}$. The position of the link $i^-(a)$ with respect to the link $i^+(a)$ we determine with the angle q_a between the x-axes of the coordinate systems $C_i \underline{\mathbf{e}}^{(i)}$ introduced, $q_a = \angle\left(\mathbf{e}_1^{i^+(a)}, \mathbf{e}_1^{i^-(a)}\right)$.

Let us replace now the revolute hinges in the motion plane through thin plates with lengths l_a having super elasticity, while the system is in a certain position $\underline{q}^*$. We assume that the rotation centers R_a are located at the centers of the lengths of the elastic hinges and that the lengths of the plates are small quantities $l_a \ll 1$ (Fig. 9).

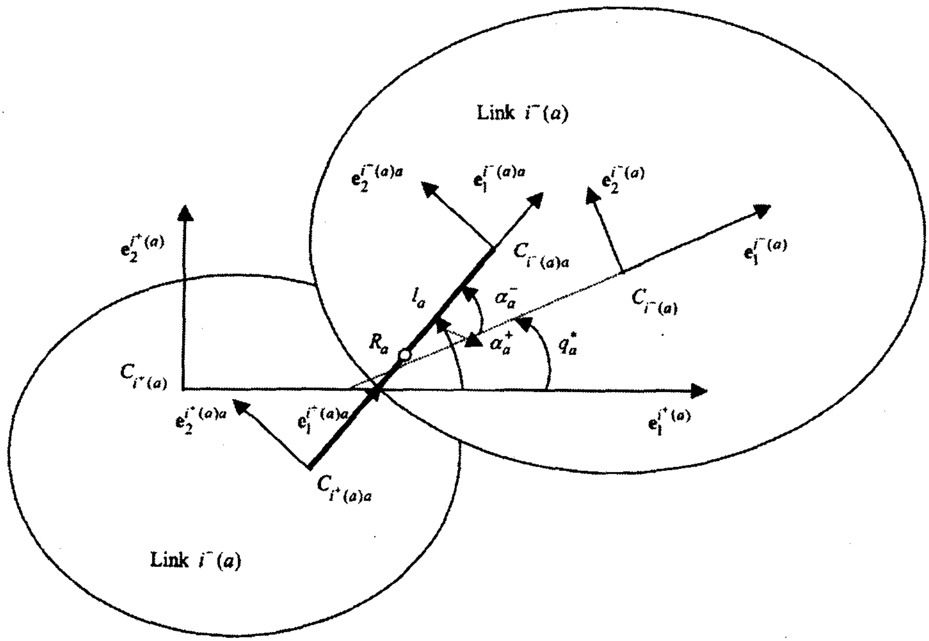

Fig. 9

3. RELATIVE DEVIATION

The basic considerations begin with the description of the relative motions in the hinges. The relative positions of the links will be determined by a method that is more complex than it is necessary when considering only a mechanism with rotational (rigid) hinges, but the method is equally applicable to the both mechanisms. This approach gives us the opportunity to realize the desired comparison. For this purpose for each hinge two *hinge points* $C_{i^\pm(a)a}$ in the corresponding contiguous links are specified and the *hinge vector* $\mathbf{z}_a = \overline{C_{i^+(a)a}C_{i^-(a)a}}$ is introduced. We denote the radius vectors of the hinge points $C_{i^\pm(a)a}$ in the corresponding bases by $\mathbf{c}_{ia} = \overline{C_iC_{ia}}$ $(i = i^\pm(a),\ a = 1,\ldots,m)$ (Fig. 10). In order to describe the relative motion in hinge a we introduce in each of the contiguous links additional reference frames $C_{i^\pm(a)a}\underline{\mathbf{e}}^{(i^\pm(a)a)}$ rigidly attached to the corresponding links $i^\pm(a)$. The position of the system $C_{i^\pm(a)a}\underline{\mathbf{e}}^{(i^\pm(a)a)}$ with respect to $C_{i^\pm(a)}\underline{\mathbf{e}}^{(i^\pm(a))}$ is determined by the position of its origin $C_{i^\pm(a)a}$ and the angle $\alpha_a^\pm$, $\alpha_a^+ = \angle\left(\mathbf{e}_1^{i^+(a)}, \mathbf{e}_1^{i^+(a)a}\right)$, $\alpha_a^- = \angle\left(\mathbf{e}_1^{i^-(a)}, \mathbf{e}_1^{i^-(a)a}\right)$. We choose as hinge points the ends of the super elastic plate. The axes $\mathbf{e}_1^{i^+(a)a}$ and $\mathbf{e}_1^{i^-(a)a}$ are directed along the plate when it is in undeformed state in position $\underline{q}^*$ and then they remain fixed in the corresponding links (Fig. 9, Fig. 10, Fig. 11, Fig. 12).

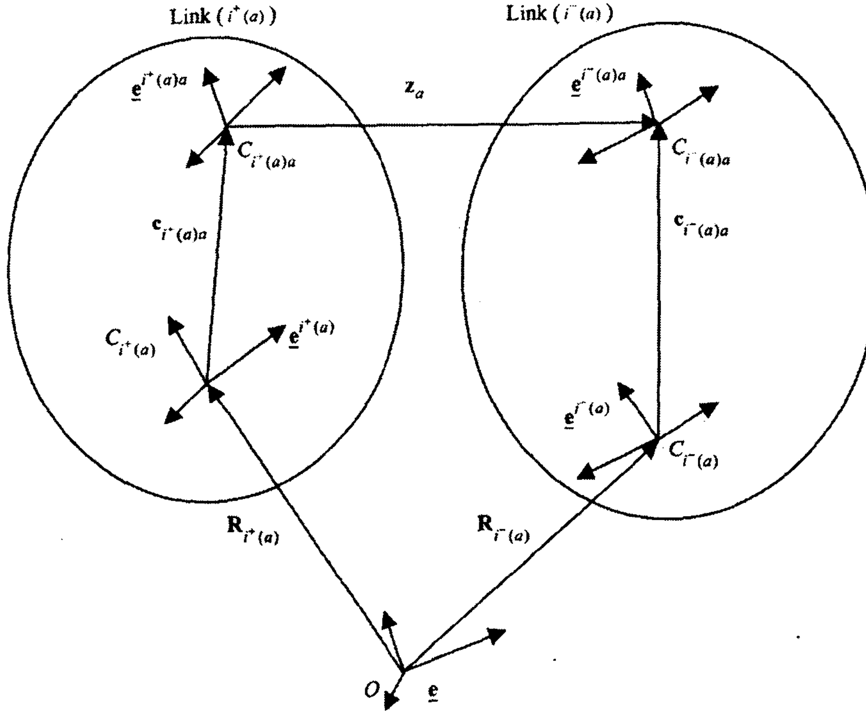

Fig. 10

The vector $\mathbf{z}_a$ and the basis $\underline{\mathbf{e}}^{i^-(a)}$ are functions of the chosen parameter of the relative motion in hinge number a, i.e.

$$\mathbf{z}_a = \mathbf{z}_a(q_a), \quad \underline{\mathbf{e}}^{(i^-(a))} = \underline{\mathbf{e}}^{(i^-(a))}(q_a).$$

The position of the plate itself in undeformed state with respect to the coordinate systems in the contiguous bodies is determined by the angles $\alpha_a^+ \angle\left(\mathbf{e}_1^{i^+(a)}, \mathbf{e}_1^{i^+(a)a}\right)$ and $\alpha_a^- \angle\left(\mathbf{e}_1^{i^-(a)}, \mathbf{e}_1^{i^-(a)a}\right)$ (Fig. 9). For the initial (rigid) mechanism the relative motion in hinge a is a rotation around the center R_a which is the middle of the plate and the vector $\mathbf{z}_a$ has the form (Fig. 12)

$$\begin{aligned} \mathbf{z}_a^r &= \overline{C_{i^+(a)a}C_{i^-(a)a}} = \overline{C_{i^+(a)a}R_a} + \overline{R_a C_{i^-(a)a}} \\ &= \left(\frac{l_a}{2} + \frac{l_a}{2}\cos q_a\right)\mathbf{e}_1^{i^+(a)a} + \left(\frac{l_a}{2}\sin q_a\right)\mathbf{e}_2^{i^+(a)a}. \end{aligned} \tag{3.1}$$

Here and further the index (r) (from rigid) is used for quantities connected with the initial (rigid) mechanism. For the corresponding quantities of the mechanism obtained after replacing the rotational hinges through super elastic plates we will use designations without index.

The relative motion in hinge number a is realized by virtue of the elastic deformation of the plate which replaces the rotation pair. The position of the system $C_{i^-(a)a}\mathbf{e}^{i^-(a)a}$ with respect to the system $C_{i^+(a)a}\mathbf{e}^{i^+(a)a}$ is determined through the angle $\theta_a = \angle\left(\mathbf{e}_1^{i^+(a)a}, \mathbf{e}_1^{i^-(a)a}\right)$ (Fig. 11, Fig. 12).

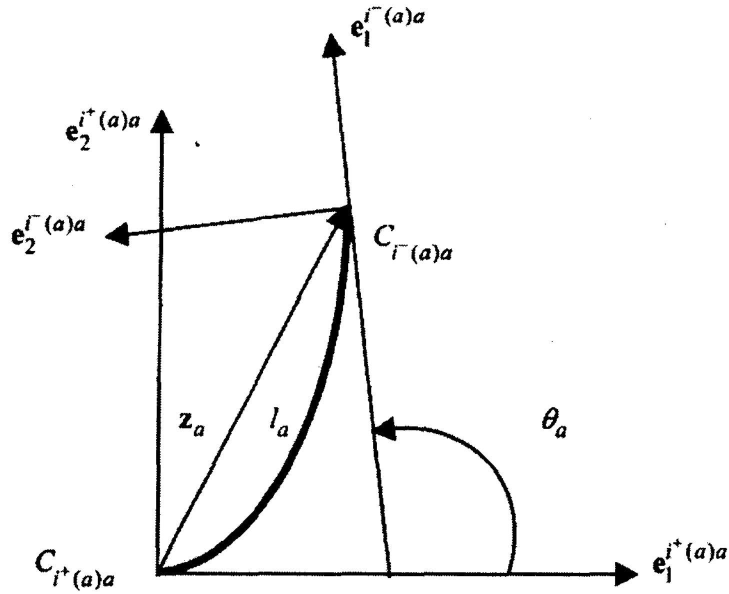

Fig. 11

Let q_a mean in the elastic mechanism again the angle $\angle\left(\mathbf{e}_1^{i^+(a)}, \mathbf{e}_1^{i^-(a)}\right)$. The following relationship is evident (Fig. 11, Fig. 12, Fig. 13):

$$q_a = \alpha_a^+ + \theta_a - \alpha_a^- = \theta_a + q_a^*.$$

The links of the both mechanisms in an undeformed state when $\theta_a = 0$ take identical position $\underline{q}^*$ in the absolute frame.

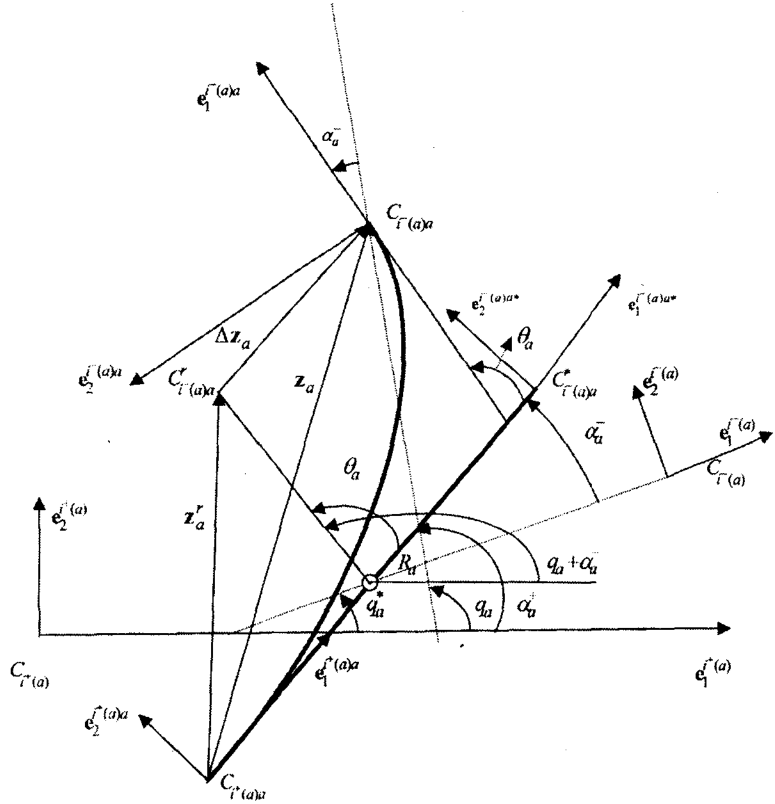

Fig. 12

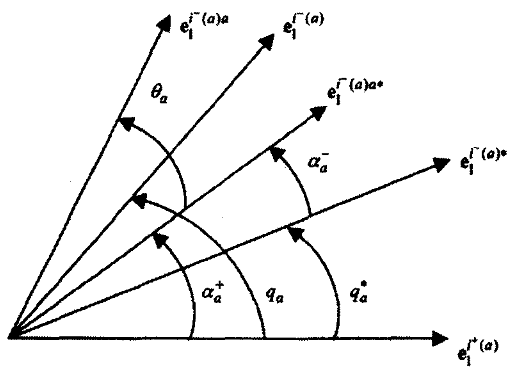

Fig. 13

We assume that the form of the deformed plate is determined through the angle θ_a. Then the translational displacement $\mathbf{z}_a$ of the coordinate system $C_{i^-(a)a}\mathbf{e}^{i^-(a)a}$

with respect to the coordinate system $C_{i^+(a)a}\mathbf{e}^{i^+(a)a}$ can be written down in the form

$$\mathbf{z}_a = \overline{C_{i^+(a)a}C_{i^-(a)a}} = f_a(\theta_a)\mathbf{e}_1^{i^+(a)a} + g_a(\theta_a)\mathbf{e}_2^{i^+(a)a}, \tag{3.2}$$

where functions $f_a(\theta_a)$ and $g_a(\theta_a)$ are known through the equation of the neutral line of the deformed plate found theoretically or from the experiment. The deviation to be determined between the links of the elastic mechanism with respect to the links of the initial mechanism is the difference between (3.1) and (3.2) (Fig. 12):

$$\begin{aligned}\Delta\mathbf{z}_a &= \mathbf{z}_a - \mathbf{z}_a^r = \\ &= \left[f_a(\theta_a) - \left(\frac{l_a}{2} + \frac{l_a}{2}\cos q_a\right)\right]\mathbf{e}_1^{i^+(a)a} + \left[g_a(\theta_a) - \frac{l_a}{2}\sin q_a\right]\mathbf{e}_2^{i^+(a)a}.\end{aligned} \tag{3.3}$$

4. DEVIATION OF THE COMPLIANT MECHANISM

We choose as an absolute coordinate system $O\underline{\mathbf{e}}$ (reference frame) the coordinate system in the fixed (zero) link $O_0\underline{\mathbf{e}}^0$. The position of each link of the mechanism in this system is determined with the help of the radius-vector $\mathbf{R}_i$ of the point C_i, fixed in this link, and the orthonormal basis $\underline{\mathbf{e}}^i$; $i = 1, \ldots, n$, introduced above (Fig. 10). The relative motion in each hinge has only one degree of freedom and in this way the position of the mechanism is determined by m generalized parameters $\underline{q} = (q_1, \ldots, q_m)^T$. We can write down for each pair of contiguous bodies the formula (Fig. 10)

$$\left(\mathbf{R}_{i^+(a)} + \mathbf{c}_{i^+(a)a}\right) - \left(\mathbf{R}_{i^-(a)} + \mathbf{c}_{i^-(a)a}\right) = -\mathbf{z}_a, \qquad a = 1, \ldots, m. \tag{4.1}$$

Taking into account the incidence matrix (2.1), we rewrite this relation in the following way:

$$\sum_{i=0}^{n} S_{ia}\left(\mathbf{R}_i + \mathbf{c}_{ia}\right) = S_{0a}\mathbf{c}_{0a} + \sum_{i=1}^{n} S_{ia}\left(\mathbf{R}_i + \mathbf{c}_{ia}\right) = -\mathbf{z}_a, \qquad a = 1, \ldots, m. \tag{4.2}$$

Let us define now with the help of the incidence matrix the following matrix

$$\underline{\mathbf{J}} = (S_{ia}\mathbf{c}_{ia}) \qquad (i = 0, 1, \ldots, n;\ a = 1, \ldots, m), \tag{4.3}$$

where the vectors $\mathbf{c}_{ia}$ are defined only for $i = i^{\pm}(a)$. We put them zero for the remaining indices. The last matrix has the same structure as the incidence matrix:

$$\underline{\mathbf{J}} = \begin{bmatrix} \overset{\vee}{\underline{\mathbf{C}}}_0 & \overset{\wedge}{\underline{\mathbf{C}}}_0 \\ \overset{\vee}{\underline{\mathbf{C}}} & \overset{\wedge}{\underline{\mathbf{C}}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{C}}_0 \\ \underline{\mathbf{C}} \end{bmatrix},$$

where

$$\overset{\vee}{\underline{\mathbf{C}}}_0 = (S_{0a}\mathbf{c}_{0a})\ (a=1,\dots,n),\quad \overset{\wedge}{\underline{\mathbf{C}}}_0 = (S_{0a}\mathbf{c}_{0a})\ (a=n+1,\dots,m)$$
$$\overset{\vee}{\underline{\mathbf{C}}} = (S_{ia}\mathbf{c}_{ia})\quad (i,a=1,\dots,n),\ \overset{\wedge}{\underline{\mathbf{C}}} = (S_{ia}\mathbf{c}_{ia})\quad (i=1,\dots,m;\ a=n+1,\dots,m),$$
$$\underline{\mathbf{C}}_0 = (S_{0a}\mathbf{c}_{0a})\ (a=1,\dots,n),\quad \underline{\mathbf{C}} = (S_{ia}\mathbf{c}_{ia})\quad (i=1,\dots,n;\ a=1,\dots,m)$$

We define in the same way the matrix

$$\underline{\mathbf{J}}^* = (S_{ia}^+\mathbf{z}_a),\qquad (i=0,1,\dots,n;\ a=1,\dots,m) \tag{4.4}$$

where S_{ia}^+ are quantities defined in (2.2). The last matrix has the form

$$\underline{\mathbf{J}} = \begin{bmatrix} \overset{\vee}{\underline{\mathbf{C}}}{}_0^* & \overset{\wedge}{\underline{\mathbf{C}}}{}_0^* \\ \overset{\vee}{\underline{\mathbf{C}}}{}^* & \overset{\wedge}{\underline{\mathbf{C}}}{}^* \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{C}}_0^* \\ \underline{\mathbf{C}}^* \end{bmatrix},$$

where

$$\overset{\vee}{\underline{\mathbf{C}}}{}_0^* = (S_{0a}^+\mathbf{z}_a)\ (a=1,\dots,n),\quad \overset{\wedge}{\underline{\mathbf{C}}}{}_0^* = (S_{0a}^+\mathbf{z}_a)\ (a=n+1,\dots,m),$$
$$\overset{\vee}{\underline{\mathbf{C}}}{}^* = (S_{ia}^+\mathbf{z}_a)\ (i,a=1,\dots,n),\ \overset{\wedge}{\underline{\mathbf{C}}}{}^* = (S_{ia}^+\mathbf{z}_a)\ (i=1,\dots,n;\ a=n+1,\dots,m),$$
$$\underline{\mathbf{C}}_0^* = (S_{0a}^+\mathbf{z}_a)\ \ (a=1,\dots,m),\ \ \underline{\mathbf{C}}^* = (S_{ia}^+\mathbf{z}_a)\ \ (i=1,\dots,n;\ a=1,\dots,m).$$

The vector $\mathbf{z}_a$ can be represented in the form:

$$\mathbf{z}_a = \sum_{i=0}^{n} S_{ia}^+\mathbf{z}_a.$$

From here for the matrix $\underline{\mathbf{z}} = (\mathbf{z}_1,\dots,\mathbf{z}_m)^T$ follows the relation

$$\underline{\mathbf{z}} = (\underline{\mathbf{J}}^*)^T\underline{1}_{n+1}, \tag{4.5}$$

where $\underline{1}_{n+1}$ is a column $[(n+1)\times 1]$-matrix of unit elements. Now, defining $\underline{\mathbf{R}} = (\mathbf{R}_1,\dots,\mathbf{R}_n)^T$ we are able to represent (4.2) in the following form:

$$\underline{I}^T\begin{bmatrix} \mathbf{0} \\ \underline{\mathbf{R}} \end{bmatrix} + \underline{\mathbf{J}}^T\underline{1}_{n+1} = -\underline{\mathbf{z}}$$

or:

$$\underline{I}^T\begin{bmatrix} \mathbf{0} \\ \underline{\mathbf{R}} \end{bmatrix} + (\underline{\mathbf{J}}+\underline{\mathbf{J}}^*)^T\underline{1}_{n+1} = \underline{\mathbf{0}}_{m\times 1}. \tag{4.6}$$

Multiplying this relation from the left with $\underline{\Psi}^T$ and taking into account the relation [7]

$$\underline{\Psi}^T \underline{I}^T = (-1_n, \underline{E}_n),$$

Where $\underline{E}_n$ is a unit $n \times n$ matrix, we find that

$$\underline{\mathbf{R}} = -\underline{\Psi}^T (\underline{\mathbf{J}} + \underline{\mathbf{J}}^\bullet)^T \underline{1}_{n+1}. \tag{4.7}$$

This expression represents the radius-vectors $\mathbf{R}_i$ $(i = 1, \ldots, n)$ of the points C_i fixed in the links with respect to the absolute coordinate system through the hinge vectors and eventually through the generalized parameters of the mechanism. Without loss of generality we can choose $C_0 = C_1$ and then $\mathbf{c}_{01} = \mathbf{0}$. Particularly, we have for the characteristic link

$$\mathbf{R}_{i^*} = \sum_{i=1}^{i^*} \mathbf{z}_i + \sum_{i=1}^{i^*-1} (\mathbf{c}_{i,i+1} - \mathbf{c}_{ii}). \tag{4.8}$$

The last relationship determines the radius-vector of point C_{i^*} of the characteristic link with respect to the absolute coordinate system. Outgoing from (4.8) and taking into account that the quantities $\mathbf{c}_{ia}$ are identical for both mechanisms and the differences in the attitudes are due to the different values of the vectors $\mathbf{z}_a$ in both mechanisms, we obtain the following expression for the deviation of the characteristic point

$$\Delta\mathbf{R}_{i^*} = \sum_{i=1}^{i^*} \Delta\mathbf{z}_i.$$

The matrices (4.3), (4.4) $\underline{\mathbf{J}}$ and $\underline{\mathbf{J}}^*$ for the four-bar mechanism have the following form (Fig. 2):

$$\underline{\mathbf{J}} = \begin{bmatrix} \mathbf{c}_{01} & \mathbf{0} & \mathbf{c}_{03} & \mathbf{0} \\ -\mathbf{c}_{11} & \mathbf{c}_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{c}_{22} & \mathbf{0} & \mathbf{c}_{24} \\ \mathbf{0} & \mathbf{0} & -\mathbf{c}_{34} & -\mathbf{c}_{34} \end{bmatrix}, \quad \underline{\mathbf{J}}^* = \begin{bmatrix} \mathbf{z}_1 & \mathbf{0} & \mathbf{z}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The formula (4.7) is now

$$\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_1 + \mathbf{c}_{01} - \mathbf{c}_{11} \\ \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{c}_{01} + \mathbf{c}_{12} - \mathbf{c}_{11} - \mathbf{c}_{22} \\ \mathbf{z}_3 + \mathbf{c}_{03} - \mathbf{c}_{33} \end{bmatrix}.$$

The radius-vector $\mathbf{R}_C$ of the characteristic point (coupler point) C is (Fig. 2)

$$\mathbf{R}_C = \mathbf{R}_1 + r_3 \mathbf{e}_1^{(2)}$$

and

$$\Delta\mathbf{R}_C = \Delta\mathbf{R}_1 = \Delta\mathbf{z}_1$$

Let us consider now the second example. For the six-bar mechanism the matrices (4.3), (4.4) $\underline{\mathbf{J}}$ and $\underline{\mathbf{J}}^*$ are (Fig. 4):

$$\underline{\mathbf{J}} = \begin{bmatrix} \mathbf{c}_{01} & \mathbf{0} & \mathbf{c}_{03} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{c}_{11} & \mathbf{c}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_{16} & \mathbf{0} \\ \mathbf{0} & -\mathbf{c}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_{27} \\ \mathbf{0} & \mathbf{0} & -\mathbf{c}_{33} & \mathbf{c}_{34} & \mathbf{c}_{35} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{44} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{47} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{55} & -\mathbf{c}_{56} & \mathbf{0} \end{bmatrix},$$

$$\underline{\mathbf{J}}^* = \begin{bmatrix} \mathbf{z}_1 & \mathbf{0} & \mathbf{z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_4 & \mathbf{z}_5 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The expression (4.7) has the form

$$\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \mathbf{R}_4 \\ \mathbf{R}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_{01} + \mathbf{z}_1 - \mathbf{c}_{11} \\ \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{c}_{01} - \mathbf{c}_{11} + \mathbf{c}_{12} - \mathbf{c}_{22} \\ \mathbf{c}_{03} + \mathbf{z}_3 - \mathbf{c}_{33} \\ \mathbf{z}_3 + \mathbf{z}_4 - \mathbf{c}_{33} + \mathbf{c}_{03} + \mathbf{c}_{34} - \mathbf{c}_{44} \\ \mathbf{z}_3 + \mathbf{z}_5 + \mathbf{c}_{03} + \mathbf{c}_{35} - \mathbf{c}_{33} - \mathbf{c}_{55} \end{bmatrix}.$$

Choosing point D as a characteristic point we have for its radius vector the expression (Fig. 4)

$$\mathbf{R}_D = \mathbf{R}_2 + h\mathbf{e}_1^{(2)}, \quad h = \left|\overline{CD}\right|,$$

or

$$\mathbf{R}_D = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{c}_{01} - \mathbf{c}_{11} + \mathbf{c}_{12} - \mathbf{c}_{22},$$

and finally

$$\Delta\mathbf{R}_D = \Delta\mathbf{R}_2 = \Delta\mathbf{z}_1 + \Delta\mathbf{z}_2.$$

The matrices (4.3), (4.4) $\underline{\mathbf{J}}$ and $\underline{\mathbf{J}}^*$ in the last example – the nine-bar mechanism (Fig. 6) are:

$$
\underline{\mathbf{J}} = \begin{bmatrix}
\mathbf{c}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_{09} & \mathbf{c}_{010} \\
-\mathbf{c}_{11} & \mathbf{c}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\mathbf{c}_{22} & \mathbf{c}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{c}_{33} & \mathbf{c}_{34} & \mathbf{0} & \mathbf{0} & \mathbf{c}_{37} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{44} & \mathbf{c}_{45} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{55} & \mathbf{c}_{56} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{66} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{68} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{77} & \mathbf{c}_{78} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{c}_{88} & \mathbf{0} & -\mathbf{c}_{810}
\end{bmatrix},
$$

$$
\underline{\mathbf{J}}^* = \begin{bmatrix}
\mathbf{z}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_9 & \mathbf{z}_{10} \\
\mathbf{0} & \mathbf{z}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_4 & \mathbf{0} & \mathbf{0} & \mathbf{z}_7 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{z}_8 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}.
$$

The formula (4.7) takes the form:

$$
\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \mathbf{R}_4 \\ \mathbf{R}_5 \\ \mathbf{R}_6 \\ \mathbf{R}_7 \\ \mathbf{R}_8 \end{bmatrix} = \begin{bmatrix}
\mathbf{z}_1+\mathbf{c}_{01}-\mathbf{c}_{11} \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{c}_{01}+\mathbf{c}_{12}-\mathbf{c}_{11}-\mathbf{c}_{22} \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3+\mathbf{c}_{01}+\mathbf{c}_{12}+\mathbf{c}_{23}-\mathbf{c}_{11}-\mathbf{c}_{22}-\mathbf{c}_{33} \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3+\mathbf{z}_4+\mathbf{c}_{01}+\mathbf{c}_{12}+\mathbf{c}_{23}+\mathbf{c}_{34}-\mathbf{c}_{11}-\mathbf{c}_{22}-\mathbf{c}_{33}-\mathbf{c}_{44} \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3+\mathbf{z}_4+\mathbf{z}_5+\mathbf{c}_{01}+\mathbf{c}_{12}+\mathbf{c}_{23}+\mathbf{c}_{34}+\mathbf{c}_{45}-\mathbf{c}_{11}-\mathbf{c}_{22}-\mathbf{c}_{33}-\mathbf{c}_{44}-\mathbf{c}_{55} \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3+\mathbf{z}_4+\mathbf{z}_5+\mathbf{z}_6+\mathbf{c}_{01}+\mathbf{c}_{12}+\mathbf{c}_{23}+\mathbf{c}_{34}+\mathbf{c}_{45}+\mathbf{c}_{36}-\left(\sum_{i=1}^{6}\mathbf{c}_{ii}\right) \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3+\mathbf{z}_7+\mathbf{c}_{01}+\mathbf{c}_{12}+\mathbf{c}_{23}+\mathbf{c}_{37}-\mathbf{c}_{11}-\mathbf{c}_{22}-\mathbf{c}_{33}-\mathbf{c}_{77} \\
\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3+\mathbf{z}_7+\mathbf{z}_8+\mathbf{c}_{01}+\mathbf{c}_{12}+\mathbf{c}_{23}+\mathbf{c}_{37}+\mathbf{c}_{78}-\mathbf{c}_{11}-\mathbf{c}_{22}-\mathbf{c}_{33}-\mathbf{c}_{77}-\mathbf{c}_{88}
\end{bmatrix}
$$

Choosing point F as a characteristic point we have for the position vector the expression (Fig. 6)

$$
\mathbf{R}_F = \mathbf{R}_2 + h_1\mathbf{e}_1^{(2)}, \qquad |BF| = h_1,
$$

or

$$
\Delta\mathbf{R}_F = \Delta\mathbf{R}_2 = \Delta\mathbf{z}_1 + \Delta\mathbf{z}_2.
$$

The availability of loops in the considered mechanisms leads to the appearance of constraints between the generalized parameters q_a imposed on the mutual motion of the links forming a system of fundamental loops. Each of the fundamental loops imposes 3 scalar constrains independent of the remaining loops. These constraints express the trivial circumstance that the radius-vector of the origin of one (no matter which) of the coordinate systems fixed in the links of the loop with respect to this origin is the zero vector and, similarly, that the angular position of this coordinate system with respect to itself is given by the unit matrix. Following from an arbitrary link of the loop in one direction and expressing these quantities through the coordinate systems of the passed links, we find a formal record of the constraints after accomplishing the cycle. Let us derive first the constraint connected with the angular attitude of the links of the loop considered. Let the loop Φ_a consist of arcs $u_{b_1}, \dots, u_{b_a}$ $(a = n+1, \dots, m)$ and the sense of direction is determined by the direction of the arc u_{b_a}. Let i and j be the contiguous links for the hinge number b_k and let α be the angle between the x-axes of the coordinate systems fixed in the contiguous links. Obviously, if the link i is chosen as a reference link and the value of the parameter q_{b_k} is α, then the value of q_{b_k} will be $(-\alpha)$ if the link j is chosen as a reference link. Therefore, starting the calculation from an arbitrary link we find that the constraint, imposed by the loop, will have the form

$$q_{j_1} + q_{j_2} + \dots + q_{j_{n_1}} = q_{k_1} + q_{k_2} + \dots + q_{k_{n_2}}, \tag{5.1}$$

where $j_1, \dots, j_{n_1}$ are the numbers of the arcs with the same sense of direction as the arc u_{b_a} and $k_1, \dots, k_{n_2}$ are the numbers of the arcs with the opposite sense of direction.

This result can be obtained in a formal way, as well. Let the transition matrix in hinge b_k be $\underline{G}_{b_k}$ and let the link number i be chosen as a reference body. Let the value of the parameter q_{b_k} be α, then the matrix $\underline{G}_{b_k}$ has the form:

$$\underline{G}_{b_k} = \underline{\mathbf{e}}^{(i)}.\underline{\mathbf{e}}^{(j)T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If we choose as a reference link the contiguous link, then the value of q_{b_k} is $(-\alpha)$ and the transition matrix is $\underline{G}_{b_k}^{-1} = \underline{G}_{b_k}^{T}$. Outgoing from the definition (2.3) for the quantities φ_{ab} we can write down the relation

$$\underline{G}_{b_1}^{\varphi_{ab_1}}, \underline{G}_{b_2}^{\varphi_{ab_2}}, \dots, \underline{G}_{b_a}^{\varphi_{ab_a}} = \underline{E}_2 \qquad (a = n+1, \dots, m), \tag{5.2}$$

where $\underline{E}_2$ is 2×2 unit matrix. Each of the matrices in this relation is an antisymmetric one. It is an easy task to prove that the product of two antisymmetric matrices is commutative

$$\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix}.$$

Consequently, in the left side of (5.2) we can first write down the matrices with positive exponents φ_{ab} and then – those with negative exponents. The product of the former matrices has the form

$$\begin{bmatrix} \cos(q_{j_1}+q_{j_2}+\cdots+q_{j_{n_1}}) & -\sin(q_{j_1}+q_{j_2}+\cdots+q_{j_{n_1}}) \\ \sin(q_{j_1}+q_{j_2}+\cdots+q_{j_{n_1}}) & \cos(q_{j_1}+q_{j_2}+\cdots+q_{j_{n_1}}) \end{bmatrix},$$

and the product of the latter – the form

$$\begin{bmatrix} \cos(q_{k_1}+q_{k_2}+\cdots+q_{k_{n_2}}) & \sin(q_{k_1}+q_{k_2}+\cdots+q_{k_{n_2}}) \\ -\sin(q_{k_1}+q_{k_2}+\cdots+q_{k_{n_2}}) & \cos(q_{k_1}+q_{k_2}+\cdots+q_{k_{n_2}}) \end{bmatrix}.$$

After multiplying these matrices the relation (5.2) obtains the form:

$$\begin{bmatrix} \cos\left[\left(\sum_{i=1}^{n_1} q_{j_i}\right)-\left(\sum_{i=1}^{n_2} q_{k_i}\right)\right] & \sin\left[\left(\sum_{i=1}^{n_1} q_{j_i}\right)-\left(\sum_{i=1}^{n_2} q_{k_i}\right)\right] \\ -\sin\left[\left(\sum_{i=1}^{n_1} q_{j_i}\right)-\left(\sum_{i=1}^{n_2} q_{k_i}\right)\right] & \cos\left[\left(\sum_{i=1}^{n_1} q_{j_i}\right)-\left(\sum_{i=1}^{n_2} q_{k_i}\right)\right] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

what leads us again to the relation (5.1).

The remaining two constraints express the equality to zero of the radius-vector of an arbitrary origin with respect to itself, or in other words, the radius-vector of the coordinate origin of the link chosen as an initial one, expressed through the sequence of vectors $\mathbf{c}_{i^\pm(a)}$ and $\mathbf{z}_a$ along the arcs u_a belonging to the considered loop, is equal to zero. We can find these constraints by multiplying the relationship (4.6) from left with the cyclomatic matrix $\underline{\Phi}$ following [7]. We find, after simple calculations, the formula

$$\underline{\Phi}\underline{\mathbf{J}}^T\underline{1}_{n+1} + \underline{\Phi}\underline{\mathbf{z}} = \underline{\mathbf{0}}_{\hat{n}\times 1}.$$

Using (4.5) we rewrite this relation in the form

$$\underline{\Phi}\underline{\mathbf{J}}^T\underline{1}_{n+1} + \underline{\Phi}\underline{\mathbf{z}} = \mathbf{0}_{\hat{n}\times 1}.$$

This relationship is fulfilled for both mechanisms, therefore

$$\underline{\Phi}(\underline{\mathbf{J}} - \underline{\mathbf{J}}^r)^T\underline{1}_{n+1} + \underline{\Phi}\Delta\underline{\mathbf{z}} = \mathbf{0}_{\hat{n}\times 1}$$

The vectors $\mathbf{c}_{ia}$ have identical values for both mechanisms, i.e. $\underline{\mathbf{J}} = \underline{\mathbf{J}}^r$ and finally

$$\underline{\Phi}\Delta\underline{\mathbf{z}} = \mathbf{0}_{\hat{n}\times 1}. \tag{5.3}$$

Projecting (5.3) on the axes x and y in the motion plane we obtain $2\hat{n}$ scalar relations which are the constrain equations together with (5.1).

As an example, let us consider again the four-bar mechanism. We have only one loop and the equations (5.3) are now (Fig. 2)

$$
\begin{aligned}
r_2 \cos q_1 + r_6 \cos(q_1+q_2) - r_4 \cos q_3 - r_1 = 0 \\
r_2 \sin q_1 + r_6 \sin(q_1+q_2) - r_4 \sin q_3 = 0.
\end{aligned} \tag{5.4}
$$

On the other hand, the relation (5.4) takes the form

$$
q_1 + q_2 - q_3 + q_4 = 0. \tag{5.5}
$$

From (5.4) we obtain

$$
q_3 = \arctan\left(\frac{r_2 \sin q_1 + r_6 \sin(q_1+q_2)}{r_2 \cos q_1 + r_6 \cos(q_1+q_2) - r}\right). \tag{5.6}
$$

Eliminating q_3 by squaring and adding the equations (5.4), we get

$$
r_4^2 - (r_1^2 + r_2^2 + r_6^2) = 2r_2 r_6 \cos q_2 - 2r_1 r_2 \cos q_1 - 2r_1 r_6 \cos(q_1+q_2).
$$

This formula allows representing q_2 as a function of q_1 and further q_3 and q_4 through (5.6) and (5.5). Finally the constraint equations (5.3) have the form

$$
\begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}_1 \\ \Delta\mathbf{z}_2 \\ \Delta\mathbf{z}_3 \\ \Delta\mathbf{z}_4 \end{bmatrix} = \mathbf{0}
$$

or

$$
\Delta\mathbf{z}_1 + \Delta\mathbf{z}_2 - \Delta\mathbf{z}_3 + \Delta\mathbf{z}_4 = \mathbf{0}.
$$

Let us now find the parameters of the displacement for the six-bar mechanism through the generalized parameters q_1, q_2, q_3, q_4, q_5, q_6 and q_7 (Fig. 4). To that end, we use the loop-closure equations. The loop-closure equations of first four-bar linkage $O_A ABO_B$ are written as:

$$
\begin{aligned}
a\cos(q_1-\alpha) - b\cos(q_1+q_6) - c\cos(q_3+\beta) - d = 0 \\
a\sin(q_1-\alpha) - b\sin(q_1+q_6) - c\sin(q_3+\beta) = 0
\end{aligned} \tag{5.7}
$$

$$
q_1 - q_3 - q_5 + q_6 = 0, \tag{5.8}
$$

where $\alpha = \angle(\overline{O_A A}, \overline{O_A C})$, $\beta = \angle(\overline{O_B E}, \overline{O_B B})$, $a = |O_A A|$, $b = |AB|$, $c = |O_B B|$, $d = |O_A O_B|$. From (5.7) we get the formula

$$
q_3 = \arctan\left(\frac{a\sin(q_1-\alpha) - b\sin(q_1+q_6)}{a\cos(q_1-\alpha) - b\cos(q_1+q_6) - d}\right) - \beta. \tag{5.9}
$$

In order to get rid of the angle q_3 in (5.7), we square and add the equations. We obtain

$$
c^2 - (a^2+b^2+d^2) = -2ab\cos(q_6+\alpha) - 2ad\cos(q_1-\alpha) + 2bd\cos(q_1+q_6). \tag{5.10}
$$

As q_1 is the input data and can be chosen as a generalized coordinate, then q_6 is determined from (5.10) as a function of q_1, hence q_3 through (5.9) is a function of q_1. The relation (5.8) gives finally $q_5 = q_1 - q_3 - q_6$ as a function of q_1.

The closure equations for the second loop – the five-bar linkage O_ACDEO_B are given as:

$$\begin{aligned} &e\cos q_1 + h\cos(q_1+q_2) - g\cos(q_3+q_4) - f\cos q_3 - d = 0\\ &e\sin q_1 + h\sin(q_1+q_2) - g\sin(q_3+q_4) - f\sin q_3 = 0 \end{aligned} \tag{5.11}$$

$$q_1 + q_2 - q_3 - q_4 + q_7 = 0 \tag{5.12}$$

where $e = |O_AC|$, $f = |O_BE|$, $g = |ED|$ and $h = |CD|$ are dimensions of the six-bar Stephenson-I mechanism.

After squaring and adding the equations (5.11) we obtain

$$\begin{aligned} h^2 - (d^2+e^2+f^2+g^2) = -2eg\cos[q_1-(q_3+q_4)] - 2ef\cos(q_1+q_3) + 2fg\cos q_4\\ - 2de\cos q_1 + 2dg\cos(q_3+q_4) + 2df\cos q_3. \end{aligned} \tag{5.13}$$

From (5.11) we get

$$q_2 = -q_1 + \arctan\frac{e\sin q_1 - g\sin(q_3+q_4) - f\sin q_3}{e\cos q_1 - g\cos(q_3+q_4) - f\cos q_3 - d}. \tag{5.14}$$

The parameter q_4 can be expressed from equation (5.13) as a function of the generalized coordinate q_1. The generelized parameter q_2 is expressed as function of q_1 from (5.14). Finally, the last parameter q_7 can be derived as a function of q_1 from (5.12).

The constraint equations (5.3) for this mechanism have the form

$$\begin{aligned} \Delta\mathbf{z}_1 - \Delta\mathbf{z}_3 - \Delta\mathbf{z}_5 + \Delta\mathbf{z}_6 = \mathbf{0}\\ \Delta\mathbf{z}_1 + \Delta\mathbf{z}_2 - \Delta\mathbf{z}_3 - \Delta\mathbf{z}_4 + \Delta\mathbf{z}_7 = \mathbf{0}. \end{aligned}$$

The nine-bar mechanism has nine links and ten revolute joints, consequently 10 generalized parameters q_i $(i = 1, \dots, 10)$ (Fig. 6). As the mechanism has two independent loops, the number of the degrees of freedom is four. We choose as generalized coordinates the first generalized parameters q_i $(i = 1, \dots, 4)$. We have from the loop $(OABFGHI)$, the following constraint equations:

$$q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - q_9 = 0 \tag{5.15}$$

$$\begin{aligned} &b_1\cos q_1 + b_2\cos(q_1+q_2) + h_1\cos(q_1+q_2+q_3) +\\ &\qquad b_4\cos\left(\sum_{i=1}^{4} q_i\right) + b_5\cos\left(\sum_{i=1}^{5} q_i\right) + b_6\cos q_9 - a = 0\\ &b_1\sin q_1 + b_2\sin(q_1+q_2) + h_1\sin(q_1+q_2+q_3) +\\ &\qquad b_4\sin\left(\sum_{i=1}^{4} q_i\right) + b_5\sin\left(\sum_{i=1}^{5} q_i\right) + b_6\sin q_9 = 0, \end{aligned} \tag{5.16}$$

where $h_1 = BF$. From relation (5.16) we get

$$q_9 = \arctan\left[\frac{b_1\cos q_1 + b_2\cos(q_1+q_2) + h_1\cos\left(\sum_{i=1}^{3} q_i\right) + b_4\cos\left(\sum_{i=1}^{4} q_i\right) + b_5\cos\left(\sum_{i=1}^{5} q_i\right) - a}{b_1\sin q_1 + b_2\sin(q_1+q_2) + h_1\sin\left(\sum_{i=1}^{3} q_i\right) + b_4\sin\left(\sum_{i=1}^{4} q_i\right) + b_5\sin\left(\sum_{i=1}^{5} q_i\right)}\right]. \tag{5.17}$$

Now, adding after squaring the equations (5.16), we get

$$\begin{aligned}
b_7^2 - (b_1^2 + b_2^2 + h_1^2 + b_4^2 + b_5^2 + a^2) = {} & \\
& 2b_1b_2\cos q_2 + 2b_1h_1\cos(q_2+q_3) + 2b_1b_4\cos\left(\sum_{i=2}^{4} q_i\right) \\
& + 2b_1b_5\cos\left(\sum_{i=2}^{5} q_i\right) + 2b_2h_1\cos q_3 + 2b_2b_4\cos(q_3+q_4) \\
& + 2b_2b_5\cos\left(\sum_{i=3}^{5} q_i\right) + 2h_1b_4\cos q_4 + 2h_1b_5\cos(q_4+q_5) \\
& + 2b_4b_5\cos q_5 - 2ab_1\cos q_1 - 2ab_2\cos(q_1+q_2) \\
& - 2ah_1\cos\left(\sum_{i=1}^{3} q_i\right) - 2ab_4\cos\left(\sum_{i=1}^{4} q_i\right) - 2ab_5\cos\left(\sum_{i=1}^{5} q_i\right).
\end{aligned}$$

This relation determines generalized parameter q_5 as a function of the generalized coordinates, $q_5 = h(q_1, q_2, q_3, q_4)$. Hence, from (5.17) we have $q_9 = f(q_1, q_2, q_3, q_4)$ and from (5.15) we get $q_6 = g(q_1, q_2, q_3, q_4)$.

The second loop ($OABCDE$) delivers the equations (Fig. 6)

$$q_1 + q_2 + q_3 + q_7 + q_8 - q_{10} = 0 \tag{5.18}$$

$$\begin{aligned}
& b_1\cos q_1 + b_2\cos(q_1+q_2) + b_3\cos\left(\sum_{i=1}^{3} q_i\right) + b_7\cos\left(q_7 + \sum_{i=1}^{3} q_i\right) + b_8\cos q_{10} - c = 0 \\
& b_1\sin q_1 + b_2\sin(q_1+q_2) + b_3\sin\left(\sum_{i=1}^{3} q_i\right) + b_7\sin\left(q_7 + \sum_{i=1}^{3} q_i\right) + b_8\sin q_{10} = 0.
\end{aligned} \tag{5.19}$$

From (5.19) we obtain the relations

$$q_{10} = \arctan\left[\frac{b_1\sin q_1 + b_2\sin(q_1+q_2) + b_3\sin\left(\sum_{i=1}^{3} q_i\right) + b_7\sin\left(q_7 + \sum_{i=1}^{3} q_i\right)}{b_1\cos q_1 + b_2\cos(q_1+q_2) + b_3\cos\left(\sum_{i=1}^{3} q_i\right) + b_7\cos\left(q_7 + \sum_{i=1}^{3} q_i\right) - c}\right], \tag{5.20}$$

$$
\begin{aligned}
b_7^2 - (b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + a^2) = \\
& 2b_1b_2\cos q_2 + 2b_1b_3\cos(q_2+q_3) + 2b_1b_7\cos(q_2+q_3+q_7) \\
& +2b_2b_3\cos q_3 + 2b_2b_7\cos(q_3+q_7) + 2b_3b_7\cos q_7 \\
& +2b_3b_4\cos q_4 + 2b_3b_5\cos(q_4+q_5) - 2b_1c\cos q_1 \\
& -2b_2c\cos(q_1+q_2) - 2b_3c\cos\left(\sum_{i=1}^{3} q_i\right) - 2b_7c\cos\left(q_7 + \sum_{i=1}^{3} q_i\right).
\end{aligned}
$$

The last relation determines the generalized parameter q_7 as a function of three generalized coordinates, $q_7 = h_1(q_1, q_2, q_3)$. Hence, from (5.20) we have $q_{10} = f_1(q_1, q_2, q_3)$ and further from (5.18) we get $q_8 = g_1(q_1, q_2, q_3)$.

The relations (5.3) for the planar platform are:

$$
\begin{aligned}
&-(\Delta\mathbf{z}_1 + \Delta\mathbf{z}_2 + \Delta\mathbf{z}_3 + \Delta\mathbf{z}_4 + \Delta\mathbf{z}_5 + \Delta\mathbf{z}_6) + \Delta\mathbf{z}_9 = \mathbf{0} \\
&-(\Delta\mathbf{z}_1 + \Delta\mathbf{z}_2 + \Delta\mathbf{z}_3 + \Delta\mathbf{z}_7 + \Delta\mathbf{z}_8) + \Delta\mathbf{z}_{10} = \mathbf{0}
\end{aligned}
$$

6. CONCLUSIONS

The displacement of the mechanism with super elastic hinges is compared with the displacement of the mechanism with traditional joints, considered as an ideal system. Using the graph theory a mathematical model is suggested and compact analytical expressions are given allowing an exact estimation of the deflections in links positions of the mechanism with super elastic hinges. The results obtained are applied on three famous mechanisms widely used in technics.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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Modeling and design of Learning Content Management Systems: UML as a tool for learning system design

KORNELIYA TODOROVA

Problems in advanced e-learning like development and delivery of dynamic flexible reusable and adaptive learning content are discussed in the paper. It is proposed an extended learning content model which purpose is to demonstrate how required flexibility can be achieved and it is offered a detailed description of design process of learning system that supports management and delivery of reusable and adaptive learning content. For this purpose advantages of UML as modeling tool for design of learning systems are discussed in the paper and elements of different types of diagrams and their implementation in the process of systems design are described. In the paper UML diagrams are used to show how necessary functionality of leaning system can be represented in clear and accurate manner and it is conducted overview analysis and of existing tools for development of UML diagrams. The model proposed is explained with several types of UML diagrams and a prototype of user interface is proposed that will provide required functionalities related to learning content manipulation and other activities related to leaning process like users profile management and their performance assessment.

Keywords: modeling, UML diagrams, state, class, system design, use case, collaborative, sequence, statechart diagrams, ontology, learning content model, reusable, flexible, adaptive learning content

2000 MSC: 68N30

1. INTRODUCTION

Delivery and development of high quality learning materials are critical for individuals and organization success in the conditions of globalization of information society. Main task of experts and specialists in the field of e-learning is to develop and support systems that allow effective learning materials to be created, edited, stored and delivered. Basic problems related to development and delivery of dynamic, flexible and adaptive to learners' needs of education learning materials are discussed in the paper and an extended and modified learning content model is proposed. Different types of learning systems are developed for the problem solution. Most complex of them are Learning Content Management Systems (LCMS). They allow learning process to be managed and their most important functions are related to learning materials manipulation. The goal in this paper is to design LCMS following the Unified Process (UP) methodology and according to it the development of high efficient system that supports e-leaning processes will achieve. Development of effective software systems that cover users' requirements and expected functionality depends on following the phases of systematic development process of learning systems. For this purpose users' requirements have to be gathered and documented and phases of analysis, planning, design, development, implementation and evaluation should be implemented. One of the most important phase and often missed or bad conducted one is design phase. Aim of the paper is to analyze and design development of Learning Content Management System that supports creation of reusable and dynamic learning content and learning process management. The definition used in this paper of LCMS is that the functionality of LCMS is an union of Learning Management Systems (LMS) and Content Management Systems (CMS) functionalities ([3]):

$$LCMS = LMS + CMS[RLOs].$$

In this paper UML model of use cases that describe system requirements is presented. Realization of the model should be analyzed and designed with class diagrams, sequence and collaboration diagrams and object and state chart diagrams. Last type of diagrams used in the paper is deployment type diagrams. Their purpose is to demonstrate the design process of LCMS as it is defined on the base of basic and users' requirements and to show how it will interpret the new model of learning content described here. User interface that supports processes in the proposed model is presented at the end of the article.

2. DESCRIPTION OF LEARNING CONTENT MANAGEMENT SYSTEM

2.1. LEARNING CONTENT MANAGEMENT SYSTEMS REQUIREMENTS-BASIC REQUIREMENTS

Basic requirements [11] to the Learning Content Management Systems are extensibility, granularity, usability and scalability . They are used in the process of LCMS deign .

A common definition of Learning Content Management System proposed by Maish Nichani in his paper [3] is following:

$$LCMS = LMS + CMS[RLO].$$

where LCMS is Learning Content Management System, LMS is Learning Management System , CMS is content Management Systems and RLO is Reusable Learning Object. Aim of RLOs is to present leaning content in the form of little independent pieces and this way leaning materials can be combined in appropriate context according to users' requirements and needs of education LMSs have to provide support for the following operations: development and delivery of course catalog, registration and management of users' accounts, assessment of students performance, score reports and support for different tools for synchronous and asynchronous communication. Content Management Systems have to allow creation, editing, approval, publishing and storage of content. On the base of combination of its capabilities, a LCMS has to offer, not only the management of the entire learning process, but the following functionalities also:

- Creation of learning content
- Storage of learning content
- Editing of learning content
- Management and delivery of learning content in different formats

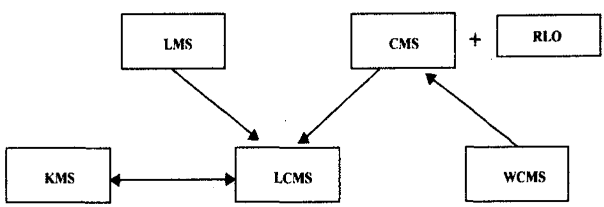

Fig.1 Types of learning systems

3. DEVELOPED MODELS

In the paper a model for delivery of learning content is developed on the base of main components of e-learning and knowledge management proposed by IMS Global Consortium: learning environment, learning components, learning objects structure, information object description and content assets as text, video, audio etc.

First the general model by IMS will be presented.

ID-based Learning Content Model is proposed by IMS Global Learning Consortium[8]. Learning environment is defined by its components, used databases and communications. Courses, curriculum and competencies are learning components of the model and each learning object is defined as a combination of objective, practice and assessments. Each information object describes different concepts, principles and procedure in a separate knowledge domain. Content asset in the terms of multimedia is described as a mix of text, video, audio and so on. All described components have to be managed by a Knowledge Management System and with addition of communications and databases that store them they provide e-learning. Dividing the entire e-learning system in small parts allows high reusability of its components to be achieved. This way flexibility and adaptability of learning materials is gained. Purpose of separation of courses in learning objects as combination of information objects and at the end as content assets is to increase reusability and adaptability of learning content. But in the basic model it is not clear how that reusability can be provided and supported. A proposal how necessary level of flexibility can be achieved is given in the developed modified model. Most important question is how all these small and independent parts to be gathered and managed. For this purpose domain ontologies can be used and this approach is presented in modified learning content model.

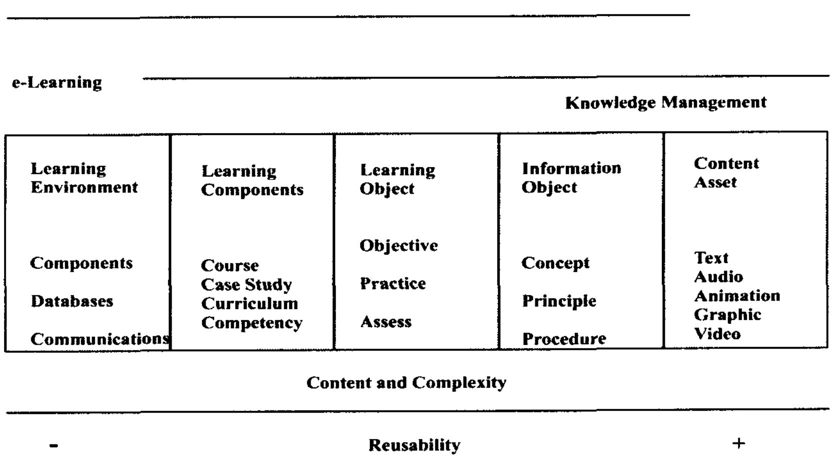

Fig.2 Types of learning systems

We improved the IMS model by replacing learning components with domain ontology concepts, relationships and rules in order to develop dynamic courses that reflect learners' preferences defined in learners profiles. The resulting improved model is presented in the next section.

For the purpose of this paper ID-based Learning Content Model has been modified. That modification replaces the learning components in the IMS model with domain ontology that is used in LCMS design.

e-Learning

Knowledge Management

Learning Environment	**Domain ontology**	**Learning Object**	**Information Object**	**Content Asset**
Components **Databases** **Communications**	**Ontology Concepts** **Rules for learning path selection**	**Objective** **Practice** **Assess**	**Concept** **Principle** **Procedure**	**Text** **Audio** **Animation** **Graphic** **Video**

Content and Complexity

- **Reusability** **+**

Fig.3 Modified Learning Content Model

Proposed modification allows elements of the learning materials to be organized in a hierarchical way in an subject domain ontology of concepts and learning objects to present information for each concept. Ontology elements allow very detailed level of learning content representation to be achieved. Each concept could be implemented as independent small part of learning content and it could be delivered with other parts of learning content in different contexts according to learners' preferences and needs of education. For this purpose technology of Reusable Learning Objects (RLO) can be used. Generated this way learning content could be searched, used in flexible form and different.

Relationships among concepts and rules are used for combination of different learning objects in different leaning paths produces on the base of learning style preferences and leaner characteristics.

The model proposed demonstrates how advantages of LO could be applied when domain ontology is used to describe basic concepts of subject matter and rules for learning path selection allow small and independent part of leaning content to be combined on the base of defined relationships among them and rules defined by subject domain expert and domain ontology specialist.

The proposed modification is necessary because in the IMS learning content model it is not specified how courses will be created. Often and very important problem in the field of e-learning is development of static content which does not allow learning materials to be used in flexible and adaptive to learner preferences manner. That is why it is proposed in that section of learning components in IMS learning model to be used domain ontologies that allow basic concepts of subject domain to be defined and described. Thus it is possible to be created metadata for each learning resource and learning resource characteristics to be compared to users' profiles and preferences and learning paths for content processing to be dynamically generated.

Described technology allows flexible learning courses to be created and proposed learning content to be most appropriate one for learning background, previous experience, level of expertise and needs of new knowledge and skills of a separate user.

4. IMPLEMENTATION OF UML DIAGRAMS IN LCMS CHARACTERISTICS DEFINITION

4.1. PHASES OF UNIFIED PROCESS

Systematic process of software development, according to UP methodology, has the following phases: analysis of gathered users' requirements, design of software system, planning of process of development, implementation, testing and evaluation of developed system. Four phases of Unified process have been followed to define and analyze users' requirements and to design LCMS according to them: inception, elaboration, construction and transition [1].

In the phase of inception [2] it is conducted identification of components of the system and needed functionality is discussed with subject matter expert. For this paper main users of the Learning Content Management System are defined: student, teacher, administrator and content developer. Necessary functionality is defined on the base of LCMS characteristics- management of learning process like users profile manipulation and features for creation, storage and delivery of learning content. In the phase of the elaboration a detailed design of subsystems and objects related to system is performed. For this purpose UML diagrams are used and they are presented in next section of the paper. In construction phase the program code has to be written. In transition phase delivery of system is conducted and it is not discussed in the paper.

The model proposed allow created learning materials in the form of LOs to be organized by domain ontology and on the base of defined rules and relations among them to be executed processes of dynamic learning content delivery. For this purpose a prototype of user interface that allow LO to be developed, edited and deleted and users characteristics to be defined, is developed. Thus adaptive and flexible leaning courses can be offered and delivered by designed LCSM.

Tools for modeling of learning systems and their advantages and disadvantages are very important for the process of design. When a software tool has to be chosen following factors influence on the decision- cost, features, scalability, and hardware platform.

Other features that are overviewed in the process of tools selection are support for the whole set of standard UML diagrams, easy navigation, multi-user support, facilities for code generation, integration with other tools. It has been conducted tools analysis according to listed criteria. Three of the most popular and common used UML tools have been analyzed.

MS Visio [4] allows different elements like classes, objects, activities and states to be created and exchanged among diagrams by drag and drop technique and it is well integrated with other application of Microsoft Office. Rational Rose [9] allows reverse engineering to be used and gives capabilities for classes and objects management and produced that way elements to be stored in a repository and change in one diagrams to affect others. Poseidon [10] visualizes systems, communicates effectively about architecture and code and allow documentation of users' requirements to be done. The tool uses reverse engineering to get a visual model of existing code and allows the model to be previewed and code to be edited within Poseidon itself. Other capabilities are to export diagrams and document requirements with UMLdoc and collaborate through standards-compliant export to XMI.

All these features are available in analyzed three most popular UML tools - MS Visio, Rational Rose and Poseidon but most powerful one and most integrated with other external tools is Microsoft Visio because it offers easy for use and intuitive GUI and for this reason it is selected for system design.

4.2. DESCRIPTION OF PROPESED MODEL USING UML DIAGRRAMS

In the process of Learning Content Management System design different types of UML diagrams will be used. UML (Unified Modeling Language) was developed in 1994 [5]. Its purpose is to support Unified Process. UML is very appropriate when a system design have to be developed and system has to be defined in different points of view. [7]

Used UML diagrams are very useful for definition of different parts of the designed LCMS and relationships among them and their place in the entire system. Different types of UML diagrams allow states and processes inside the system and interfaces that students use to exchange information with the system to be decribed.

Use case diagram is used to describe main participants and the entire system architecture and relationships among them - LCMS and processes like registration of users with their characteristics like professional background, level of competence in subject domain, needs of education and preferred learning styles on one hand and content development as RLOs and capabilities for their editing, updating and deletion so proposed information to be accurate and up-to-date on the other. Another functionality is grades recording for assessment of learners performance. Main participants in the learning process are defined in their different roles: teacher, student, administrator and content developer according to their duties and responsibilities..

The most important relation is interaction of learning content developer that creates and edits learning materials as LOs and dynamic generated courses could be used by teachers and students as it is proposed in developed learning content model. Class diagrams are used to define user class (main participants in the system) with their attributes (characteristics) and operations (relationships with other classes and subcalsses). It can be used generalization to summarize common characteristics like it is shown on Diagram 1.

In this paper there is a class "User " with following attributes : username and password and other personal characteristics -name, e-mail address and so on.

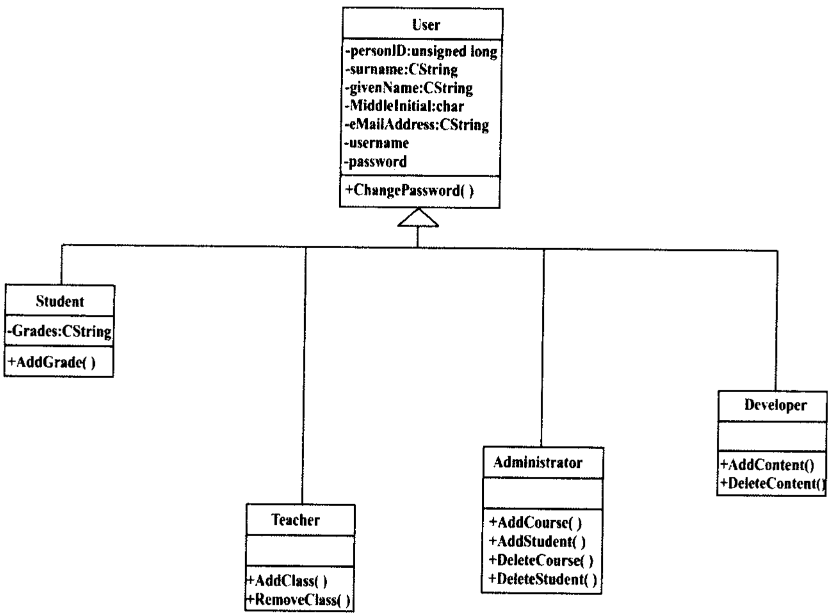

Diagram 1. User class and its generalized subclasses – student, teacher, developer and administrator with their common and specific attributes and operations

Class diagram is used to define attributes and operation for each class- teacher, student, administrator, developers. Each of them has username and password. Entities of teacher subclass has classes that have to be taught as attribute and operations for adding and removing classes. Student subclass has grades. Relationships between teacher, student and classes (courses) are defined by name of the association (user role) and multiplicity (1:1,1:M, N:M).

Databases that will be used are described with composition and generalization and types of learning content are defined as aggregation of text, video, audio and graphic components. That way it is applied recommendation of the proposed model learning resources to be created in different formats so they can be used in different contexts and that way to allow users with disabilities to have access to the proposed learning materials.

Diagrams of interaction like sequence and collaboration diagrams are used to describe messages that are sent between users and different parts of the system : database, learning objects repository (LOR), web interface and log files. They represent interactivity and communication among different parts presented in the first section of proposed model- learning environment with its databases, different types of communication among participant and LCMS components and subsystems - web interface, LOR and learning materials.

Another requirement to advanced LCMS is to deliver high quality learning content. For this purpose a view of systematic process of creation and publishing of learning content is presented to assure the necessary quality. Activity diagram on Diagram 2 defines the process of learning content submitting. For this purpose developed content should be created and after the process of approval it has to be published or edited. The processes described should be supported by each LCMS and diagram represents the flow of processes that should be implemented for development of high quality learning content.

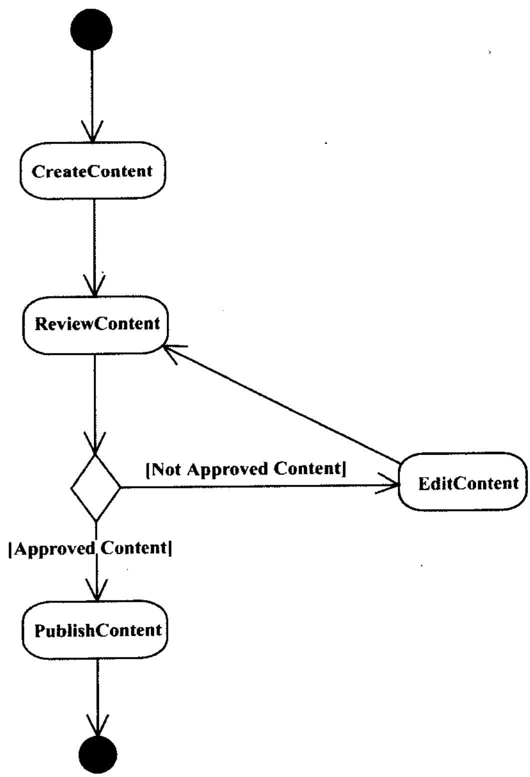

Diagram 2. Activity diagram describes processes related to content management.

Proper navigation is critical for effective education. It allows users to interact with different components of LCMS and with learning content. This way needed flexibility and adaptability are achieved. State chart diagram shown on Diagram 3 represents system menu and possibilities for navigation. It allows different users to manipulate with system components so they can execute their duties and fully used systems features and supported capabilities.

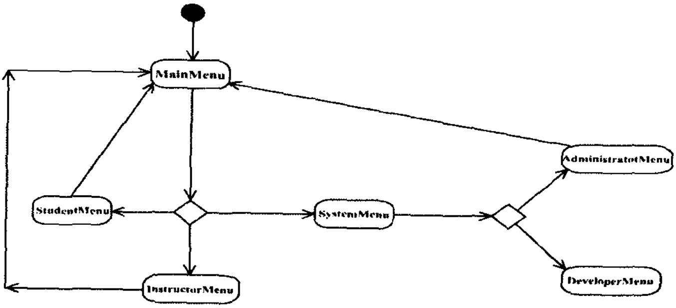

Diagram 3. Statechart diagram defines system menu

Following the proposed model a prototype of user interface has been developed that will provide support for LOs manipulation. The main functionalities related to learning content creation, arrangement and publishing are included in proposed menu of user interface. A menu that will be used by content developer is presented in Fig. 4 and in Fig. 5 – a menu that facilitates activities related to learning process management: creation, editing and deletion of courses and students.

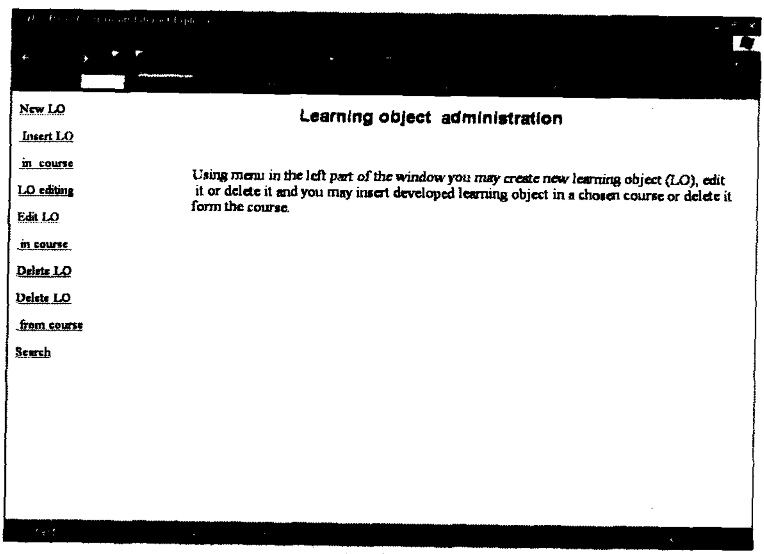

Fig. 4. Developer menu.

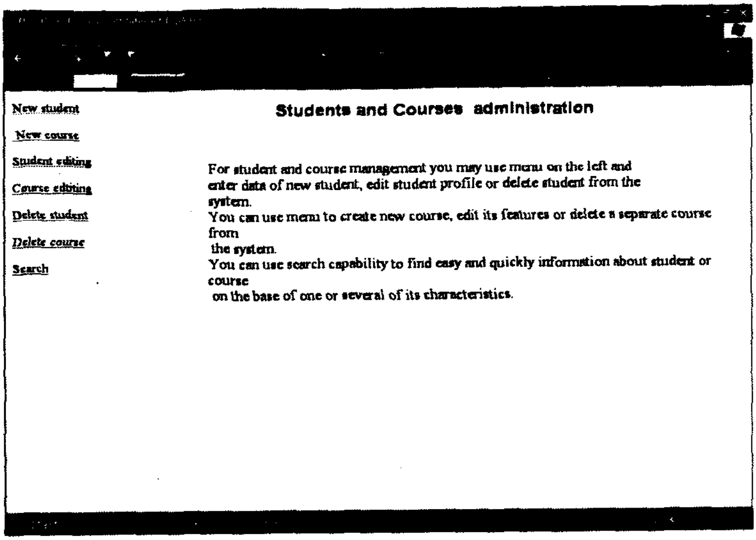

Fig.5 Administrator menu.

5. CONCLUSION

In this paper new learning content model is presented and discussed. Unified Process is applied to design a system that supports the model proposed. Modifications and advantages of developed model and their implementation in designed system are described by UML diagrams because UML is defined as the most appropriate and useful tool that has to be used in all phases of the UP. The conducted analysis of existing software tools for systems design and development of UML diagrams is presented.

A prototype system is designed following the UP methodology. Well designed systems that cover users' requirements and offer expected and necessary functionalities are critical components for effective education.

Different types of static and dynamic diagrams help system design to be executed so precisely and accurate defining of processes, states and relationships of system components and user interfaces to be gained. UML diagrams are used to describe proposed modified model and to document basic and users requirement to advanced LCMS. Learning Content Management System has to offer many functionalities and its parts, participant and communications among them can be well described and presented to users of the system and developers. Thus high quality of developed system and proper documentation for future improvements is achieved. Developed learning content model presents the idea of domain ontology use in learning objects management and description and how they can facilitate delivery of flexible and adaptive learning materials which is one of the most important tasks of advanced e-leaning. For providing described functionalities it is developed appropriate user interface.

Future work will be related to design and development of learning systems that support high interactivity with delivered learning materials, easy and useful collaboration among participants in the learning process and development and management of dynamic, reusable , flexible and adaptive learning content which can be used in different contexts according to learners needs of education. Another filed for future research is development of new models for learning content manipulation and implementation of learning systems that support proposed models.

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